Chapter 3

The Kerr solution

As we have seen, the solution of Einstein's equations describing the exterior of an isolated, spherically symmetric object (the Schwarzschild solution) is quite simple. Indeed, it has been found in 1916, immediately after the derivation of Einstein's equation. In the case of a rotating body, instead, the problem (which is very relevant: astrophysical bodies **do** rotate) is much more difficult: we don't know any analytic, exact solution describing the exterior of a rotating star (even if we know approximate solutions).

But we know the exact solution describing a rotating, stationary, axially symmetric *black hole*. It is the **Kerr solution**, derived in 1963 by R. Kerr. We say that this metric describes a black hole, because it is a solution of Einstein equations in vacuum ($T_{\mu\nu} = 0$) and it has a curvature singularity covered by an *horizon*: like in the case of Schwarzschild spacetime, everything falling inside the hole cannot escape.

We stress that while, thanks to Birkhoff theorem, the Schwarzschild metric for $r > 2M$ describes the exterior of any spherically symmetric isolated object (a star, a planet, a stone, etc.), the Kerr metric outside the horizon can only describe the exterior of a black hole.[1]

[1]Actually, there is no proof that it cannot exist a stellar model matching with Kerr metric at the surface of the star, but such a model has never been found, and it is common belief that it is unlikely to exist.

3.1 The Kerr metric in Boyer-Lindquist coordinates

The explicit form of the metric is the following:

$$\begin{aligned} ds^2 &= -dt^2 + \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) + (r^2 + a^2)\sin^2\theta d\phi^2 \\ &\quad + \frac{2Mr}{\Sigma}(a\sin^2\theta d\phi - dt)^2 \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} \Delta(r) &\equiv r^2 - 2Mr + a^2 \\ \Sigma(r,\theta) &\equiv r^2 + a^2\cos^2\theta\,. \end{aligned} \tag{3.2}$$

The coordinates (t, r, θ, ϕ) in which it is expressed the Kerr metric in (3.1) are called *Boyer-Lindquist coordinates*.

The Kerr metric depends on two parameters, M and a; comparing (3.1) with the far field limit metric of an isolated object (18.3), we see that M represents the mass of the black hole, and Ma its angular momentum, as measured from infinity.

Some properties of the Kerr metric can be directly seen by looking at the line element (3.1):

- It is *stationary*: it does not depend explicitly on time.
- It is *axisymmetric*: it does not depend explicitly on ϕ.
- It is not static: it is not invariant for time reversal $t \to -t$.
- It is invariant for simultaneous inversion of t and ϕ,

$$\begin{aligned} t &\to -t \\ \phi &\to -\phi\,, \end{aligned} \tag{3.3}$$

  as can be expected: the time reversal of a rotating object produces an object which rotates in the opposite direction.
- In the limit $r \to \infty$, the Kerr metric (3.1) reduces to Minkowski metric in polar coordinates; then, the Kerr spacetime is *asymptotically flat*.

- In the limit $a \to 0$ (with $M \neq 0$), it reduces to the Schwarzschild metric: $\Delta \to r(r-2M)$, $\Sigma \to r^2$ then

$$ds^2 \to -(1-2M/r)dt^2 + (1-2M/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{3.4}$$

- In the limit $M \to 0$ (with $a \neq 0$), it reduces to

$$ds^2 = -dt^2 + \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}dr^2 + (r^2 + a^2\cos^2\theta)d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2 \tag{3.5}$$

which is the metric of flat space in spheroidal coordinates:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{3.6}$$

where

$$\begin{aligned} x &= \sqrt{r^2 + a^2}\sin\theta\cos\phi \\ y &= \sqrt{r^2 + a^2}\sin\theta\sin\phi \\ z &= r\cos\theta\,. \end{aligned} \tag{3.7}$$

Indeed,

$$\begin{aligned} dx &= \frac{r}{\sqrt{r^2 + a^2}}\sin\theta\cos\phi dr + \sqrt{r^2 + a^2}\cos\theta\cos\phi d\theta - \sqrt{r^2 + a^2}\sin\theta\sin\phi d\phi \\ dy &= \frac{r}{\sqrt{r^2 + a^2}}\sin\theta\sin\phi dr + \sqrt{r^2 + a^2}\cos\theta\sin\phi d\theta + \sqrt{r^2 + a^2}\sin\theta\cos\phi d\phi \\ dz &= \cos\theta dr - r\sin\theta d\theta \end{aligned} \tag{3.8}$$

thus

$$\begin{aligned} & dx^2 + dy^2 + dz^2 = \left(\frac{r^2}{r^2 + a^2}\sin^2\theta + \cos^2\theta\right)dr^2 \\ & + \big((r^2 + a^2)\cos^2\theta + r^2\sin^2\theta\big)\,d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2 \\ & = \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}dr^2 + (r^2 + a^2\cos^2\theta)d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2\,. \end{aligned} \tag{3.9}$$

- The metric (3.1) is singular for $\Delta = 0$ and for $\Sigma = 0$. By computing the curvature invariants (like for instance $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$) one finds that they are regular at $\Delta = 0$, and singular at $\Sigma = 0$. Thus $\Sigma = 0$ is a true, curvature singularity of the manifold, whereas (as we will show) $\Delta = 0$ is a coordinate singularity.

Notice that in the Schwarzschild limit ($a = 0$), $\Sigma = r^2 = 0$ gives the curvature singularity, while (for $r \neq 0$) $\Delta = r(r - 2M) = 0$ gives the coordinate singularity at the horizon.

The metric has the form

$$g_{\mu\nu} = \begin{pmatrix} g_{tt} & 0 & 0 & g_{t\phi} \\ 0 & \frac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ g_{t\phi} & 0 & 0 & g_{\phi\phi} \end{pmatrix} \tag{3.10}$$

with

$$\begin{aligned} g_{tt} &= -\left(1 - \frac{2Mr}{\Sigma}\right) \\ g_{t\phi} &= -\frac{2Mr}{\Sigma} a \sin^2\theta \\ g_{\phi\phi} &= \left[r^2 + a^2 + \frac{2Mra^2}{\Sigma}\sin^2\theta\right]\sin^2\theta \,. \end{aligned} \tag{3.11}$$

The $g_{\phi\phi}$ component can be rewritten in a different way, which will be useful later:

$$\begin{aligned} g_{\phi\phi} &= (r^2 + a^2)\sin^2\theta + \frac{2Mra^2\sin^4\theta}{\Sigma} \\ &= \frac{\sin^2\theta}{\Sigma}\left[(r^2 + a^2\cos^2\theta)(r^2 + a^2) + 2Mra^2\sin^2\theta\right] \\ &= \frac{\sin^2\theta}{\Sigma}\left[(r^2 + a^2)^2 - (r^2 + a^2)a^2\sin^2\theta + 2Mra^2\sin^2\theta\right] \\ &= \frac{\sin^2\theta}{\Sigma}\left[(r^2 + a^2)^2 - a^2\sin^2\theta\Delta\right] \,. \end{aligned} \tag{3.12}$$

Let us compute the inverse metric. To get $g^{\mu\nu}$, we only have to invert the $t\phi$ block in (3.10), while the inversion of the $r\theta$ part is trivial. The metric in the $t\phi$ block is

$$\tilde{g}_{ab} = \begin{pmatrix} g_{tt} & g_{t\phi} \\ g_{t\phi} & g_{\phi\phi} \end{pmatrix} \tag{3.13}$$

and its determinant is

$$\tilde{g} = g_{tt}g_{\phi\phi} - g_{t\phi}^2$$

$$
\begin{aligned}
&= -\left(1-\frac{2Mr}{\Sigma}\right)\left[r^2+a^2+\frac{2Mra^2}{\Sigma}\sin^2\theta\right]\sin^2\theta - \frac{4M^2r^2a^2}{\Sigma^2}\sin^4\theta \\
&= -\left[r^2+a^2+\frac{2Mra^2}{\Sigma}\sin^2\theta\right]\sin^2\theta + (r^2+a^2)\frac{2Mr}{\Sigma}\sin^2\theta \\
&= -(r^2+a^2)\sin^2\theta + \frac{2Mr}{\Sigma}\sin^2\theta\left[-a^2\sin^2\theta+r^2+a^2\right] \\
&= -(r^2+a^2)\sin^2\theta + 2Mr\sin^2\theta = -\Delta\sin^2\theta
\end{aligned} \tag{3.14}
$$

therefore

$$
\tilde{g}^{ab} = -\frac{1}{\Delta\sin^2\theta}\begin{pmatrix} g_{\phi\phi} & -g_{t\phi} \\ -g_{t\phi} & g_{tt} \end{pmatrix} \tag{3.15}
$$

and

$$
g^{\mu\nu} = \begin{pmatrix} g^{tt} & 0 & 0 & g^{t\phi} \\ 0 & \frac{\Delta}{\Sigma} & 0 & 0 \\ 0 & 0 & \frac{1}{\Sigma} & 0 \\ g^{t\phi} & 0 & 0 & g^{\phi\phi} \end{pmatrix} \tag{3.16}
$$

with

$$
\begin{aligned}
g^{tt} &= -\frac{1}{\Delta}\left[r^2+a^2+\frac{2Mra^2}{\Sigma}\sin^2\theta\right] \\
g^{t\phi} &= -\frac{2Mr}{\Sigma\Delta}a \\
g^{\phi\phi} &= \frac{\Delta - a^2\sin^2\theta}{\Sigma\Delta\sin^2\theta}
\end{aligned} \tag{3.17}
$$

where we have used the fact that

$$
\frac{\Sigma - 2Mr}{\Sigma\Delta\sin^2\theta} = \frac{r^2+a^2\cos^2\theta - 2Mr}{\Sigma\Delta\sin^2\theta} = \frac{\Delta - a^2\sin^2\theta}{\Sigma\Delta\sin^2\theta}\,. \tag{3.18}
$$

3.2 Symmetries of the metric

Being stationary and axisymmetric, the Kerr metric admits two Killing vector fields:

$$
\underline{k} \equiv \frac{\partial}{\partial t} \qquad \underline{m} \equiv \frac{\partial}{\partial \phi} \tag{3.19}
$$

or equivalently, in coordinates (t, r, θ, ϕ),

$$
k^\mu \equiv (1,0,0,0) \qquad m^\mu \equiv (0,0,0,1)\,. \tag{3.20}
$$

In the motion of a particle with four-velocity u^μ, there are then two conserved quantities:

$$E \equiv -u^\mu k_\mu = -u_t \qquad L = u^\mu m_\mu = u_\phi \,. \tag{3.21}$$

In the case of massive particles, for which the four-momentum is $P^\mu = mu^\mu$, they are the energy at infinity per mass unit and the angular momentum per mass unit, respectively. In the case of massless particle, we can choose properly the affine parameter (as we will always do in the following) so that the four-momentum coincides with the four-velocity: $P^\mu = u^\mu$; thus, for massless particles E is the energy at infinity and L the angular momentum.

It can be shown that k^μ, m^μ are the only Killing vector fields of the Kerr metric; thus, any Killing vector field is a linear combination of them.

3.3 Frame dragging and ZAMO

Let us consider an observer, with timelike four-velocity u^μ, which falls into the black hole with zero angular momentum

$$L = u_\phi = 0 \,. \tag{3.22}$$

This implies that at $r \to \infty$, where the metric becomes flat, also $u^\phi = 0$, and its angular velocity is zero. Such observer is conventionally named ZAMO, which stands for "zero angular momentum observer". The *contravariant* ϕ component of the velocity does not vanish (except in the limit $r \to \infty$):

$$u^\phi = g^{\phi t} u_t \neq 0 \tag{3.23}$$

then the trajectory of the ZAMO has a non-zero angular velocity:

$$\Omega \equiv \frac{d\phi}{dt} = \frac{\frac{d\phi}{d\tau}}{\frac{dt}{d\tau}} = \frac{u^\phi}{u^t} \neq 0 \,. \tag{3.24}$$

To compute Ω in terms of the metric (3.1), which is given in *covariant* form, we use the fact that

$$u_\phi = 0 = g_{\phi\phi} u^\phi + g_{\phi t} u^t \tag{3.25}$$

thus

$$\Omega = \frac{u^\phi}{u^t} = -\frac{g_{\phi t}}{g_{\phi\phi}} \,. \tag{3.26}$$

We have

$$g_{\phi t} = -\frac{2Mra}{\Sigma}\sin^2\theta \tag{3.27}$$

and, due to (3.12),

$$g_{\phi\phi} = \frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)^2 - a^2\sin^2\theta\Delta\right] , \tag{3.28}$$

therefore the angular velocity of a ZAMO is

$$\Omega = \frac{2Mar}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta} . \tag{3.29}$$

Notice that

$$(r^2+a^2)^2 > a^2\sin^2\theta(r^2+a^2-2Mr) \tag{3.30}$$

thus we always have $\Omega/(Ma) > 0$: the angular velocity has the same sign as the angular momentum Ma of the black hole, namely, the motion of the ZAMO is corotating with the black hole.

We can conclude that an observer which approaches a Kerr black hole with a trajectory which has zero angular velocity at infinity (and then zero angular momentum) is *dragged* by the gravitational field of the black hole, acquiring an angular velocity corotating with the black hole.

3.4 Horizon structure of the Kerr metric

3.4.1 Removal of the singularity at $\Delta = 0$

To show that $\Delta = 0$ is a coordinate singularity, we make a coordinate transformation that brings the metric into a form which is not singular at $\Delta = 0$, and then extend the spacetime; such coordinates are called *Kerr coordinates*. They are the generalization, to rotating black holes, of the Eddington-Finkelstein coordinates derived in Schwarzschild spacetime. To begin with, we need to find two families of null geodesics, one ingoing and one outgoing, and to determine the corresponding null coordinates (u, v), i.e. the quantities which are constant in any of these geodesics. In the case of Kerr geometry, the spacetime cannot be decomposed in a product of two-dimensional manifolds, thus the study of null geodesics is more complex than in the Schwarzschild case. The Kerr metric admits two special families of null geodesics, named *principal null geodesics*, given by

$$u^\mu = \frac{dx^\mu}{d\lambda} = \left(\frac{dt}{d\lambda}, \frac{dr}{d\lambda}, \frac{d\theta}{d\lambda}, \frac{d\phi}{d\lambda}\right) = \left(\frac{r^2+a^2}{\Delta}, \pm 1, 0, \frac{a}{\Delta}\right) , \tag{3.31}$$

where the sign plus (minus) corresponds to outgoing (ingoing) geodesics. In the Schwarzschild limit these are the usual outgoing and ingoing geodesics $u^\mu = (1/(1-2M/r), \pm 1, 0, 0)$, but in the Kerr case they acquire an angular velocity $d\phi/d\lambda$ proportional to a and diverging when $\Delta = 0$.

We will show explicitly that (3.31) are geodesics later, in the coordinate frame we are going to define; here we check that they are null:

$$g_{\mu\nu} u^\mu u^\nu = 0\,. \tag{3.32}$$

We have

$$\begin{aligned} g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} &= -\left(\frac{dt}{d\lambda}\right)^2 + \Sigma\left(\frac{1}{\Delta}\left(\frac{dr}{d\lambda}\right)^2 + \left(\frac{d\theta}{d\lambda}\right)^2\right) \\ &\quad +(r^2+a^2)\sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2 + \frac{2Mr}{\Sigma}\left(a\sin^2\theta\frac{d\phi}{d\lambda} - \frac{dt}{d\lambda}\right)^2 . \end{aligned} \tag{3.33}$$

First, we notice that

$$\frac{dt}{d\lambda} - a\sin^2\theta\frac{d\phi}{d\lambda} = \frac{r^2+a^2-a^2\sin^2\theta}{\Delta} = \frac{\Sigma}{\Delta}\,. \tag{3.34}$$

Then,

$$\begin{aligned} g_{\mu\nu}u^\mu u^\nu &= -\frac{(r^2+a^2)^2}{\Delta^2} + \frac{\Sigma}{\Delta} + (r^2+a^2)\sin^2\theta\frac{a^2}{\Delta^2} + \frac{2Mr\Sigma}{\Delta^2} \\ &= \frac{1}{\Delta^2}\big[-(r^2+a^2)(r^2+a^2) + (r^2+a^2\cos^2\theta)(r^2+a^2-2Mr) \\ &\quad + \sin^2\theta a^2(r^2+a^2) + (r^2+a^2\cos^2\theta)2Mr\big] = 0 \end{aligned} \tag{3.35}$$

and the tangent vector (3.31) is null.

Let us consider the ingoing geodesics, whose tangent vector we call

$$l^\mu = \left(\frac{r^2+a^2}{\Delta}, -1, 0, \frac{a}{\Delta}\right) ; \tag{3.36}$$

let us parametrize the geodesics in terms of r:

$$\frac{dt}{dr} = -\frac{r^2+a^2}{\Delta} \qquad \frac{d\phi}{dr} = -\frac{a}{\Delta}\,. \tag{3.37}$$

We want these geodesics to be coordinate lines of our new system; thus, one of our coordinates is r, while the others are quantities which are constant along a geodesic of the family. One of these is θ, which is constant along the considered geodesics; the remaining two coordinates are given by

$$\begin{aligned} v &\equiv t + T(r) \\ \bar{\phi} &\equiv \phi + \Phi(r) \end{aligned} \tag{3.38}$$

where $T(r)$ and $\Phi(r)$ are solutions of[2]

$$\begin{aligned} \frac{dT}{dr} &= \frac{r^2+a^2}{\Delta} \\ \frac{d\Phi}{dr} &= \frac{a}{\Delta} \end{aligned} \tag{3.39}$$

so that, along a geodesic of the family,

$$\frac{dv}{dr} = \frac{d\bar{\phi}}{dr} \equiv 0 \tag{3.40}$$

and the tangent vector of the ingoing principal null geodesics (3.36) is, in the new coordinates, simply

$$l^\mu = (0,-1,0,0)\,. \tag{3.41}$$

We can now compute the metric tensor in the coordinate system $(v, r, \theta, \bar{\phi})$. We recall that, in Boyer-Lindquist coordinates,

$$ds^2 = -dt^2 + \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) + (r^2+a^2)\sin^2\theta d\phi^2 + \frac{2Mr}{\Sigma}(a\sin^2\theta d\phi - dt)^2\,. \tag{3.42}$$

We have

$$\begin{aligned} dv &= dt + \frac{r^2+a^2}{\Delta}dr \quad ; \quad dt = dv - \frac{r^2+a^2}{\Delta}dr \\ d\bar{\phi} &= d\phi + \frac{a}{\Delta}dr \quad ; \quad d\phi = d\bar{\phi} - \frac{a}{\Delta}dr\,, \end{aligned} \tag{3.43}$$

[2]Notice that (3.39) have an unique solution, with the only arbitrariness of the choice of the origins of v and $\bar{\phi}$, because the right-hand sides of (3.39) depend on r only.

then

$$
\begin{aligned}
-dt^2 &= -dv^2 - \frac{(r^2+a^2)^2}{\Delta^2}dr^2 + 2\frac{r^2+a^2}{\Delta}dvdr \\
(r^2+a^2)\sin^2\theta d\phi^2 &= (r^2+a^2)\sin^2\theta d\bar{\phi}^2 + (r^2+a^2)\frac{a^2}{\Delta^2}\sin^2\theta dr^2 \\
&\quad -2(r^2+a^2)\frac{a}{\Delta}\sin^2\theta dr d\bar{\phi}\,, \qquad (3.44)
\end{aligned}
$$

$\frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$ do not change (r, θ are also coordinates in the new frame), the parenthesis in the last term of (3.42) reduces to

$$
\begin{aligned}
dt - a\sin^2\theta d\phi &= dv - a\sin^2\theta d\bar{\phi} - \frac{r^2+a^2-a^2\sin^2\theta}{\Delta}dr \\
&= dv - a\sin^2\theta d\bar{\phi} - \frac{\Sigma}{\Delta}dr\,, \qquad (3.45)
\end{aligned}
$$

thus

$$
\begin{aligned}
&\frac{2Mr}{\Sigma}(dt - a\sin^2\theta d\phi)^2 = \frac{2Mr}{\Sigma}dv^2 + \frac{2Mr}{\Sigma}a^2\sin^4\theta d\bar{\phi}^2 \\
&-\frac{4Mr}{\Delta}dvdr + \frac{4Mr}{\Delta}a\sin^2\theta d\bar{\phi}dr + \frac{2Mr\Sigma}{\Delta^2}dr^2 - \frac{4Mr}{\Sigma}a\sin^2\theta dv d\bar{\phi}
\end{aligned}
\qquad (3.46)
$$

and, putting all together, we have that the coefficient of $dvdr$ is

$$
2\frac{r^2+a^2}{\Delta} - \frac{4Mr}{\Delta} = 2\,, \qquad (3.47)
$$

the coefficient of $d\bar{\phi}dr$ is

$$
-2(r^2+a^2)\frac{a}{\Delta}\sin^2\theta + \frac{4Mr}{\Delta}a\sin^2\theta = -2a\sin^2\theta\,, \qquad (3.48)
$$

and the coefficient of dr^2 is

$$
\begin{aligned}
&\frac{\Sigma}{\Delta} - \frac{(r^2+a^2)^2}{\Delta^2} + \frac{r^2+a^2}{\Delta^2}a^2\sin^2\theta + \frac{2Mr}{\Delta^2}(r^2+a^2\cos^2\theta) \\
&= \frac{\Sigma}{\Delta} - \frac{(r^2+a^2)(r^2+a^2-2Mr)}{\Delta^2} + \frac{r^2+a^2-2Mr}{\Delta^2}a^2\sin^2\theta \\
&= \frac{\Sigma}{\Delta} - \frac{r^2+a^2-a^2\sin^2\theta}{\Delta} = 0\,. \qquad (3.49)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
ds^2 &= -\left(1-\frac{2Mr}{\Sigma}\right)dv^2+2dvdr+\Sigma d\theta^2 \\
&\quad +\frac{(r^2+a^2)^2-\Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta d\bar\phi^2 \\
&\quad -2a\sin^2\theta drd\bar\phi-\frac{4Mra}{\Sigma}\sin^2\theta dvd\bar\phi\,. \qquad (3.50)
\end{aligned}
$$

The coordinates $(v,r,\theta,\bar\phi)$ are called **Kerr coordinates**. They reduce to the Eddington-Finkelstein coordinates for $a=0$. In this frame, it is easy to show that l^μ are tangent vector to geodesics; indeed, since $l^\mu=(0,-1,0,0)$,

$$
l^\nu l^\mu{}_{;\nu}=l^\nu l^\alpha\Gamma^\mu_{\nu\alpha}=\Gamma^\mu_{rr}=0 \quad\Leftrightarrow\quad \Gamma_{rr\mu}=0 \quad\Leftrightarrow\quad g_{\mu r,r}=g_{rr,\mu}=0 \qquad (3.51)
$$

and this is the case, because in (3.50) $g_{rr}=0$ and g_{vr}, $g_{\bar\phi r}$ do not depend on r.

In the Kerr coordinates, differently from the the Boyer-Lindquist coordinates, the metric is not singular at $\Delta=0$. Thus, after changing coordinates to the Kerr frame, we can *extend* the manifold, to include also the submanifold $\Delta=0$, and removing the corresponding coordinate singularity.

We note, for later use, that being

$$
g_{vr}=1 \qquad g_{rr}=g_{\theta r}=0 \qquad g_{\bar\phi r}=-a\sin^2\theta\,, \qquad (3.52)
$$

we have

$$
l_\mu=(-1,0,0,a\sin^2\theta)\,. \qquad (3.53)
$$

Notice also that, as we have shown above,

$$
\begin{aligned}
g_{\bar\phi\bar\phi} &= \frac{(r^2+a^2)^2-\Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta \\
&= (r^2+a^2)\sin^2\theta+\frac{2Mr}{\Sigma}a^2\sin^4\theta \qquad (3.54)
\end{aligned}
$$

and

$$
\frac{2Mr}{\Sigma}(dv-a\sin^2\theta d\bar\phi)^2=\frac{2Mr}{\Sigma}\left[dv^2+a^2\sin^4\theta d\bar\phi^2-2a\sin^2\theta dvd\bar\phi\right] \qquad (3.55)
$$

therefore the metric in Kerr coordinates can also be written in the simpler form

$$\begin{aligned} ds^2 &= -dv^2 + 2dvdr + \Sigma d\theta^2 + (r^2+a^2)\sin^2\theta d\bar{\phi}^2 - 2a\sin^2\theta dr d\bar{\phi} \\ &\quad + \frac{2Mr}{\Sigma}(dv - a\sin^2\theta d\bar{\phi})^2 \,. \end{aligned} \tag{3.56}$$

If we want an explicit time coordinate, we can define

$$\bar{t} \equiv v - r \tag{3.57}$$

so that the metric (3.56) becomes

$$\begin{aligned} ds^2 &= -d\bar{t}^2 + dr^2 + \Sigma d\theta^2 + (r^2+a^2)\sin^2\theta d\bar{\phi}^2 - 2a\sin^2\theta dr d\bar{\phi} \\ &\quad + \frac{2Mr}{\Sigma}(d\bar{t} + dr - a\sin^2\theta d\bar{\phi})^2 \,. \end{aligned} \tag{3.58}$$

3.4.2 The horizon

Here we study the submanifold

$$\Delta = r^2 + a^2 - 2Mr = 0 \,, \tag{3.59}$$

which is a coordinate singularity in Boyer-Lindquist coordinates

$$ds^2 = -dt^2 + \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) + (r^2+a^2)\sin^2\theta d\phi^2 + \frac{2Mr}{\Sigma}(a\sin^2\theta d\phi - dt)^2 \,. \tag{3.60}$$

When $a^2 > M^2$, the equation $\Delta = 0$ has no real solution. In this case there is no horizon, and the Kerr solution does not describe a black hole. In this situation, the singularity $\Sigma = 0$ is not "covered" by any horizon ("naked singularity"), and this would bring to paradoxes in our universe. For this reason, and for the reason that known astrophysical processes are believed to give rise to black holes with $|a| < M$, this situation is generally considered unphysical. Here and in the following, then, we will restrict our analysis to the case

$$a^2 \leq M^2 \tag{3.61}$$

(the limiting case $a^2 = M^2$ is called *extremal black hole*).

We have

$$\Delta(r) = (r - r_+)(r - r_-) \tag{3.62}$$

with

$$\begin{aligned} r_+ &\equiv M + \sqrt{M^2 - a^2} \\ r_- &\equiv M - \sqrt{M^2 - a^2} \end{aligned} \tag{3.63}$$

solutions of Eq. (3.60). The surfaces of the coordinate singularity $\Delta = 0$ are then $r = r_+$ and $r = r_-$.

Let us consider now the surfaces $\Theta \equiv r - constant = 0$, whose normal is

$$n_\mu = \Theta_{,\mu} = (0, 1, 0, 0) \, . \tag{3.64}$$

From (3.64), (3.16)

$$n_\mu n_\nu g^{\mu\nu} = g^{rr} = \frac{\Delta}{\Sigma} \, . \tag{3.65}$$

Thus, on the surfaces $r = r_+$ and $r = r_-$, where $\Delta = 0$, $n_\mu n^\mu = 0$, and these surfaces are *null hypersurfaces*, i.e. horizons. Being $r_+ > r_-$, we can say that $r = r_+$ is the *outer horizon*, and $r = r_-$ is the *inner horizon*. Actually, we should have used the Kerr coordinates to make this computation, but the result would be the same; indeed, the surfaces $r = const.$ are the same in the two frames, and the sign of $n_\mu n^\mu$ is the same as well. It can be easily checked that the result it the same by computing $n_\mu n^\mu$ in Kerr coordinates.

The two horizons separate the spacetime in three regions:

I. The region with $r > r_+$. Here the $r = const.$ hypersurfaces are timelike. The $r \to \infty$ limit, where the metric becomes flat, is in this region; so we can consider this region the exterior of the black hole.

II. The region with $r_- < r < r_+$. Here the $r = const.$ hypersurfaces are spacelike. An object which falls inside the outer horizon, can only continue falling to decreasing values of r, until it reaches the inner horizon and pass to region III.

III. The region with $r < r_-$. Here the $r = const.$ hypersurfaces are timelike. This region contains the singularity, which we will study in section 3.6.

In the case of extremal black holes, when $a^2 = M^2$, the two horizons coincide, and region II disappears.

If we consider the outer horizon r_+ as a sort of "surface" of the black hole, then we could conventionally consider the angular velocity at $r = r_+$ of an observer which falls radially from infinity - i.e., an observer with zero angular momentum, or ZAMO - as a sort of "angular velocity" of the black hole. The angular velocity of a ZAMO is given by (3.29):

$$\Omega = \frac{d\phi}{dt} = \frac{2Mar}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}\,. \tag{3.66}$$

At $r = r_+$, $\Delta = 0$ thus

$$\Omega = \frac{2Mar_+}{(r_+^2+a^2)^2} \equiv \Omega_H \tag{3.67}$$

which is a constant. In this sense, we can say that a black hole *rotates rigidly.*

The quantity $\Omega_H = \Omega(r_+)$ can be expressed in a simpler way. We have

$$(r_+ - M)^2 = M^2 - a^2 \tag{3.68}$$

therefore

$$r_+^2 + a^2 = 2Mr_+ \tag{3.69}$$

and

$$\Omega_H = \frac{a}{2Mr_+} = \frac{a}{r_+^2+a^2}\,. \tag{3.70}$$

3.5 The infinite redshift surface and the ergosphere

While in Schwarzschild spacetime the horizon is also the surface where g_{tt} changes sign, in Kerr spacetime these surfaces do not coincide. We have that

$$\begin{aligned} g_{tt} &= -1 + \frac{2Mr}{\Sigma} = -\frac{1}{\Sigma}\left(r^2 - 2Mr + a^2\cos^2\theta\right) \\ &= -\frac{1}{\Sigma}(r - r_{S+})(r - r_{S-}) = 0 \end{aligned} \tag{3.71}$$

when $r = r_{S+}$ and when $r = r_{S-}$, where

$$r_{S\pm} \equiv M \pm \sqrt{M^2 - a^2\cos^2\theta}\,. \tag{3.72}$$

These surfaces are called *infinite redshift surfaces*, because if a source located on a point P_{em} near the black hole emits a light signal with frequency ν_{em}, it will be observed at infinity with frequency

$$\nu_{obs} = \sqrt{\frac{g_{tt}(P_{em})}{g_{tt}(P_{obs})}}\nu_{em} \tag{3.73}$$

thus if at P_{em} $g_{tt} = 0$, $\nu_{obs} = 0$.

The coefficient of r^2 in (3.71) is negative, so $g_{tt} < 0$ outside $[r_{S_-}, r_{S_+}]$, and $g_{tt} > 0$ inside that interval. On the other hand, being $\sqrt{M^2 - a^2\cos^2\theta} \geq \sqrt{M^2 - a^2}$, the horizons, located at

$$r_\pm = M \pm \sqrt{M^2 - a^2}\,, \tag{3.74}$$

fall inside the interval $[r_{S_-}, r_{S_+}]$:

$$r_{S-} \leq r_- < r_+ \leq r_{S+}\,. \tag{3.75}$$

They coincide at $\theta = 0, \pi$, i.e. on the symmetry axis, while at the

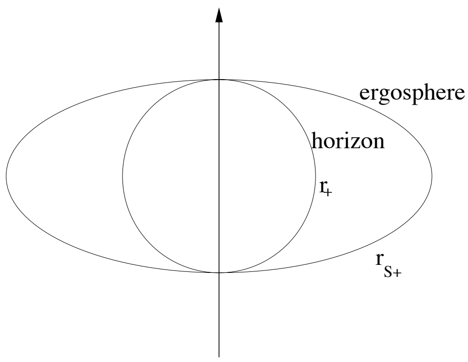

Figure 3.1: The ergosphere and the outer horizon

equatorial plan $r_{S+} = 2M$ and $r_{S-} = 0$.

Therefore, there is a region *outside the outer horizon* where $g_{tt} > 0$ [3]. This region, i.e.

$$r_+ < r < r_{S+} \tag{3.76}$$

is called *ergoregion*, and its outer boundary $r = r_{S_+}$ is called *ergosphere*. Notice that, being the ergosphere outside the outer horizon,

[3]This does not happen in Schwarzschild spacetime, where $g_{tt} > 0$ only inside the horizon

an observer from infinity can go inside the ergoregion and come back to infinity.

In the ergoregion the killing vector $k^\mu = (1, 0, 0, 0)$ becomes spacelike:

$$k^\mu k^\nu g_{\mu\nu} = g_{tt} > 0\,. \tag{3.77}$$

We define a *static observer* an observer (i.e. a timelike curve) with tangent vector proportional to k^μ. The coordinates r, θ, ϕ are constant along its worldline, therefore this observer is still in the Boyer-Lindquist coordinate system (3.60). Such an observer *cannot exist* inside the ergosphere, because k^μ is spacelike there; in other words, an observer inside the ergosphere cannot stay still, but is forced to move.

A *stationary observer* is an observer which does not see the metric change in its motion. Then, its tangent vector must be a killing vector, i.e. it must be a combination of the two killing vectors of the Kerr metric, $\underline{k} = \partial/\partial t$ and $\underline{m} = \partial/\partial\phi$:

$$u^\mu = \frac{k^\mu + \omega m^\mu}{|\underline{k} + \omega\underline{m}|} = (u^t, 0, 0, u^\phi) = u^t(1, 0, 0, \omega) \tag{3.78}$$

where we have defined the angular velocity of the observer

$$\omega \equiv \frac{d\phi}{dt} = \frac{u^\phi}{u^t}\,. \tag{3.79}$$

In other words, the worldline has constant r and θ. The observer can only move along a circle, with angular velocity ω. Indeed, in such orbits it does not see the metric change, being the spacetime axially symmetric.

A stationary observer can exist provided

$$u^\mu u^\nu g_{\mu\nu} = (u^t)^2\left[g_{tt} + 2\omega g_{t\phi} + \omega^2 g_{\phi\phi}\right] = -1 \tag{3.80}$$

i.e.

$$\omega^2 g_{\phi\phi} + 2\omega g_{t\phi} + g_{tt} < 0\,. \tag{3.81}$$

To solve (3.81), let us consider the equation

$$\omega^2 g_{\phi\phi} + 2\omega g_{t\phi} + g_{tt} = 0 \tag{3.82}$$

whose solutions are

$$\omega_\pm = \frac{-g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}{g_{\phi\phi}}\,. \tag{3.83}$$

The discriminant is (using eq.(3.14))

$$g_{t\phi}^2 - g_{tt}g_{\phi\phi} = \Delta \sin^2\theta \,. \tag{3.84}$$

Thus, a stationary observer cannot exist when $\Delta < 0$, i.e. inside the horizon $r_- < r < r_+$.

Being (see (3.30))

$$g_{\phi\phi} = \frac{\sin^2\theta}{\Sigma}[(r^2+a^2)^2 - a^2\sin^2\theta\Delta] > 0 \,, \tag{3.85}$$

the coefficient of ω^2 in (3.81) is positive, and the inequality (3.81) is satisfied, outside the outer horizon (where $r > r_+$, so $\Delta > 0$ and then $\omega_- < \omega_+$), for

$$\omega_- < \omega < \omega_+ \,. \tag{3.86}$$

On the outer horizon $r = r_+$, $\Delta = 0$ and $\omega_- = \omega_+$, so (3.81) has no solution, whereas equation (3.82), corresponding to a stationary null worldline (for instance, a photon), has one solution only; the only possible stationary null worldline on the horizon has

$$\omega = -\frac{g_{t\phi}}{g_{\phi\phi}} = \Omega_H \tag{3.87}$$

i.e. it has the ZAMO angular velocity. This is another reason why the angular velocity of the ZAMO at the horizon is considered as the black hole angular velocity: it is the only possible angular velocity of a stationary particle on the outer horizon.

On the infinite redshift surface, $g_{tt} = 0$ so (being $g_{t\phi} < 0$)

$$\omega_- = \frac{-g_{t\phi} - \sqrt{g_{t\phi}^2}}{g_{\phi\phi}} = 0 \,. \tag{3.88}$$

As expected, for $r \geq r_{S_+}$ $\omega_- \leq 0$, and $\omega = 0$ belongs to the interval (3.86), thus the static observer (which has $\omega = 0$) is allowed, while for $r < r_{S_+}$ $\omega_- > 0$ and the static observer is not allowed.

3.6 The singularity of the Kerr metric

Let us consider the curvature singularity

$$\Sigma = r^2 + a^2\cos^2\theta = 0 \,. \tag{3.89}$$

If we interpret the Boyer-Lindquist coordinates t, r, θ, ϕ as spherical polar coordinates, like in Schwarzschild spacetime, it is not clear at all *where* is the singularity: $\Sigma = 0$ at $r = 0$, $\theta = \pi/2$, not at $r = 0$, $\theta \neq \pi/2$, but this has no meaning in polar coordinates; we need a coordinate system which has not the coordinate singularity $r = 0$, so that we can distinguish and analyze the curvature singularity.

3.6.1 The Kerr-Schild coordinates

In order to understand the singularity structure, we now change coordinate frame, to the so-called Kerr-Schild coordinates, which are well defined in $r = 0$. Let us start with the metric in Kerr coordinates $(\bar{t}, r, \theta, \bar{\phi})$, given in eq. (3.58):

$$\begin{aligned} ds^2 &= -d\bar{t}^2 + dr^2 + \Sigma d\theta^2 + (r^2 + a^2)\sin^2\theta d\bar{\phi}^2 - 2a\sin^2\theta dr d\bar{\phi} \\ &\quad + \frac{2Mr}{\Sigma}(d\bar{t} + dr - a\sin^2\theta d\bar{\phi})^2 \,. \end{aligned} \tag{3.90}$$

The Kerr-Schild coordinates $(\bar{t}, x, y, z)$ are defined by

$$\begin{aligned} x &= \sqrt{r^2 + a^2}\sin\theta\cos\left(\bar{\phi} + \arctan\frac{a}{r}\right) \\ y &= \sqrt{r^2 + a^2}\sin\theta\sin\left(\bar{\phi} + \arctan\frac{a}{r}\right) \\ z &= r\cos\theta \,. \end{aligned} \tag{3.91}$$

In the next section we will derive the form of the metric in Kerr-Schild coordinates, showing that the coordinate singularity $r = 0$ can be removed in this frame; therefore, in this frame we only have the curvature singularity; to understand the structure of the curvature singularity, then, we must consider it in this frame.

We have

$$\begin{aligned} x^2 + y^2 &= (r^2 + a^2)\sin^2\theta \\ z^2 &= r^2\cos^2\theta \end{aligned} \tag{3.92}$$

thus

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1 \,, \tag{3.93}$$

then the surfaces with constant r are ellipsoids (Figure 3.2), and

$$\frac{x^2 + y^2}{a^2\sin^2\theta} - \frac{z^2}{a^2\cos^2\theta} = 1 \,, \tag{3.94}$$

then the surfaces with constant θ are half-hyperboloids (Figure 3.3).

In Figures 3.2, 3.3 we have represented the $r = const$, $\theta = const$

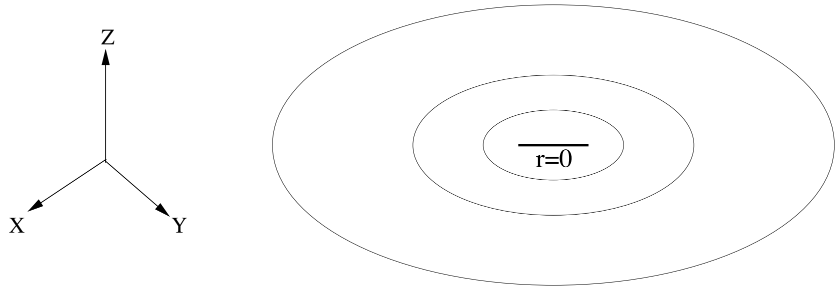

Figure 3.2: $r = const$ ellipsoidal surfaces in the Kerr-Schild frame; the thick line represents the $r = 0$ disk.

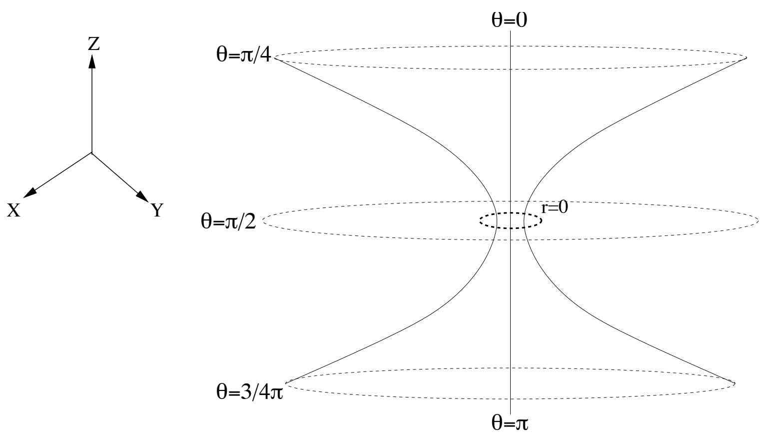

Figure 3.3: $\theta = const$ half-hyperboloidal surfaces in the Kerr-Schild frame; the thick ring represents the $r = 0, \theta = \pi/2$ singularity.

surfaces in the Kerr-Schild $(\bar{t}, x, y, z)$ frame. This means that x, y, z are represented as Euclidean coordinates, and r, θ are considered as functions of x, y, z.

Notice that if we look at Kerr spacetime where r is sufficiently large, the r, θ coordinates behave like ordinary polar coordinates. But closer to the black hole, their nature changes: $r = 0$ is not a single point but a disk,

$$x^2 + y^2 \leq a^2\,, \qquad z = 0 \tag{3.95}$$

and this disk is parametrized by the coordinate θ. In particular,

$$r = 0 \qquad \theta = \frac{\pi}{2} \tag{3.96}$$

corresponds to the *ring*

$$x^2 + y^2 = a^2\,, \qquad z = 0\,. \tag{3.97}$$

This is the structure of the singularity of the Kerr metric: it is a ring singularity. Inside the ring, the metric is perfectly regular.

3.6.2 The metric in Kerr-Schild coordinates

By calling $\alpha = \arctan a/r$, we have

$$r^2 \sin^2\alpha = a^2 \cos^2\alpha \tag{3.98}$$

thus

$$\begin{aligned} r^2 &= (r^2 + a^2)\cos^2\alpha \\ a^2 &= (r^2 + a^2)\sin^2\alpha \end{aligned} \tag{3.99}$$

and, rewriting (3.91) as

$$\begin{aligned} x &= \sin\theta\sqrt{r^2+a^2}(\cos\bar{\phi}\cos\alpha - \sin\bar{\phi}\sin\alpha) \\ y &= \sin\theta\sqrt{r^2+a^2}(\sin\bar{\phi}\cos\alpha + \cos\bar{\phi}\sin\alpha) \\ z &= r\cos\theta \end{aligned} \tag{3.100}$$

and substituting (3.99) we have

$$\begin{aligned} x &= \sin\theta(r\cos\bar{\phi} - a\sin\bar{\phi}) \\ y &= \sin\theta(r\sin\bar{\phi} + a\cos\bar{\phi}) \\ z &= r\cos\theta\,. \end{aligned} \tag{3.101}$$

Differentiating,

$$\begin{aligned} dx &= \cos\theta(r\cos\bar{\phi} - a\sin\bar{\phi})d\theta + \sin\theta\cos\bar{\phi}dr - \sin\theta(r\sin\bar{\phi} + a\cos\bar{\phi})d\bar{\phi} \\ dy &= \cos\theta(r\sin\bar{\phi} + a\cos\bar{\phi})d\theta + \sin\theta\sin\bar{\phi}dr + \sin\theta(r\cos\bar{\phi} - a\sin\bar{\phi})d\bar{\phi} \\ dz &= -r\sin\theta d\theta + \cos\theta dr \end{aligned} \tag{3.102}$$

thus

$$
\begin{aligned}
dx^2+dy^2+dz^2 &= dr^2+\left(r^2\sin^2\theta+(r^2+a^2)\cos^2\theta\right)d\theta^2 \\
&\quad +(r^2+a^2)\sin^2\theta d\bar{\phi}^2-2\sin^2\theta adrd\bar{\phi} \\
&= dr^2+\Sigma d\theta^2+(r^2+a^2)\sin^2\theta d\bar{\phi}^2-2a\sin^2\theta drd\bar{\phi}\,.
\end{aligned}
\tag{3.103}
$$

Then, the metric (3.90) is the Minkowski metric plus the term

$$\frac{2Mr}{\Sigma}(d\bar{t}+dr-a\sin^2\theta d\bar{\phi})^2\,. \tag{3.104}$$

Being

$$\Sigma=r^2+a^2\cos^2\theta=r^2+\frac{a^2z^2}{r^2}\,, \tag{3.105}$$

the factor $2Mr/\Sigma$ is easily expressed in Kerr-Schild coordinates:

$$\frac{2Mr}{\Sigma}=\frac{2Mr^3}{r^4+a^2z^2}\,. \tag{3.106}$$

The one-form $d\bar{t}+dr-a\sin^2\theta d\bar{\phi}$ is more complicate to transform. We will prove that

$$d\bar{t}+dr-a\sin^2\theta d\bar{\phi}=d\bar{t}+\frac{r(xdx+ydy)-a(xdy-ydx)}{r^2+a^2}+\frac{zdz}{r}\,. \tag{3.107}$$

First of all, let us express the differentials (3.102) as

$$
\begin{aligned}
dx &= \frac{\cos\theta}{\sin\theta}xd\theta+\sin\theta\cos\bar{\phi}dr-yd\bar{\phi} \\
dy &= \frac{\cos\theta}{\sin\theta}yd\theta+\sin\theta\sin\bar{\phi}dr+xd\bar{\phi} \\
dz &= -r\sin\theta d\theta+\cos\theta dr\,.
\end{aligned}
\tag{3.108}
$$

We have

$$
\begin{aligned}
xdx+ydy &= \frac{\cos\theta}{\sin\theta}(x^2+y^2)d\theta+\sin\theta(x\cos\bar{\phi}+y\sin\bar{\phi})dr \\
&= \sin\theta\cos\theta(r^2+a^2)d\theta+\sin^2\theta rdr
\end{aligned}
\tag{3.109}
$$

$$
\begin{aligned}
ydx-xdy &= -(x^2+y^2)d\bar{\phi}+\sin\theta(y\cos\bar{\phi}-x\sin\bar{\phi})dr \\
&= -(r^2+a^2)\sin^2\theta d\bar{\phi}+\sin^2\theta adr
\end{aligned}
\tag{3.110}
$$

$$zdx = -r^2\sin\theta\cos\theta d\theta+r\cos^2\theta dr \tag{3.111}$$

then

$$
\begin{aligned}
& (xdx+ydy)\frac{r}{r^2+a^2}+(ydx-xdy)\frac{a}{r^2+a^2}+\frac{zdz}{r} \\
= & \left(r\sin\theta\cos\theta d\theta+\frac{r^2}{r^2+a^2}\sin^2\theta dr\right) \\
& +\left(-a\sin^2\theta d\bar{\phi}+\frac{a^2}{r^2+a^2}\sin^2\theta dr\right) \\
& +\left(-r\sin\theta\cos\theta d\theta+\cos^2\theta d\theta\right) \\
= & \; dr-a\sin^2\theta d\bar{\phi}
\end{aligned}
\tag{3.112}
$$

which proves (3.107). The metric in Kerr-Schild coordinates is then

$$
\begin{aligned}
ds^2 = & -d\bar{t}^2+dx^2+dy^2+dz^2 \\
& +\frac{2Mr^3}{r^4+a^2z^2}\left[d\bar{t}+\frac{r(xdx+ydy)-a(xdy-ydx)}{r^2+a^2}+\frac{zdz}{r}\right]^2 .
\end{aligned}
\tag{3.113}
$$

Notice that the metric has the form

$$
g_{\mu\nu}=\eta_{\mu\nu}+Hl_\mu l_\nu \tag{3.114}
$$

with

$$
H\equiv\frac{2Mr^3}{r^4+a^2z^2} \tag{3.115}
$$

and, in Kerr-Schild coordinates,

$$
l_\mu dx^\mu=-\left(d\bar{t}+\frac{r(xdx+ydy)-a(xdy-ydx)}{r^2+a^2}+\frac{zdz}{r}\right) \tag{3.116}
$$

while in Kerr coordinates

$$
l_\alpha dx^\alpha=-\left(d\bar{t}+dr-a\sin^2\theta d\bar{\phi}\right)=-dv+a\sin^2\theta d\bar{\phi} \tag{3.117}
$$

thus l_μ is exactly the null vector (3.53), i.e. the generator of the principal null geodesics which have been used to define the Kerr coordinates. The form (3.114), called Kerr-Schild form, has been the starting point for Kerr to derive his solution.

3.6.3 Some strange features of the inner region of the Kerr metric

If we took seriously the Kerr metric and its ring singularity, we find some really weird features. We should keep in mind that what we are going to discuss has no direct link with astrophysical observations, since only the region $r > r_+$ is causally connected to us. Furthermore, we are considering an ethernal black hole, and it is unlikely that these properties apply also to astrophysical Kerr black holes, originating (at finite time) from gravitational collapse. It is only for completeness in our discussion on the Kerr metric that I briefly discuss such features, which, although fascinating, should not be taken too seriously.

Maximal extension of the Kerr metric

As we have discussed above, the Kerr metric in Kerr-Schild coordinates is

$$\begin{aligned} ds^2 &= -d\bar{t}^2 + dx^2 + dy^2 + dz^2 \\ &+ \frac{2Mr^3}{r^4 + a^2z^2}\left[d\bar{t} + \frac{r(xdx + ydy) - a(xdy - ydx)}{r^2 + a^2} + \frac{zdz}{r}\right]^2 \end{aligned} \tag{3.118}$$

where r is a function of $(\bar{t}, x, y, z)$, defined implicitly by

$$r^4 - (x^2 + y^2 + z^2 - a^2)r^2 - a^2z^2 = 0\,. \tag{3.119}$$

This metric is not singular inside the ring, i.e. at $r = 0$, $\theta \neq \pi/2$, or, equivalently,

$$r = 0\,,\; x^2 + y^2 < a^2\,. \tag{3.120}$$

It is singular at the ring $r = 0$, $\theta = \pi/2$, i.e. $r = 0$, $x^2 + y^2 = a^2$; this ring is a true curvature singularity: indeed the scalar invariant $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ diverges there.

We can then extend the spacetime manifold to $r = 0$, $\theta \neq \pi/2$, removing the coordinate singularity at the interior of the ring. As discussed in the case of Schwarzschild spacetime, we have to extend the spacetime manifold so that the geodesics can be extended across the ring itself. But this extension cannot simply consist in the inclusion of the hypersurface corresponding to the coordinate singularity (in this case, the interior of the ring), as we did to remove the horizon singularity.

To understand this problem, let us consider an observer falling to the center of the ring through the $\theta = 0$ axis; along its geodesic, $x = y = 0$ and $r = z$. It arrives at $z = r = 0$ (which is not

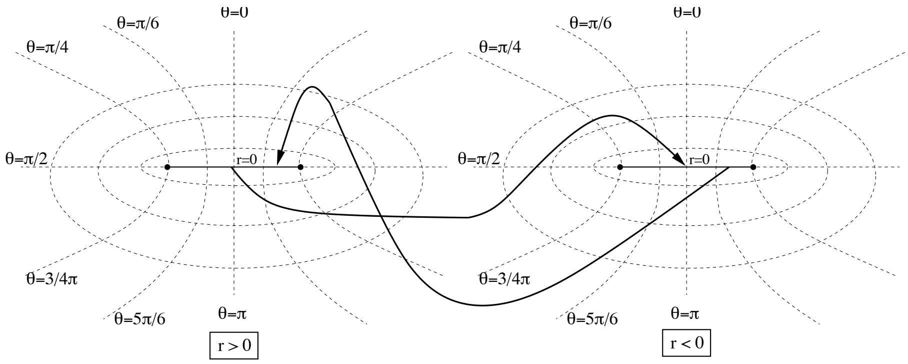

Figure 3.4: Removal of the disc singularity. The top of the disc in the spacetime with $r \geq 0$ (left panel) is identified with the bottom of the spacetime with $r \leq 0$ (right panel), and viceversa. Crossing the ring, an observer can pass from the $r > 0$ region to the $r < 0$ region (and, crossing again, come back to the $r > 0$ region) avoiding discontinuities in $dr/d\lambda$ and in θ.

a singularity of the spacetime) with $\theta = 0$ and a finite value of $dz/d\lambda = dr/d\lambda$; then, θ jumps to π and $dr/d\lambda$ changes sign. One could object that we should not worry about r, θ, since x, y, z, which are the coordinates in this frame, behave regularly; on the other hand, if we compute the curvature scalars, we find that they are discontinuous as we pass through the ring. Therefore, we have not really removed the coordinate singularity inside the ring.

Let us consider, now, the equation (3.119) for r: it admits two real solutions for r (there are other two, but they are complex conjugate), one positive and one negative. Therefore, for each set of Kerr-Schild coordinates there are two different real values of r, with opposite signs. We have then two different (asymptotically flat) spacetimes described by the metric (3.118), one with $r > 0$ and one with $r < 0$. The spacetime with $r < 0$ has no horizon, as can be easily verified studying the surfaces $r = const.$.

If we identify the top of the disc $x^2 + y^2 < a^2$, $z = 0$ in the spacetime with $r > 0$ with the bottom of the disc $x^2 + y^2 < a^2$, $z = 0$ in the spacetime with $r < 0$, and viceversa, as in Figure 3.4, we have really removed the coordinate singularity of the disc. Our observer, falling in the disc with r positive but decreasing, emerges on the top of the disc of the space with $r < 0$, $\theta = 0$. The observer, at this point, can escape to the asymptotically flat limit $r \to -\infty$.

Actually, the maximal extension of Kerr spacetime is even larger. A detailed study of geodesic completeness would go far beyond these lectures, we only give the final result. Requiring that all (timelike or null) geodesics which do not hit the curvature singularity can be extended, forward and backwards, for an infinite amount of the affine parameter, one finds that it is necessary to patch together an infinity of copies of Kerr spacetime, both with $r > 0$ and with $r < 0$. A schematic structure of the maximally extended spacetime (considered, for simplicity, along the $\theta = 0$ axis) is shown in Figure 3.5, where the regions I, II, III correspond to:

$$\begin{array}{ccll} I & : & r_+ < r < +\infty & \text{(exterior of the black hole)} \\ II & : & r_- < r < r_+ & \text{(where the } r = const. \text{ surfaces are spacelike)} \\ III & : & -\infty < r < r_- & \text{(ring singularity and } r < 0 \text{ space)}\,. \end{array}$$

The dashed hyperbolic curves corresponds to $r = 0$.

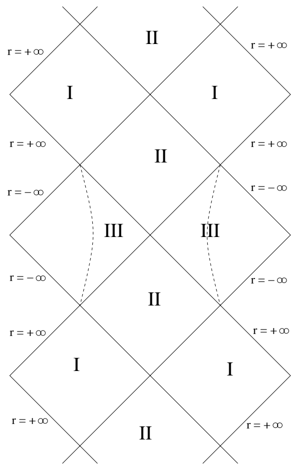

Figure 3.5: Schematic representation of the maximal extension of the Kerr metric, along the $\theta = 0$ axis. The dashed hyperbolic curves correspond to $r = 0$. We denote with I the exterior of the black hole, with II the regions between the inner and outer horizons, with III the inner regions where the $r = 0$ disc is located and the asymptotically flat region with $r < 0$.

The situation, then, is very different from that of the Schwarzschild spacetime, where the two asymptotically flat regions are causally disconnected; in Kerr spacetime an observer which falls inside the inner horizon can either cross the ring, escaping to the asymptotically flat region $r < 0$, or reach another copy of the region II, and then another copy of the region I, which is asymptotically flat with $r > 0$, and so on. Such copies of the region I are causally connected.

On the other hand, we should remind that this scheme only describes an eternal black hole. In the case of a black hole originating from a gravitational collapse this multiplication of spacetimes disappears; indeed, our region I cannot receive signals from a region II, because they should come from $t \to -\infty$, when the black hole was not yet born.

Causality violations

To conclude the discussion of the ring singularity of Kerr spacetime, we show another weird feature of the region close to the singularity.

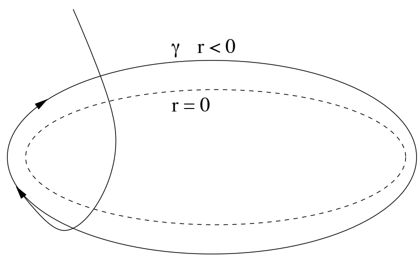

Figure 3.6: Close timelike curve in Kerr spacetime.

Let us consider a curve γ consisting in a ring just outside the singular ring, in the spacetime with $r < 0$:

$$\gamma : \quad \left\{ \bar{t} = const., \theta = \frac{\pi}{2}, 0 \le \bar{\phi} \le 2\pi, |r| \ll a, M, r < 0 \right\} . \quad (3.121)$$

The curve γ belongs to the inner region of the black hole, and can be reached by an observer that crosses the two horizons, pass through the $r = 0$ ring, and goes around it up to the $z = 0$ plane, just outside the ring (see Fig. 3.6).

The norm of the tangent vector to this curve is (since $\theta = \pi/2$ and then $\Sigma = r^2$)

$$m^\mu m^\nu g_{\mu\nu} \quad = \quad g_{\bar{\phi}\bar{\phi}} = g_{\phi\phi} = \frac{1}{r^2} \left((r^2 + a^2)^2 - a^2 (r^2 + a^2 - 2Mr) \right)$$

$$= \frac{1}{r^2}(r^4+r^2a^2+2Mra^2)=r^2+a^2+\frac{2Ma^2}{r} < 0 \quad (3.122)$$

since $r < 0$ and $|r| \ll a, M$. Therefore the curve γ is a timelike curve, and then can be interpreted as the worldline of an observer (even if it is not a geodesic), but it is also a closed curve; its existence is a causality violation: the observer meets itself in its own past.

The occurrence of closed timelike curves (CTC) in some solutions of Einstein's equations was first found by Kurt Godel, but Godel's solution was considered as unphysical. In the present case, instead, the CTC appears in a solution in some sense related to a physical process, i.e. the gravitational collapse.

Actually in a "real" rotating black hole, born in a gravitational collapse, the structure of ring singularity (and then the occurrence of CTCs) could be destroyed by the presence of the fluid and of an initial time of the singularity, but presently there is no definitive proof that this is the case. Therefore, while the presence of a collapsing fluid surely eliminates the multiple copies of spacetime in the maximal extension, it is not clear if it also eliminates the causality violations inherent to the ring singularity.

A possible point of view could be that of considering the problem of causality violation, together with the problem of the existence of a singularity (where some timelike geodesics end, in a finite amount of proper time), as inconsistencies of the theory of general relativity, which could disappear once a more fundamental theory (unifying general relativity with quantum field theory) will take its place. Indeed, quantum gravity effects are expected to be significant near the singularities.

In any case, we should not worry about this problem, since these CTCs occur inside the horizon, and then cannot be observed (at least, as long as we do not fall into a Kerr black hole); this is a further motivation for the cosmic censorship conjecture, which then protects us (and the consistency of the observable universe) not only from future singularities, but also from causality violations associated to the ring singularity.

3.7 General black hole solutions

In general, we can define a black hole as an asymptotically flat solution of Einstein's equations in vacuum with an horizon, and a curvature singularity inside the horizon. Black holes form in the gravitational collapse of stars, if they are sufficiently massive.

When a star has collapsed producing a black hole, we can expect that, after some time, it settles down to a stationary state as a result of gravitational waves emission. It is then reasonable to consider **stationary** black holes (i.e. black holes admitting a killing vector field which is timelike at r sufficiently large).

There are some remarkable theorems on stationary black holes, derived by S. Hawking, W. Israel, B. Carter, which prove the following:

- A stationary black hole is axially symmetric.
- Any stationary, axially symmetric black hole, without electric charge, is described by the Kerr solution.
- Any stationary, axially symmetric black hole is described by the so-called **Kerr-Newman solution**, which is a generalization of Kerr solution with nonvanishing electric charge and nonvanishing electromagnetic fields, characterized by the mass M, the angular momentum J, and the charge Q of the black hole.

Furthermore, we remark that any *static* black hole is spherically symmetric, and, if it has no electric charge, it is described by the Schwarzschild solution; in presence of electric charge, it is described by the **Reissner-Nordstrom** solution, which is the non-rotating limit of the Kerr-Newman solution, and is characterized by the mass M and the charge Q. We have not considered in these lectures the Reissner-Nordstrom and the Kerr-Newman solutions because it is widely believed that they are not astrophysically relevant; indeed, if an astrophysical black hole has an electric charge, it would likely lose it in a very short timescale, due to the interactions with the surrounding matter.

We can conclude that a general stationary black hole is characterized by three quantities only: the mass M, the angular momentum J, and the charge Q. All other features of the star which has collapsed to the black hole are not features of the final black hole. This result has been summarized with the sentence: "**A black hole has no hair**". For this reason, the unicity theorems are also called *no hair theorems.*