Chapter 5

Black hole thermodynamics

The study of black holes in general relativity has allowed to discover that there are deep links among different fields of physics: general relativity, thermodynamics, and quantum field theory.

In this Chapter we use the normalization convention $E = -P^\mu k_\mu$, which we have followed in Section 4.1.3 when discussing the Penrose process.

5.1 Limits for energy extraction from a Kerr black hole

Let us consider a Kerr black hole. As we have shown in Section 4.1.3, it is possible to extract energy from the black hole through the Penrose process.

In the process described in Section 4.1.3, a massive particle with energy $E$ and angular momentum $L$ falls from infinity into the black hole; in the ergoregion, it decays in two particles, with

\begin{align*}
E &= E_1 + E_2 \\
L &= L_1 + L_2
\end{align*}

(5.1)

and

\begin{align*}
V_+(r) < E_1 < 0 & \quad L_1 < 0.
\end{align*}

(5.2)

The particle 2 comes back to infinity, while the photon 1 falls into the black hole. In this way, the mass $M$ and the angular momentum $J = Ma$ of the black hole have changed by

\begin{align*}
\delta M &= E_1 = E - E_2 < 0
\end{align*}
\[ \delta J = L_1 = L - L_2 < 0; \quad (5.3) \]

these quantities have been extracted from the black hole.

If the angular momentum of the particle 1 is fixed, the energy which can be extracted is

\[ \delta M = E_1 \geq V_+(r) \geq V_+(r_+) = L_1 \frac{2 Mr_+}{(r_+^2 + a^2)^2} = L_1 \Omega_H = \delta J \Omega_H \quad (5.4) \]

where the inequality is saturated if the particle 1 is produced at \( r = r_+ \) with \( E_1 = V_+(r_+) \), i.e. it is produced at the horizon and released with infinitesimally small radial velocity.\(^1\)

We remind that the angular velocity of the horizon can be expressed as

\[ \Omega_H = \frac{a}{r_+^2 + a^2}. \quad (5.5) \]

thanks to the relation

\[ r_+^2 + a^2 = 2 Mr_+ \quad (5.6) \]

thus

\[ \delta M \geq \delta J \frac{a}{r_+^2 + a^2}. \quad (5.7) \]

The meaning of this relation is clear: the larger is the angular momentum extracted by the black hole, the larger is its slowing down, and the larger is the energy which can be extracted by the black hole.

Being \( J = Ma \), we have

\[ M \delta M \geq \frac{J \delta J}{r_+^2 + a^2} \quad (5.8) \]

therefore

\[ \delta \left( M^2 - \frac{j^2}{r_+^2 + a^2} \right) \geq 0. \quad (5.9) \]

Indeed,

\[ \delta \left( M^2 - \frac{j^2}{r_+^2 + a^2} \right) = 2 \left( M \delta M - \frac{j \delta J}{r_+^2 + a^2} \right) + \frac{2J^2}{(2Mr_+)^2} \delta (Mr_+) \quad (5.10) \]

\(^1\)It can be shown that the inequality (5.4) and the fact that it saturates when the particle is released from the horizon with \( \dot{r} = 0 \) remain true when the particle 1 is massive, but the proof is more complicate.
and

\[
\delta(Mr_+) = \delta(M^2 + \sqrt{M^4 - J^2}) = 2M\delta M + \frac{2M^3\delta M - J\delta J}{\sqrt{M^4 - J^2}} = 2M\delta M + \frac{2M^2\delta M - J\delta J/M}{r_+ - M} = \frac{2r_+}{r_+ - M} \left( M\delta M - \frac{J\delta J}{2Mr_+} \right).
\]

(5.11)

therefore

\[
\delta \left( M^2 - \frac{J^2}{r_+^2 + a^2} \right) = \left( 2 + \frac{4J^2r_+}{(r_+ - M)(2Mr_+)^2} \right) \left( M\delta M - \frac{J\delta J}{r_+^2 + a^2} \right) \geq 0.
\]

(5.12)

We can conclude that there is a quantity which always increases in these transformations. This quantity can be expressed in a simpler way, by using (5.6):

\[
M^2 - \frac{J^2}{r_+^2 + a^2} = M^2 \left( 1 - \frac{a^2}{r_+^2 + a^2} \right) = \frac{M^2r_+^2}{r_+^2 + a^2} = \frac{r_+^2 + a^2}{4}.
\]

(5.13)

therefore if we define the irreducible mass

\[
M_{irr} \equiv \sqrt{\frac{r_+^2 + a^2}{2}}.
\]

(5.14)

we have

\[
\delta M_{irr} \geq 0
\]

(5.15)

and

\[
M_{irr}^2 = M^2 - \frac{J^2}{4M_{irr}^2}.
\]

(5.16)

The irreducible mass cannot decrease; the only possible process (of sending a particle into the black hole) which leaves it constant is a process in which the particle is slowly released from the horizon; such processes can be reversed (by sending another particle with positive energy and angular momentum, leaving \(M_{irr} \) constant), i.e. are reversible processes, while all the others are irreversible processes.
If we want to extract the maximum possible amount of energy from a black hole with has initially mass $M$ and angular momentum $J$, we must perform a sequence of reversible Penrose processes, until all the angular momentum has radiated away, and the black hole has become Schwarzschild. The mass of the final black hole (which has $J = 0$) is equal to its irreducible mass, which has not changed:

$$M_{f_{in}}^2 = M_{irr}^2 = \frac{r_+^2 + a^2}{4} = \frac{Mr_+}{2}$$

(5.17)

therefore the extracted mass-energy, divided by the initial mass, is

$$\frac{\Delta M}{M} = \frac{M - M_{f_{in}}}{M} = 1 - \sqrt{\frac{r_+}{2M}}$$

(5.18)

(if the processes are not reversible, $M_{irr}$ increases, thus $M_{f_{in}}$ is larger and the extracted energy is smaller than in the case of reversible processes.)

Being $r_+$ a decreasing function of $a$, the energy extracted is larger as the initial black hole angular momentum is larger. If $a = 0$ (i.e. the initial black hole is Schwarzschild), $r_+ = 2M$ and $\Delta M/M = 0$; on the contrary, if $a = M$ (i.e. the initial black hole is extremal), $r_+ = M$ and

$$\frac{\Delta M}{M} = 1 - \frac{1}{\sqrt{2}} = 0.29 :$$

(5.19)

the 29% of the mass-energy of the initial extremal black hole has been extracted!

A deeper physical understanding of the quantity $M_{irr}$ is given by computing the area of the black hole, i.e. of the 2-surface with $t = const.$ and $r = r_+$. The metric restricted on this surface is

$$d\sigma^2 = (r_+^2 + a^2 \cos^2 \theta) d\theta^2 + \frac{(r_+^2 + a^2 \sin^2 \theta)}{r_+^2 + a^2 \cos^2 \theta} d\phi^2$$

(5.20)

therefore the area is

$$A = \int d\theta d\phi (r_+^2 + a^2) \sin \theta = 4\pi (r_+^2 + a^2) = 16\pi M_{irr}^2 .$$

(5.21)

Therefore, the irreducible mass represents the area of the black hole. We can conclude that the area of a Kerr black hole (but the same
results easily generalizes to all black hole or multi-black hole space-times) can only increase, or remain constant under a reversible process:
\[ \delta A \geq 0. \]  
(5.22)

### 5.2 The laws of black hole thermodynamics

The notion of reversible and irreversible processes, following from the fact that there is a quantity which can only increase or remain constant (eq. (5.22)), is also a characteristic of thermodynamical processes: (5.22) corresponds to the II\textsuperscript{nd} law of thermodynamics. This correspondence, which is, at this point of our discussion, only a formal analogy, can be extended to the I\textsuperscript{st} law of thermodynamics, as we are going to show.

Let us consider the relation
\[ \delta M \geq \Omega_H \delta J. \]  
(5.23)

It relates the change in mass-energy to the change of another variable characterizing the black hole, i.e. the angular momentum. The missing term, needed to make (5.23) an equality, should be related to change in the irreducible mass, i.e. in the area, which can be considered as another variable characterizing the black hole. We can look for a differential relation between \( M, J \) and \( A \), which can be found by differentiating
\[ M^2 = M^2_{\text{irr}} + \frac{J^2}{4M_{\text{irr}}^2} = \frac{A}{16\pi} + \frac{4\pi J^2}{A} \]  
(5.24)

getting
\[ dM = \frac{\partial M}{\partial J} dJ + \frac{\partial M}{\partial A} dA. \]  
(5.25)

We have
\[ \frac{\partial M}{\partial J} = \frac{1}{2M} \frac{\partial M^2}{\partial J} = \frac{J 4\pi}{M A} = \frac{a}{r^2_a + a^2} = \Omega_H \]  
(5.26)

and
\[ \frac{\partial M}{\partial A} = \frac{1}{2M} \frac{\partial M^2}{\partial A} = \frac{1}{2M} \left( \frac{1}{16\pi} - \frac{4\pi J^2}{A^2} \right) \]
\begin{align*}
= \frac{1}{32\pi M} \left(1 - \frac{4M^2a^2}{(r_+^2 + a^2)^2}\right) &= \frac{1}{32\pi M} \left(1 - \frac{a^2}{r_+^2}\right) \\
= \frac{1}{32\pi M} \left(2 - \frac{r_+^2 + a^2}{r_+^2}\right) &= \frac{1}{32\pi M} \left(2 - \frac{2M}{r_+}\right) \\
= \frac{1}{8\pi} \frac{r_+ - M}{2Mr_+} &= \frac{1}{8\pi} \frac{\sqrt{M^2 - a^2}}{r_+^2 + a^2} = \frac{\kappa}{8\pi} \text{ (5.27)}
\end{align*}

where we have defined the surface gravity \(\kappa\) as

\[\kappa \equiv \frac{\sqrt{M^2 - a^2}}{r_+^2 + a^2}.\text{ (5.28)}\]

(It can be shown that \(\kappa\) is the acceleration of a ZAMO at the horizon; notice that \(\kappa\) is a decreasing function of \(a\): \(\kappa = 0\) for extremal Kerr black holes, \(\kappa = \frac{1}{4M}\) for Schwarzschild black holes.)

Therefore, we have

\[\delta M = \Omega_H \delta J + \frac{\kappa}{8\pi} \delta A \text{ (5.29)}\]

which closely resembles the 1st law of thermodynamics. Notice that the term \(\Omega_H \delta J\) is analogous to the work term \(-P \delta V\) in the first law of thermodynamics; indeed, in the case of a rotating fluid, a term \(\Omega \delta J\) appears together with the work term in the first law.

The relations (5.29), (5.22) correspond to the first and second laws of thermodynamics, if we consider the area of the black hole as a sort of entropy related to the black hole, i.e. we define

\[S_{BH} \equiv \alpha A \text{ (5.30)}\]

where \(\alpha\) is a constant with dimensions of length to the minus two (taking units in which the Boltzmann constant is 1, so that the entropy is dimensionless); following the analogy, we define a temperature of the black hole as

\[T_{BH} \equiv \frac{\kappa}{8\pi \alpha} \text{ (5.31)}\]

so that

\[\delta M = \Omega_H \delta J + T_{BH} \delta S_{BH}. \text{ (5.32)}\]

This is the 1st law of black hole thermodynamics, while

\[\delta S_{BH} \geq 0 \text{ (5.33)}\]
is the II$^{nd}$ law of black hole thermodynamics.

Summarizing, the correspondence between properties of black holes and of thermodynamical systems is the following:

\[-PdV \leftrightarrow \Omega_H dJ\]
\[dU \leftrightarrow dM\]
\[TdS \leftrightarrow \frac{\kappa}{8\pi} dA = T_{BH} dS_{BH}.\]

The process of reduction of $J$ corresponds to the expansion of a fluid; if this is done with a sequence of a reversible processes (in which $S_{BH}$ is constant), it corresponds to an adiabatic expansion, which determines a reduction of the internal energy; if, instead, $S_{BH}$ increases, it corresponds to an expansion in which we give heat to the fluid, and then the reduction of internal energy is smaller.

We have shown these laws for the particular case of a Kerr black hole, but they can be proved under much more general assumption: they hold for any black hole solution, even for multi-black hole solutions. In particular, in the coalescence of two black holes, the area of the final black hole is always greater than the sum of the areas of the initial black holes: this is analogous to what happens to the entropy of a thermodynamical system made up from two subsystems.

We mention that it has been shown that also the III$^{rd}$ law of thermodynamics has a black hole analogue: it is impossible to reach $T_{BH} = 0$ by any physical process, i.e. by accreting matter or energy onto the black hole. We remark that $T_{BH} \propto \kappa$, thus (see eq. (5.28)) $T_{BH}$ is a decreasing function of $a$, and $T_{BH} = 0$ for extremal Kerr black holes; therefore, astrophysical black holes cannot be spun up to extremality by accretion.

More trivial is the case of the zeroth law of thermodynamics, which states that a body in thermal equilibrium has uniform temperature; it corresponds, for black holes, to the statement that $T_{BH}$ is constant over the horizon of a stationary black hole (indeed, the surface gravity $\kappa$ does not depend on $\theta, \phi$).

5.3 The generalized II$^{nd}$ law of thermodynamics

When it was first proposed, the analogy (discussed in the previous Section) between the quantities $S_{BH}, T_{BH}$ and the thermodynamical entropy and temperature seemed to be a mere formal analogy. Then,
in the 60’s, Bekenstein conjectured that this correspondence is based on real physical motivations.

The Bekenstein reasoning is the following. Let us consider a certain quantity of a fluid having thermodynamical entropy \( S_{\text{fluid}} \). This entropy has a statistical interpretation: it is the logarithm of the number of different microscopical states of the fluid corresponding to the same (macroscopical) thermodynamical state. On the other hand, when the fluid falls into the black hole, there are no more different microscopical states: as stated by the no-hair theorem (see Section 3.7), a black hole is characterized only by its mass and angular momentum (and, eventually, by its electric charge); the entropy \( S_{\text{fluid}} \) has disappeared, and the total entropy of the universe is apparently decreased, violating the \( \text{II}^{\text{nd}} \) law of thermodynamics.

The only possible solution to this paradox is to assign an entropy to the black hole (i.e. \( S_{\text{BH}} \) defined in (5.30)), which increases when the fluid falls inside, and to conjecture that the \( \text{II}^{\text{nd}} \) law of thermodynamics must be generalized to

\[
\delta S_{\text{fluid}} + \delta S_{\text{BH}} \geq 0.
\]

The entropy \( S_{\text{BH}} \), then, is a measure of the information inside the black hole, which cannot be seen from outside.

This conjecture doesn’t tell us which is the value of the constant \( \alpha \), but some further reasoning can suggest at least its order of magnitude.

Let us consider a Penrose process. As we have seen, the inequality \( \delta S_{\text{BH}} \geq 0 \) can be saturated in this process only if the particle is released from the horizon with zero radial velocity. But quantum mechanics tells us that we cannot locate the particle (with \( \dot{r} = 0 \)) exactly at \( r = r_+ \); there is an uncertainty in the position, which is of the order of the Compton wavelength

\[
\lambda = \frac{\hbar}{E_{\text{part}}}.\]

Therefore, when a particle is thrown into the black hole, the process cannot be reversible: there is a minimum amount of area increase due to the fact that the particle has been released at \( r \neq r_+ \), at a proper distance from the horizon of the order \( \sim \lambda \)

\[
\delta A \geq \delta A_{\text{min}}.
\]
and therefore there is a minimum amount of entropy increase

\[ \delta S_{BH} \geq \delta S_{BH}^{\min} = \alpha \delta A_{\min}. \]  

(5.37)

If we identify this minimum amount of entropy increase, intrinsically related to the fact that a particle has been swallowed by the black hole, with the minimum entropy of the particle itself, which is of the order of \( \ln 2 \) (i.e. a bit of information\(^2\)), we get a qualitative estimate \( \alpha \sim \ln 2 / \delta A_{\min} \).

Let us compute \( \delta A_{\min} \), i.e. the area increase when a particle of mass \( m \) falls into the black hole, starting with \( \dot{r} = 0 \) at \( r = r_+ + \delta r \), where \( \delta r \) is given by

\[ \lambda = \int_{r_+}^{r_+ + \delta r} \sqrt{g_{rr}} \, dr. \]  

(5.38)

We assume for simplicity that the motion of the particle is equatorial \( \theta = \pi/2 \), and that \( L = 0 \); then the particle is a ZAMO, and its energy measured by a ZAMO is \( m \), thus \( \lambda = h/m \).

We have

\[ g_{rr} = \frac{\Sigma}{\Delta} = \frac{r^2}{(r - r_+)(r - r_-)} \]  

(5.39)

and defining \( r = r_+ + x \) (with \( x \ll r_+ \)),

\[ g_{rr} = \frac{r_+^2}{r_+ - r_-} \frac{1}{x} + \ldots \]  

(5.40)

thus

\[ \lambda = \frac{r_+}{\sqrt{r_+ - r_-}} \int_0^{\delta r} \frac{dx}{\sqrt{x}} + \ldots = 2 \sqrt{\delta r} \frac{r_+}{\sqrt{r_+ - r_-}} + \ldots \]  

(5.41)

and then

\[ \delta r = \frac{1}{4} \left( \frac{r_+ - r_-}{r_+^2} \right) \lambda^2 + O(\lambda^3). \]  

(5.42)

Since \( L = 0 \), \( V_\pm = 0 \), and the massive particle follows an equatorial geodesic; then the condition \( \dot{r} = 0 \) gives

\[ r^2 = \frac{C}{r^2} \left( \frac{E}{m} \right)^2 - \frac{\Delta}{r^2} = 0 \]  

(5.43)

\(^2\)If we add to the system a bit of information, namely, a subsystem which can have two possible configurations, the space of possible states is doubled, and the logarithm of the number of states is increased by \( \ln 2 \).
(where we divide $E$ by $m$ because we have a different normalization of $E$ from Chapter 4), and the increase of the mass of the black hole is

$$
\delta M = E = m\sqrt{\frac{\Delta}{C}}. \tag{5.44}
$$

Let us compute $\Delta$ and $C$ by replacing $r = r_+ + \delta r$. We have

$$
\Delta(r) = (r-r_+)(r-r_-) = (r_+ - r_-)\delta r + O(\delta r^2) = \left(\frac{r_+ - r_-}{2r_+}\right)^2 \lambda^2 + O(\lambda^3)
$$

and

$$
C = r^2 + a^2 + \frac{2Ma^2}{r} = r_+^2 + a^2 + \frac{2Ma^2}{r_+} + O(\lambda^2)
= 2Mr_+ + \frac{2Ma^2}{r_+} + O(\lambda^2)
= \frac{2M}{r_+}(r_+^2 + a^2) + O(\lambda^2) = 4M^2 + O(\lambda^2). \tag{5.46}
$$

Therefore,

$$
\delta M \simeq m\frac{r_+ - r_-}{4Mr_+} \lambda = \frac{1}{2} \frac{r_+ - r_-}{r_+^2 + a^2} m\lambda
= \frac{\sqrt{\lambda^2 - a^2}}{r_+^2 + a^2} m\lambda = \kappa m\lambda = \kappa \hbar. \tag{5.47}
$$

The increase of the area in this process is then (since $L = 0$ and then $\delta J = 0$)

$$
\delta A = \frac{1}{\delta M} \frac{\delta M}{\delta A} \simeq \frac{\kappa \hbar}{\kappa/8\pi} = 8\pi \hbar \tag{5.48}
$$

and then the order of magnitude of the minimum amount of the area increase is

$$
\delta A_{\text{min}} \sim 8\pi \hbar. \tag{5.49}
$$

We have then

$$
\alpha \sim \frac{\ln 2}{8\pi \hbar} \sim \frac{1}{\hbar}. \tag{5.50}
$$

Note that in the units $c = G = 1$, $[\hbar] = l^2$.

Actually, we could have derived the estimate (5.50) in a much simpler (even if less well founded) way, through dimensional considerations: the constant $\alpha$, as we said above, has dimensions $l^{-2}$; the
only fundamental dimensionful constant in the theory\(^3\) is \(\hbar\), therefore it must be (5.50); note that in these dimensional considerations we are assuming that all dimensionless constants in our theory are of the order of unity.

If we define the Planck length

\[
l_P = \sqrt{\hbar} = 1.61 \times 10^{-33} \text{cm}
\]

(5.51)

we have \(\alpha \sim 1/l_P^2\). We can express the constant \(\alpha\) in terms of a dimensionless constant \(\hat{\alpha}\) as \(\alpha = \hat{\alpha}/\hbar\), therefore

\[
S_{BH} = \hat{\alpha} \frac{A}{\hbar} = \hat{\alpha} \frac{A}{l_P^2}.
\]

(5.52)

In the Bekenstein conjecture, the (still undetermined) dimensionless constant \(\hat{\alpha}\) must be of the order of unity.

The black hole temperature, in terms of \(\hat{\alpha}\), is

\[
T_{BH} \equiv \frac{\hbar \kappa}{8\pi \hat{\alpha}}.
\]

(5.53)

Note that Bekenstein’s conjecture gives a sort of convincing “thermodynamical” interpretation of \(S_{BH}\), but no interpretation of \(T_{BH}\): how is it possible to assign a temperature to an object which, by definition, has no emission?

Indeed, the generalized II\(^{nd}\) law of thermodynamics presents a problem. If there is a black hole with \(M, T_{BH}\) surrounded by radiation with \(U, T\), and some radiation enters into the black hole with no angular momentum,

\[
dU = TdS, \quad dM = T_{BH}dS_{BH}
\]

(5.54)

with \(dU = -dM\). Therefore, due to the generalized II\(^{nd}\) law of thermodynamics,

\[
dS_{BH} + dS = \frac{dM}{T_{BH}} + \frac{dU}{T} = \left(\frac{1}{T_{BH}} - \frac{1}{T}\right)dM \geq 0.
\]

(5.55)

This means that if \(T_{BH} < T\), \(dM \geq 0\), as expected; but if \(T_{BH} > T\), then \(dM \leq 0\), i.e. there is emission of energy from the black hole.

How can a black hole emit?

\(^3\)Actually, in the geometric units which are most commonly used, it is assumed that \(\hbar = c = 1\), while \(G\) has dimensions \(l^2\).
5.4 The Hawking radiation

A decisive proof of Bekenstein’s conjecture, together with a precise determination of the constant $\hat{\alpha}$, came from the discovery, due to S. Hawking, that if quantum mechanics is taken into account, a black hole can radiate, the so-called Hawking radiation.

Furthermore, the spectrum of the Hawking radiation is

$$N_\omega = \frac{1}{e^{\frac{2\pi\omega}{\hbar c}} - 1}$$

which is a blackbody spectrum

$$N_\omega = \frac{1}{e^{\frac{\hbar \omega}{\kappa T_{BH}}} - 1}$$

with temperature

$$T_{BH} \equiv \frac{\hbar \kappa}{2\pi}.$$ 

This result provides a convincing interpretation of the black hole temperature, and determines the precise value of the constant $\hat{\alpha}$, which, as supposed by Bekenstein, is of the order of unity:

$$\hat{\alpha} = \frac{1}{4}.$$  

The subscript $BH$ in the black hole entropy and temperature does not stands for “black hole”, but for “Bekenstein-Hawking”.

Summarizing, if quantum mechanics is taken into account, black holes radiate, with blackbody spectrum at temperature $T_{BH}$. When matter and energy enter into the black hole, their entropy becomes inaccessible from the exterior, but it manifests as an increase in the area of the black hole, with the law

$$S_{BH} = \frac{1}{4} A.$$ 

If a black hole interacts with the surrounding matter and energy, the generalized II\textsuperscript{nd} law of thermodynamics holds:

$$\delta S_{BH} + \delta S \geq 0.$$ 

For practical purposes, the thermal emission of astrophysical black holes is negligible, since

$$T_{BH} \sim 10^{-6} \frac{M_{\odot}}{M} K \ll T_{CMB}$$
where $T_{CMB}$ is the temperature of the cosmic microwave background. Only an extremely small black hole could have $T_{BH} > T_{CMB}$.

If the black hole is hotter than the surrounding radiation, the Hawking emission determines a decrease of the black hole mass; then, the area of a black hole decreases, but the entropy of the emitted radiation guarantees that the generalized $\text{II}^{\text{nd}}$ law of thermodynamics (5.61) is not violated. For astrophysical black holes, this doesn’t happen since they are colder than the surrounding radiation, but a mini-black hole would be hotter enough to decrease its mass, becoming then hotter and hotter until complete evaporation. This could create in principle a problem: the quantum information on the state entering the black hole would disappear once the black hole evaporates, but this “information loss” would be in conflict with the unitarity of the evolution in quantum mechanics, which is a fundamental principle, necessary for the consistence of quantum mechanics itself. It is still under debate whether black holes evaporation really determines information loss, and it is not at all clear which is the solution of this problem.

Anyway such small, evaporating black holes cannot be originated by gravitational collapse. In principle, they could have been produced in the big bang; in current cosmological model these primordial black holes are not expected, but they are not excluded.

A rigorous derivation of the Hawking radiation spectrum would be beyond the scope of these notes. In the rest of the Chapter we describe the main passages and concepts in this derivation, showing where does the blackbody spectrum comes from.

### 5.4.1 Quantum fields in Minkowski spacetime

Let us consider a classical massless scalar field $\phi(x)$ (we denote by $x \equiv x^\mu = (t, \mathbf{x})$), defined in Minkowski spacetime, satisfying the Klein-Gordon equation:

$$\Box_F \phi \equiv \eta^\mu{}^\nu \partial_\mu \partial_\nu \phi = 0. \quad (5.63)$$

The procedure to quantize $\phi$ is the following. First, we write the general solution of the classical field in terms of its Fourier components,

$$\phi(x) = \int_0^\infty \left[ \phi_\omega(x) e^{-i\omega t} + \phi_\omega^*(x) e^{i\omega t} \right] d\omega \quad (5.64)$$

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where we have written separately the components with positive frequencies $e^{-i\omega t}$ and with negative frequencies $e^{i\omega t}$ (by definition, $\omega \geq 0$).

Second, we build, at each $t$, a complete basis \{\(f_r\)\} of the time-independent equation \(\nabla^2 \phi_\omega + \omega^2 \phi_\omega = 0\); here \(r\) is a discrete or a continuum index, depending on the boundary conditions. For simplicity of notation, we consider the case in which it is a discrete index. A typical choice is the plane wave basis

\[
\phi(t, x) = \sum_k \left[ a_k f_k(x) e^{-i\omega_k t} + a_k^* f_k^*(x) e^{i\omega_k t} \right] (5.66)
\]

where we have defined

\[
g_k(x) \equiv f_k(x) e^{-i\omega_k t} . \tag{5.67}
\]

Notice that \(\{g_r, g_r^*\}\) is a complete basis of the solution space of \(\Box \phi = 0\).

Third, we quantize the field \(\phi\), namely, we consider \(\phi\) to be an operator field, acting on the Hilbert space \(\mathcal{H}\) of the physical states. We follow the Heisenberg convention, in which the states are constant and the operators depend on time. We require that the field satisfies the equal time commutation relations

\[
[\phi(t, x), \partial_t \phi(t, y)] = i\hbar \delta^{(3)}(x - y) . \tag{5.68}
\]

The observables are computed by taking expectation values of the corresponding operators in a given state. For instance, the value of the scalar field in the state \(|\alpha\rangle\) is

\[
< \alpha | \phi | \alpha > . \tag{5.69}
\]

The expansion (5.66) becomes

\[
\phi(x) = \sum_k \left[ a_k g_k(x) + a_k^\dagger g_k^*(x) \right] (5.70)
\]
where now $a_k$, $a_k^\dagger$ are operators.

The equal time commutation relations (5.68) are equivalent to

$$\begin{align*}
[a_k, a_{k'}^\dagger] &= \delta_{kk'} \\
[a_k, a_k] &= 0 \\
[a_{k'}^\dagger, a_{k'}^\dagger] &= 0.
\end{align*}$$

(5.71)

The relations (5.71) allow the interpretation of $a_k^\dagger, a_k$ as creation and annihilation operators respectively, which, when applied to a state, create or annihilate a particle with momentum $\hbar k$ and energy $\hbar \omega = \hbar |k|$.

The vacuum $|0 \rangle$ is, by definition, the state which is annihilated by all operators $a_k$

$$a_k|0 \rangle = 0 \quad \forall k.$$  

(5.72)

The physical space is then constructed by applying arbitrary products of creation operators $a_k^\dagger$ to $|0 \rangle$. For instance, a single particle state is given by

$$|1_k \rangle = a_k^\dagger |0 \rangle.$$  

(5.73)

A state with $n$ particles having momentum $\hbar k$ and $m$ particles having momentum $\hbar k'$ is

$$|n_k, m_{k'} \rangle = \frac{(a_k^\dagger)^n (a_{k'}^\dagger)^m}{\sqrt{n! m!}} |0 \rangle.$$  

(5.74)

The number density of particles with momentum $\hbar k$ in a given state can be found by taking the expectation value of the number operator

$$N_k \equiv a_k^\dagger a_k.$$  

(5.75)

The total number of particles is

$$N = \sum_k N_k.$$  

(5.76)

For instance, by applying (5.71) it is easy to show that

$$< n_k, m_{k'} | N | n_k, m_{k'} > = n + m.$$  

(5.77)

The expectation value of the field $\phi$ in a one-particle state is

$$< 1_k | \phi(x) | 1_k > = g_k(x) = f_k(x) e^{-i\omega t}.$$  

(5.78)
We stress that any $|\alpha \rangle \in \mathcal{H}$ is a state encoding the field configuration in the entire spacetime.

It can be shown that the energy expectation value of the state $|1_k \rangle$ is $\hbar \omega$. Therefore, we are forced to associate positive frequency solutions with annihilation operators $a$, and negative frequency solutions with creation operators $a^\dagger$: if there is an annihilation operator associated to a negative frequency solution, there would exist a negative energy particle; a state with more and more of such particles would be energetically favoured with respect to the vacuum state; the theory would then be unstable.

We remark that the concept of particle (and then the concept of vacuum, i.e. the state without particles) depends in principle on the way the creation and annihilation operators $a^\dagger, a$ have been defined. Therefore, it depends on the choice of the basis of solutions $\{g_r\}$ of the Klein-Gordon equation. We can choose two different sets of solutions $\{g_r\}, \{g'_s\}$ (where $r, s$ are arbitrary indexes, which can be different from $k$), such that both $\{g_r, g^*_r\}$ and $\{g'_s, g^*_s\}$ are bases for the solution space of the Klein-Gordon equation. We expand the field $\phi$ as

$$
\phi(x) = \sum_r (a_r g_r + a^\dagger_r g^*_r) = \sum_s (b_s g'_s + b^\dagger_s g^{*'}_s) . \tag{5.79}
$$

The operators $a^\dagger_r, a_r, b^\dagger_s, b_s$ satisfy the relations

\[
\begin{align*}
[a_r, a^\dagger_{r'}] &= \delta_{rr'} \\
[a_r, b_{r'}] &= 0 \\
[a^\dagger_r, a^\dagger_{r'}] &= 0 \\
[b_s, b^\dagger_{s'}] &= \delta_{ss'} \\
[b_s, b_{s'}] &= 0 \\
[b^\dagger_s, b^\dagger_{s'}] &= 0
\end{align*}
\tag{5.80}
\]

therefore can be interpreted as creation and annihilation operators. Thus, they define the vacuum states $|0 \rangle_g, |0 \rangle_{g'}$, by

$$
a_r |0 \rangle_g = 0 \quad \forall r \\
b_s |0 \rangle_{g'} = 0 \quad \forall s . \tag{5.81}
$$
But there is a constraint: as explained above, the functions \( \{g_r\} \), \( \{g'_s\} \) must have positive frequency, namely, they must be combinations of the positive frequency modes \( \{u_j\} \) of the Klein-Gordon equation, defined by
\[
\frac{\partial u_j}{\partial t} = -i\omega_j u_j \tag{5.82}
\]
with \( \omega_j > 0 \). For simplicity we assume they are energy eigenfunctions, thus they satisfy
\[
\frac{\partial g_r}{\partial t} = -i\omega_r g_r, \quad \frac{\partial g'_s}{\partial t} = -i\omega_s g'_s. \tag{5.83}
\]
This implies that we can write
\[
g'_s = \sum_r \alpha_{rs} g_r \tag{5.84}
\]
and therefore
\[
a_r = \sum_s \alpha_{rs} b_s. \tag{5.85}
\]
Thus, the state \(|0 >_{g'}\), satisfying \(b_s|0 >_{g'} = 0 \) \(\forall s\), also satisfies \(a_r|0 >_{g'} = 0 \) \(\forall r\) and vice versa; then
\[
|0 >_g = |0 >_{g'}. \tag{5.86}
\]
Furthermore, it can be shown that if we make a Lorentz transformation, changing the reference frame to another inertial frame with time coordinate \(t'\), the positive frequency solutions remain positive frequency solutions, the negative frequency solutions remain negative frequency solutions. The result (5.86), then, still holds if the two expansion bases have been defined in two different inertial frames.

We can conclude that in Minkowski spacetime the definition of the vacuum state is unique, and Lorentz invariant.

### 5.4.2 Quantum fields in a general spacetime

Let us consider a general, non-Minkowskian spacetime.

It is simple to generalize the Klein-Gordon equation (5.63):
\[
\Box \phi \equiv g^{\mu\nu} \phi_{,\mu\nu} = 0. \tag{5.87}
\]
The subsequent steps, instead, are not trivial, because the equal time commutation relations (5.68) involve the time, and in a general
spacetime there is no global definition of time. We do not discuss here the way this problem is handled. We only remark that also the definition of positive frequency modes, (5.83), is based on the notion of time, and then we cannot give a precise definition of a particle in presence of a gravitational field.

Still, if a spacetime is asymptotically flat, we can define positive frequency modes in the regions where it approaches Minkowski spacetime. In these regions, then, we can define particles, and a vacuum state. But these definitions need not to coincide in the different Minkowskian regions.

Let us consider, for instance, a spacetime with two asymptotically flat regions, $I$ (in the past) and $II$ (in the future). It can happen that the solutions of the Klein-Gordon equation which have positive frequency in region $I$, have not positive frequency in region $II$. Therefore, if $|0 \rangle_I$ is the vacuum of region $I$, and $|0 \rangle_{II}$ is the vacuum of region $II$, in general

$$|0 \rangle_I \neq |0 \rangle_{II}. \quad (5.88)$$

If there are no particles in the past, this means that the state is $|0 \rangle_I$, but then there can be particles in the future, i.e. in region $II$. In this way, the gravitational field can create particles.

Let us state this more precisely. Given two sets of functions $\{g_r\}$, $\{g'_s\}$, such that both $\{g_r, g^*_r\}$ and $\{g'_s, g^*_s\}$ are bases for the solution space of the Klein-Gordon equation, and such that both $\{g_r\}$, $\{g'_s\}$, have positive frequency if considered in two different Minkowskian regions of spacetime,

$$g'_s = \sum_r [\alpha_{rs} g_r + \beta_{rs} g^*_r]. \quad (5.89)$$

If we expand the quantum field $\phi$ as in (5.79),

$$\phi(x) = \sum_r (a_r g_r + a^+_r g^*_r)$$

$$= \sum_s (b_s g'_s + b^+_s g'^*_s) \quad (5.90)$$

then

$$a_r = \sum_s [\alpha_{rs} b_s + \beta^*_{rs} b^+_s]. \quad (5.91)$$
The coefficients $\alpha_{rs}, \beta_{rs}$ are called Bogoliubov coefficients.

The vacuum state related to the expansion $\{g_r\}, |0>_g$, in general is not a vacuum state with respect to the expansion $\{g'_s\}$: in this state there are particles created by $b^+_s$. The number of particles for each $s$ is

$$ g < 0 |b^+_s a_s| 0>_g = \sum_r |\beta_{rs}|^2. \tag{5.92} $$

To prove (5.92), the following properties of the Bogoliubov coefficients (which we give without proof)

$$ \sum_k (\alpha_{ik} \alpha^*_{jk} - \beta_{jk} \beta^*_{ik}) = \delta_{ij} $$
$$ \sum_k (\alpha_{ik} \beta^*_{jk} - \beta^*_{ik} \alpha_{jk}) = 0 \tag{5.93} $$

must be used, in order to invert the transformation (5.91), thus finding

$$ b_s = \alpha^*_{ks} a_k - \beta^*_{ks} a^+_k. \tag{5.94} $$

Then,

$$ g < 0 |b^+_s a_s| 0>_g = \sum_k |\beta_{ks}|^2 $$
$$ g < 0 |a_k a^+_k| 0>_g = \sum_k |\beta_{ks}|^2. \tag{5.95} $$

Therefore, if in the past (region $I$) there are no particles, i.e. the state is $|0>_g$, then in the future (region $II$), where the correct vacuum state is $|0>_g'$, there is a nonvanishing distribution of particles; we can consider that this distribution has been created by the gravitational field.

The Unruh effect

A similar phenomenon happens in flat space with an accelerated observer. In this case, we have two observers, one inertial and one accelerated, each of them with a globally defined time coordinate. This allows to define solutions of the Klein-Gordon equation with positive frequency with respect to the inertial observer, $\{g_r\}$, and solutions with positive frequency with respect to the accelerated observer, $\{g'_s\}$.

Therefore, it is possible to define particles in both reference frames, through (5.90), but the two definitions are different.
In the state $|0 >_g$ (vacuum as seen by the inertial observer), the accelerated observer detects a non-vanishing particle distribution, given by (5.92). By performing the explicit calculation, it can be shown that this distribution is a thermal particle spectrum. This is called *Unruh effect*.

### 5.4.3 Particle creation in Schwarzschild spacetime

Let us consider the case of Schwarzschild spacetime. Due to spherical symmetry, the solutions of the Klein-Gordon equation have the form

$$\phi(x) = \frac{1}{r} f(t, r) Y_{lm}(\theta, \phi)$$

with $Y_{lm}$ spherical harmonic with indexes $l, m$, and $f(t, r)$ solution of

$$\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial r^2} + V(r)f = 0,$$

where

$$V(r) \equiv \left(1 - \frac{2M}{r}\right) \left[\frac{(l + 1)}{r^2} + \frac{2M^2}{r^3}\right],$$

and

$$r_* = r + 2M \ln \left(\frac{r}{2M} - 1\right)$$

is the usual tortoise coordinate.

At $r_* \to \pm \infty$ (i.e. $r \to \infty$ and $r \to 2M$), $V(r) \to 0$, and the Klein-Gordon equation reduces to

$$\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial r_*^2} = 0$$

whose general solution is a function of the form

$$f = f_1(u) + f_2(v) \quad (r_* \to \pm \infty)$$

where $u, v$ are the usual null outgoing and ingoing coordinates:

$$u = t - r_*$$
$$v = t + r_*$$

(at finite $r_*$, instead, $f = f(t, r)$ or, equivalently, $f = f(u, v)$).

In the Schwarzschild spacetime there are two null asymptotically flat limits. One, the null past infinity, corresponding to $u \to -\infty$;
the other, the null future infinity, corresponding to \( v \to +\infty \). Conventionally, one denotes the asymptotic 3-dimensional manifold at \( u \to -\infty \) (described by the coordinates \( v, \theta, \phi \)) by \( I^- \), and the asymptotic 3-dimensional manifold at \( v \to +\infty \) (described by the coordinates \( u, \theta, \phi \)) by \( I^+ \). The Eddington-Finkelstein diagram of the Schwarzschild spacetime with the asymptotic 3-surfaces \( I^+, I^- \) is shown in Figure 5.1.

On the surface \( I^- \), the classical field \( f(u,v) \) depends only on \( v = t + r_* \), i.e. it describes an ingoing wave, while on the surface \( I^+ \), the field \( f(u,v) \) depends only on \( u = t - r_* \), i.e. it describes an outgoing wave. In particular, if \( \{g_r\} \) is a positive frequency basis \( (\sim e^{-i\omega t}) \) in the Minkowskian region \( u \to -\infty \), and \( \{g'_s\} \) is a positive frequency basis in the Minkowskian region \( v \to +\infty \), then

\[
\begin{align*}
g_r &\sim e^{-i\omega v} \quad \text{on } I^- \quad \text{on } I^- \\
g'_s &\sim e^{-i\omega u} \quad \text{on } I^+ \quad \text{on } I^+,
\end{align*}
\]

because they both should depend on \( t \) as \( \sim e^{-i\omega t} \), and \( g_r \), on \( I^- \), is a function of \( v = t + r_* \), whereas \( g'_s \), on \( I^+ \), is a function of
\[ u = t - r^*. \] We stress that \( \{g_r\}, \{g'_r\} \) are functions defined in the entire spacetime; eqns. (5.103) give their asymptotic behaviours at the two null infinity limits.

The surface \( I^- \) is what is called a \textit{Cauchy hypersurface}. This means that if we know the value of the fields at \( I^- \) we know the fields in the entire spacetime. Indeed, in the far past we can have ingoing fields coming from \( I^- (u \to -\infty) \), but we cannot have fields coming from \( v \to -\infty \), because nothing can come from the horizon. Thus, the evolution of the field is completely determined by its value at \( I^- \). Therefore, \( \{g_r, g'_r\} \) is a complete basis for the field \( f(x) \):

\[ f = \sum_r [g_r a_r + g'_r a_r^*], \quad (5.104) \]

and \( g_r \) tend to purely ingoing waves on \( I^- \).

On the other hand, the surface \( I^+ \) is not a Cauchy hypersurface: in the future, there is in general a non-vanishing field on \( I^+ \), but there is also a non-vanishing field on the horizon. Therefore, we cannot expand \( f \) in terms on the functions \( g'_s \) (which are outgoing waves on \( I^+ \)) only: we have, more generally,

\[ f = \sum_s \left[ g'_s b_s + g''_s b_s^* + g''_s c_s + g''_s c_s^* \right], \quad (5.105) \]

where \( g''_s \) are functions which vanish on \( I^+ \), and are nonvanishing on the horizon.

Now let’s quantize the field. We have

\[ f = \sum_r \left[ g_r a_r + g'_r a_r^* \right] = \sum_s \left[ g'_s b_s + g''_s b_s^* + g''_s c_s + g''_s c_s^* \right], \quad (5.106) \]

where \( a_r, b_s, c_s \) are now annihilation operators.

We take the state of the black hole to be the vacuum of the ingoing particles \( a_r \), i.e. \( |0 >_{in} \) defined by

\[ a_r |0 >_{in} = 0 \quad \forall r. \quad (5.107) \]

In order to compute the number of particles \( b_s \), i.e. of outgoing
particles at $I^+$, one must expand $g'_s$ in terms of $g_r$: \(^{4}\)

$$g'_s = \sum_r [\alpha_{rs} g_r + \beta_{rs} g'_r] .$$  \hspace{1cm} (5.108)

Then,

$$N_s = \langle 0 | b^\dagger_s b_s | 0 \rangle_{in} = \sum_r |\beta_{rs}|^2 \neq 0 .$$  \hspace{1cm} (5.109)

Therefore, even if there are no ingoing waves from past null infinity, there is a nonvanishing particle distribution at the asymptotically flat future null infinity $I^+$. This is the so-called Hawking radiation.

As we are going to show, the explicit computation gives

$$N_s = \frac{\Gamma_s}{e^{\frac{2\pi \kappa}{\omega_s}} - 1} .$$  \hspace{1cm} (5.110)

Here $\kappa$ is the surface gravity, defined in (5.28). In the case, which we are presently considering, of a Schwarzschild black hole,

$$\kappa = \frac{1}{4M} .$$  \hspace{1cm} (5.111)

The factor $\Gamma_s$ is the probability amplitude that an ingoing scalar wave coming from $I^-$, with frequency $\omega_s$, is absorbed by the black hole; indeed, an incoming wave is partially reflected, partially absorbed (with amplitude $\Gamma_s$) by the black hole, due to the scattering potential $V(r)$ in the wave equation (5.97).

In the limit $\omega M \gg 1$ (high frequency wave) all the wave is absorbed, $\Gamma_s \rightarrow 1$, and

$$N_s = \frac{1}{e^{\frac{2\pi \kappa}{\omega_s}} - 1}$$  \hspace{1cm} (5.112)

which is the blackbody Planck spectrum of thermal radiation at temperature

$$T_{BH} = \frac{\hbar \kappa}{2\pi} ,$$  \hspace{1cm} (5.113)

as anticipated in (5.58).

In general, since an incoming wave is partially reflected, the black hole is not a black body, and then the spectrum (5.110) is not a

\(^{4}\)Notice that we cannot write the inverse relation, i.e. $g_r$ in terms of the $g'_r$: all particles in $I^+$ come from particles in $I^-$, but not all particles on $I^-$ go to $I^+$; as we have discussed some of them, corresponding to particles created by $c^\dagger_s$, fall into the black hole.
perfect Planck spectrum: it is modulated by the factor $\Gamma_s$, named greybody factor. The distribution (5.110) can be interpreted as the thermal spectrum, with temperature $T_{BH}$, emitted by a grey body.

In order to show that
\[ \sum_r |\beta_{rs}|^2 = \frac{\Gamma_s}{e^{\frac{2\pi \omega_s}{\kappa}} - 1} \] (5.114)
we need to find the Bogoliubov coefficients $\beta_{rs}$ relating the functions $g_r$ and $g'_s$
\[ g'_s = \sum_r [\alpha_{rs}g_r + \beta_{rs}g^*_r] \] (5.115)
which are defined in the entire spacetime, but have the following limiting values on two different asymptotic null surfaces: by choosing appropriately the normalizations,
\[ g_r = e^{-i\omega_r v} \quad \text{on} \quad I^- \]
\[ g'_s = e^{-i\omega_s u} \quad \text{on} \quad I^+ . \] (5.116)

To find the relationship between these two sets of functions we must compare them on the same region of spacetime; if we know, for instance, given a function $g'_s$ (which tends to $e^{-i\omega_s u}$ on $I^+$) its expression on $I^-$ (where we already know that $g_r = e^{-i\omega_r v}$), then we can work out the expansion of $g'_s$ in terms of $g_r$.

In order to determine the expression of $g'_s$ on $I^-$ we must consider the actual spacetime of a Schwarzschild black hole produced in a gravitational collapse. In Figure 5.2 we show the corresponding Eddington-Finkelstein diagram, including one further space dimension (the angle $\phi$).

The null geodesics $v = \text{const.}$ come from the past infinity surface $I^-$, and enter into the collapsing fluid. If $v$ is smaller than a certain value $v_0$, then the corresponding geodesic avoids the horizon, escaping to the future infinity surface $I^+$ through an $u = \text{const.}$ geodesic; if, instead, $v > v_0$, then the geodesic crosses the horizon and is trapped into the black hole, becoming an $u = \text{const.}$ geodesic which ends up on the singularity; the value $v_0$ corresponds to the geodesic (which we call $\gamma_H$) which goes to the horizon, and remains there.

Therefore, each geodesics $u = \text{const.}$ on $I^+$ comes from a geodesic $v = \text{const.} < v_0$ on $I^-$, thus
\[ u = G(v) \] (5.117)
Figure 5.2: Eddington-Finkelstein diagram of the stellar collapse to a Schwarzschild black hole, with the asymptotic 3-surfaces $I^+$, $I^-$. A further space dimension has been included.
where the function $G$ still has to be determined. Since, as $v \to v_0$ (i.e. approaching the horizon), the corresponding $u \to +\infty$, the geodesics with $v < v_0$ pile up close to the horizon.

Let us consider a null geodesic, which we call $\gamma$, starting from $I^{-}$ with $v$ (smaller than $v_0$) very close to $v_0$. It can be shown that, under some assumption of symmetry of the spacetime (which are fulfilled by the spherically symmetric collapse), if we join the geodesics $\gamma$ and $\gamma_H$ by orthogonal null geodesics (see Fig. 5.2), they extend over the same range $\lambda$ of the affine parameter. We remark that if we define, as in the construction of the geodesic deviation equation, the “separation vector” $n^\mu$ as the tangent vector of these orthogonal geodesics, then the separation $\lambda n^\mu$ does not remain constant, because $n^\mu$ changes, but $\lambda$ itself remains constant.

At $I^{-}$ the orthogonal geodesic between $\gamma$ and $\gamma_H$ is a $u = \text{const}$ geodesic parametrized by $v$; the affine parameter of this geodesic is $v$, since in this limit the spacetime becomes flat and in Minkowski space $u, v$ are affine parameters, thus

$$\lambda = v - v_0 . \quad (5.118)$$

At the horizon (just outside the stellar fluid) the orthogonal geodesic between $\gamma$ and $\gamma_H$ is a $v = \text{const}$ geodesic parametrized by $u$; the affine parameter of this geodesic is, modulo an arbitrary multiplication factor, the Kruskal coordinate

$$U = -e^{-\kappa u} = -e^{-\kappa u} \quad (5.119)$$

and, on $\gamma_H$, $U = 0$; therefore,

$$\lambda = -Ce^{-\kappa u} \quad (5.120)$$

with $C > 0$ unknown constant (we must introduce $C$ because the affine parameter is not unique, it can be rescaled). We can conclude that

$$v - v_0 = -Ce^{-\kappa u} \quad (5.121)$$

thus

$$u = G(v) = -\frac{1}{\kappa} \ln \left( \frac{v_0 - v}{C} \right) . \quad (5.122)$$

Since the geodesics pile up close to the horizon, the phase of the scalar waves in this region oscillates fastly, and we can use the geometric optics approximation, in which the lines of constant phase
of the wave \( g'_s \) (solution of the classical Klein-Gordon equation) are the null geodesics; therefore, since on \( I^+ \) \( g'_s = e^{-i\omega_s u} \), on \( I^- \)

\[
\begin{align*}
  g'_s &= Ke^{-i\omega_s G(v)} \quad \text{if } v < v_0 \\
  g'_s &= 0 \quad \text{if } v > v_0
\end{align*}
\]

i.e.

\[
\begin{align*}
  g'_s &= Ke^{i\frac{\omega_s}{2} \ln \frac{v_0 - v}{v_0}} \quad \text{if } v < v_0 \\
  g'_s &= 0 \quad \text{if } v > v_0
\end{align*}
\]

with \( K, C \) unknown constants.

Now we know, on \( I^- \) (which is a three-surface described by \( v, \theta, \phi \)) both the functions \( g_r = e^{-i\omega_r u} \) and \( g'_s \) (given by (5.124)), and we can invert the expansion (5.115). By applying on \( I^- \) the properties of the Bogoliubov coefficients, it can be shown that

\[
\begin{align*}
  \alpha_{rs} &= \int_{-\infty}^{+\infty} dvg'_r g'_s \\
  \beta_{rs} &= \int_{-\infty}^{+\infty} dvg_r g'_s.
\end{align*}
\]

(5.125)

The coefficients \( \alpha_{rs} \) are

\[
\begin{align*}
  \alpha_{rs} &= K \int_{-\infty}^{v_0} dv e^{i\frac{\omega_s}{2} \ln \frac{v_0 - v}{v_0}} e^{i\omega_r v} \\
  &= K e^{i\omega_r v_0} \int_{0}^{\infty} dv e^{i\frac{\omega_s}{2} \ln \frac{v_0 - v}{v_0}} e^{-i\omega_r v}
\end{align*}
\]

(5.126)

where we have changed variable \( v \to v_0 - v \). We remark that in general \( \alpha_{rs} \neq 0 \) for \( \omega_r \neq \omega_s \), thus the frequency of the wave changes in the process. The coefficients \( \beta_{rs} \) are

\[
\begin{align*}
  \beta_{rs} &= K \int_{-\infty}^{v_0} dv e^{i\frac{\omega_s}{2} \ln \frac{v_0 - v}{v_0}} e^{i\omega_r v} \\
  &= K e^{-i\omega_r v_0} \int_{0}^{\infty} dv e^{i\frac{\omega_s}{2} \ln \frac{v_0 + v}{v_0}} e^{i\omega_r v}.
\end{align*}
\]

(5.127)

The integral (5.126) can be expressed in terms of the integral (5.127) as follows. Let’s us complexify \( v \); the integrand of (5.126) is analytic in the complex plane with the exception of the branch cut \( Im v = 0 \),
Re \(v \leq 0\). Since \(e^{-i\omega r v} \to 0\) as \(Im v \to -\infty\), the integral on closed path \(C\) in Fig. 5.3 vanishes:

\[
\oint_C dve^i\omega r v e^{i\ln v} = 0
\]  

(5.128)

thus

\[
\alpha_{rs} = Ke^{i\omega r v_0} \int_0^\infty dve^i\omega r v e^{i\frac{\ln v}{v}} = -Ke^{i\omega r v_0} \int_{-\infty}^0 dve^{-i\omega r v} e^{i\frac{\ln v}{v}} \ln(\frac{v}{v} - i\epsilon)
\]

(5.129)

where \(\epsilon \to 0\). Using the relation

\[
\ln(-a - i\epsilon) = -i\pi + \ln(a)
\]  

(5.130)

we have

\[
\alpha_{rs} = -Ke^{i\omega r v_0} \int_0^\infty dve^{i\omega r v} e^{i\frac{\ln v}{v}} (-i\pi + \ln(\frac{v}{v} - i\epsilon)) = -Ke^{i\omega r v_0} \int_0^\infty dve^{i\omega r v} e^{2\frac{\ln v}{v} \cdot e^{i\frac{\ln v}{v} \ln(\frac{v}{v} - i\epsilon)}} = -e^{2\omega r v_0} e^{\frac{\pi}{v}} \beta_{rs}.
\]

(5.131)
The completeness relation
\[
\sum_r \left( |\alpha_{rs}|^2 - |\beta_{rs}|^2 \right) = 1 \tag{5.132}
\]
does not hold in this case, since the particles created by \( c_s^\dagger \) should be taken into account. If \( 1 - \Gamma_s \) is the probability amplitude that a particle coming from infinity does not fall into the black hole but is back-scattered by the potential \( V(r) \), it is also the probability that some of the particles in our picture, coming from the region close to the horizon, does not reach \( I^+ \) because they are back-scattered by the potential \( V(r) \) towards the black hole. We have to subtract these particles from the completeness relation, getting
\[
\sum_r \left( |\alpha_{rs}|^2 - |\beta_{rs}|^2 \right) = \Gamma_s . \tag{5.133}
\]
Replacing (5.131),
\[
\sum_r \left( e^{2\pi\omega_s / \kappa} - 1 \right) |\beta_{rs}|^2 = \Gamma_s \tag{5.134}
\]
thus
\[
\sum_r |\beta_{rs}|^2 = \frac{\Gamma_s}{e^{2\pi\omega_s / \kappa} - 1} \tag{5.135}
\]
which proves Hawking’s formula.

A final remark to conclude the discussion on Hawking radiation. The approach used to derive these results is *quantum field theory in curved space* (also called semiclassical gravity). It is not quantum gravity because, although the effects of gravity on quantum fields are taken into account, the gravitational field itself is treated classically (i.e. neglecting quantum effects). This approximation is appropriate in the case we have been considering, since the quantum effects of gravity become relevant only for phenomena with lengthscales of the order of the Planck length
\[
l_P = 1.6 \cdot 10^{-33} \text{cm} . \tag{5.136}
\]
Today, the only phenomena we can imagine involving such a small lengthscale are the black hole singularity and the cosmological singularity. We are not able to describe such phenomena, because we do not have quantum gravity: we don’t know any consistent theory unifying quantum mechanics and general relativity.