

Addendum 1: gravitational waves generation and retarded potentials

222 ■ General Relativity: the physical theory of gravity

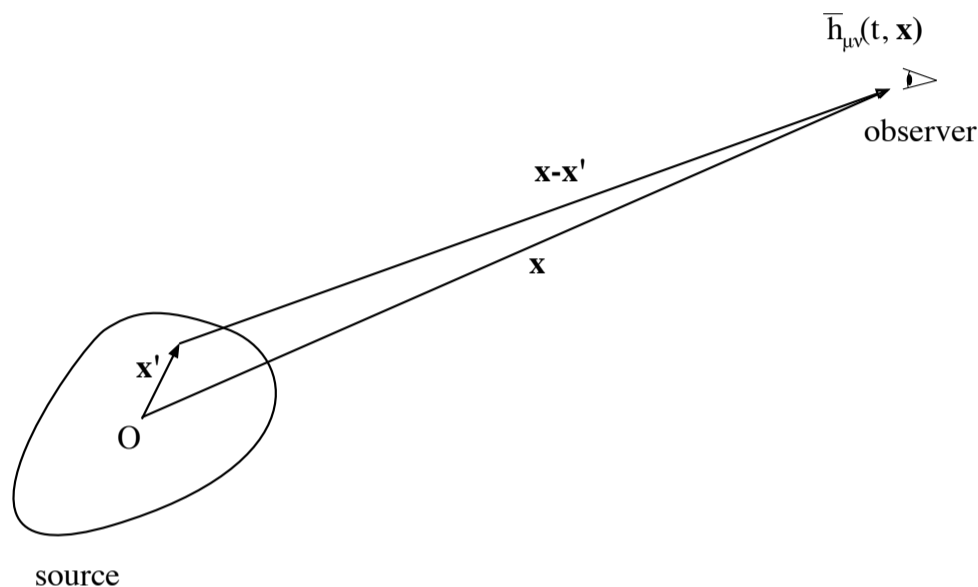


Figure 12.1: An observer located in \mathbf{x} , at the time t receives a wave $\bar{h}(t, \mathbf{x})$ which is the sum of the contributions emitted from each source element located in \mathbf{x}' at the retarded time $t - |\mathbf{x} - \mathbf{x}'|$.

As in electrodynamics, the solution to Eqs. 12.33 can be written in terms of retarded potentials [56]

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = \frac{4G}{c^4} \int_V \frac{T_{\mu\nu}(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3x', \quad (12.36)$$

where V is the three-dimensional source volume, \mathbf{x}' is the distance of an element of the emitting source from the origin of a frame centered in some point of the source, \mathbf{x} is the distance of the observer from the source (see Fig. 12.1).

Remarkably, the retarded potentials solution in Eq. 12.36 automatically satisfies the harmonic gauge condition

$$\frac{\partial}{\partial x^\mu} \bar{h}^{\mu\nu} = 0. \quad (12.37)$$

Indeed, if we define the Green function

$$g(\vec{x} - \vec{x}') \equiv \frac{4G}{c^5} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \delta \left[t' - \left(t - \frac{|\mathbf{x} - \mathbf{x}'|}{c} \right) \right], \quad (12.38)$$

where $\vec{x} = (ct, \mathbf{x})$ and $\vec{x}' = (ct', \mathbf{x}')$, Eq. 12.36 can be written as a four-dimensional integral as follows

$$\bar{h}_{\mu\nu}(\vec{x}) = \int_{\Omega} T_{\mu\nu}(\vec{x}') g(\vec{x} - \vec{x}') d^4x', \quad (12.39)$$

where $\Omega \equiv V \times I$, and I is a time interval to be taken such that $g(\vec{x} - \vec{x}')$ vanishes at the extrema of I . This condition is satisfied if I is so large that, for all $\mathbf{x}' \in V$, the instant $t' = t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}$ is contained in I ; indeed, from Eq. 12.38, the Green function g is different from zero only for $t' = t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}$.

Since g is a function of the difference $(\vec{x} - \vec{x}')$, then

$$\frac{\partial}{\partial x^\mu} [g(\vec{x} - \vec{x}')] = -\frac{\partial}{\partial x'^\mu} [g(\vec{x} - \vec{x}')] . \quad (12.40)$$

Consequently,

$$\frac{\partial}{\partial x^\mu} \bar{h}^{\mu\nu}(\vec{x}) = \int_{\Omega} T^{\mu\nu}(\vec{x}') \frac{\partial}{\partial x^\mu} g(\vec{x} - \vec{x}') d^4x' = - \int_{\Omega} T^{\mu\nu}(\vec{x}') \frac{\partial}{\partial x'^\mu} g(\vec{x} - \vec{x}') d^4x'. \quad (12.41)$$

The last term can be integrated by parts and gives

$$\begin{aligned} \int_{\Omega} T^{\mu\nu}(\vec{x}') \frac{\partial}{\partial x'^\mu} g(\vec{x} - \vec{x}') d^4x' = \\ \int_{\Omega} d^4x' \frac{\partial}{\partial x'^\mu} [T^{\mu\nu}(\vec{x}') g(\vec{x} - \vec{x}')] - \int_{\Omega} d^4x' \left[\frac{\partial}{\partial x'^\mu} T^{\mu\nu}(\vec{x}') \right] g(\vec{x} - \vec{x}') d^4x' = 0. \end{aligned} \quad (12.42)$$

By Gauss' theorem, the first integral over the 4-volume Ω on the right-hand side of Eq. 12.42 is equal to the flux of $[T^{\mu\nu}(\vec{x}') g(\vec{x} - \vec{x}')] g(\vec{x} - \vec{x}')$ across the surface $\partial\Omega$ enclosing that volume, i.e.

$$\int_{\Omega} d^4x' \frac{\partial}{\partial x'^\mu} [T^{\mu\nu}(\vec{x}') g(\vec{x} - \vec{x}')] = \int_{\partial\Omega} [T^{\mu\nu}(\vec{x}') g(\vec{x} - \vec{x}')] dS_\mu, \quad (12.43)$$

where dS is the surface element on $\partial\Omega$, with normal vector n^μ , and $dS_\mu = n_\mu dS$. This integral vanishes since $T^{\mu\nu} = 0$ on the boundary of V and $g = 0$ on the boundary of I . The second integral on the right-hand side of Eq. 12.42 also vanishes, since the stress-energy tensor satisfies the conservation law $T^{\mu\nu}{}_{,\nu} = 0$. Consequently

$$\frac{\partial}{\partial x^\mu} \bar{h}^{\mu\nu}(\vec{x}) = 0, \quad (12.44)$$

as previously stated.

As discussed in Section 12.2 the general solution to the sourced D'Alembertian equation 13.4 can be written in terms of *retarded potentials*,

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = \frac{4G}{c^4} \int_V \frac{T_{\mu\nu}(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}, \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3x', \quad (13.6)$$

where $t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}$ is the retarded time, i.e. the time that the wave takes to travel from the source element at \mathbf{x}' to the observer located at \mathbf{x} (see Fig. 12.1). The above integral is performed over the entire three-dimensional source volume V .

By defining the Fourier transforms of both \bar{h} and $T_{\mu\nu}$,

$$\begin{aligned} T_{\mu\nu}(t, \mathbf{x}) &= \int_{-\infty}^{+\infty} T_{\mu\nu}(\omega, \mathbf{x}) e^{-i\omega t} d\omega, \\ \bar{h}(t, \mathbf{x}) &= \int_{-\infty}^{+\infty} \bar{h}(\omega, \mathbf{x}) e^{-i\omega t} d\omega, \end{aligned} \quad (13.7)$$

Eq. 13.6 becomes

$$\bar{h}(\omega, \mathbf{x}) = \frac{4G}{c^4} \int_V d^3x' T_{\mu\nu}(\omega, \mathbf{x}') \frac{e^{i\omega \frac{|\mathbf{x}-\mathbf{x}'|}{c}}}{|\mathbf{x}-\mathbf{x}'|}. \quad (13.8)$$

Since in the slow-motion approximation $\lambda_{GW} = \frac{2\pi c}{\omega} \gg \epsilon$, within the source $\omega|\mathbf{x}'|/c \ll 2\pi$. Therefore,¹

$$\frac{e^{i\omega \frac{|\mathbf{x}-\mathbf{x}'|}{c}}}{|\mathbf{x}-\mathbf{x}'|} \simeq \frac{e^{i\omega \frac{|\mathbf{x}|}{c}}}{|\mathbf{x}|} \simeq \frac{e^{i\omega \frac{r}{c}}}{r}, \quad (13.9)$$

where we have set $r = |\mathbf{x}|$. Thus, Eq. 13.8 can be written as

$$\bar{h}(\omega, r) = \frac{4G}{c^4} \frac{e^{i\omega \frac{r}{c}}}{r} \int_V T_{\mu\nu}(\omega, x'^i) d^3x'. \quad (13.10)$$

The solution in the time domain is the inverse Fourier transform of the above expression,

$$\bar{h}(t, r) = \frac{4G}{rc^4} \int_V T_{\mu\nu} \left(t - \frac{r}{c}, x'^i \right) d^3x'. \quad (13.11)$$

This is the gravitational wave signal emitted by the source to leading order in the slow-motion, weak-field approximation.

In the next section we shall further simplify the integral in Eq. 13.11. Meanwhile, we recall that, as shown in Section 12.2, the solution 13.6 (and thus the solution 13.11) automatically satisfies the harmonic gauge condition

$$\frac{\partial}{\partial x^\mu} \bar{h}^\mu{}_\nu = 0. \quad (13.12)$$

We also note that, in order to extract the physical components of the wave, we still have to transform to the TT gauge. This will be done explicitly in Sec. 13.3.

Finally, we note that Eq. 13.11 has been derived on *two* very strong assumptions: weak field ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$) and slow motion ($v_{\text{typical}} \ll c$). For this reason that expression has to be considered as an estimate of the emitted radiation by the system, and it will fail in various cases of interest, namely when the sources are relativistic and in the spacetime regions where the gravitational field is strong.

¹Note that while $|\mathbf{x}-\mathbf{x}'| \simeq |\mathbf{x}|$ is always satisfied as long as $|\mathbf{x}'| \ll |\mathbf{x}|$, the approximation $e^{i\omega \frac{|\mathbf{x}-\mathbf{x}'|}{c}} \simeq e^{i\omega \frac{|\mathbf{x}|}{c}}$ requires the stronger condition $\omega|\mathbf{x}'|/c \ll 2\pi$. Indeed, given two real numbers a, A with $a \ll A$, we always have $A+a \simeq A$, but $\sin(A+a) \simeq \sin(A)$ only if the condition $a \ll 2\pi$ is also satisfied.