17.3 MASS AND ANGULAR MOMENTUM OF AN ISOLATED OBJECT

As shown in Section 17.2, if the gravitational field is weak everywhere, close to and far from the source, the constants M and J emerge as the mass and angular momentum of the source as defined in Special Relativity. If the field is *not* weak inside the source and in its neighborhood, then M and J arise as integration constants of the general solution to the far-field equations. In this case, we need to assess the physical meaning of these constants.

One possibility is to provide an *operational definition* of mass and angular momentum for an isolated, stationary object, based on the following argument. Suppose that a test body is moving in the spacetime described by the metric 17.84, far away from the source; the study of its motion does not allow to distinguish whether the gravitational field is weak or strong near the source, because the metric in the far-field limit is the same in both cases. Thus, if the source is a strong-gravity one, we can *define* its mass and angular momentum as the quantities M and J, related to the integration constants appearing in the far-field metric by Eq. 17.83. The mass M can be measured from the orbital frequency of the test mass through Kepler's third law, and the angular momentum J can be inferred by measuring the precession of gyroscopes orbiting around the source (see Section 17.4 below).

A more rigorous way to assess the physical meaning of M and J for a strong-gravity source uses the stress-energy pseudotensor $t^{\mu\nu}$, which we defined in Chapter 13. We remind that $t^{\mu\nu}$ describes the energy and momentum carried by the gravitational field, and satisfies, together with the stress-energy tensor of matter and fields $T^{\mu\nu}$, the conservation law

$$[(-g)(T^{\mu\nu}+t^{\mu\nu})]_{,\nu}=0\,. \tag{17.85}$$

As shown in Sec. 13.6.1, the quantity $(-g)(T^{\mu\nu}+t^{\mu\nu})$ can be defined in terms of the spacetime metric as follows:

$$(-g)(T^{\mu\nu}+t^{\mu\nu})=\frac{\partial\zeta^{\mu\nu\alpha}}{\partial x^\alpha} \tag{17.86}$$

where

$$\zeta^{\mu\nu\alpha}=\frac{\partial\lambda^{\mu\nu\alpha\beta}}{\partial x^\beta} \tag{17.87}$$

and

$$\lambda^{\mu\nu\alpha\beta}\equiv\frac{1}{16\pi}\left[(-g)(g^{\mu\nu}g^{\alpha\beta}-g^{\mu\alpha}g^{\nu\beta})\right]\,. \tag{17.88}$$

Since we are considering a stationary spacetime, Eq. 17.86 becomes

$$(-g)(T^{\mu\nu}+t^{\mu\nu})=\frac{\partial\zeta^{\mu\nu k}}{\partial x^k}\,,\qquad k=1,2,3\,, \tag{17.89}$$

and $\zeta^{\mu\nu\alpha}=\frac{\partial\lambda^{\mu\nu\alpha i}}{\partial x^i}$.

Let us now consider a spherical, three-dimensional volume V centered on the source, with radius r much larger than the source size (we choose an asymptotically flat coordinate frame (t,x^i)). As discussed in Sec. 13.6.1, the total four-momentum P^μ enclosed in the volume V is given by

$$P^\mu=\int_V d^3x(-g)(T^{0\mu}+t^{0\mu}) \tag{17.90}$$

(see Eq. 13.120) and it is contributed *both by the source and by the gravitational field*. By substituting Eq. 17.89 in Eq. 17.90, P^μ can be expressed as

$$P^\mu=\int_V d^3x\frac{\partial\zeta^{0\mu k}}{\partial x^k}\,. \tag{17.91}$$

Using Gauss' theorem (see Box 7-A) the integral of a three-divergence of a vector over the volume V can be written as the flux of the vector across the spherical surface ∂V surrounding the volume

$$P^\mu = \int_{\partial V} \zeta^{0\mu k} dS_k \,, \tag{17.92}$$

where $dS_k = n^k dS = n^k r^2 d\Omega$ and $n^k = \frac{x^i}{r}$ is the unit vector orthogonal to the surface element. In Sec. 17.1 we computed the mass and the angular momentum of a stationary, isolated source – on the assumption that the gravitational field it generates is weak – by integrating suitable components of the stress-energy tensor on the volume of the source (see Eqs. 17.15 and 17.30). We are now considering a stationary, isolated source whose gravitational field is not weak and, in order to compute the *total mass-energy* and the *total angular momentum* which include the contribution of the gravitational field, we need to evaluate the integral in Eq. 17.92 over a volume much larger than the source volume; in this way the surface ∂V is in the far-field region where the metric is known, and $\zeta^{0\mu k}$ can be computed using Eqs. 17.87 and 17.88. The total mass-energy is

$$M_{\rm tot} = P^0 = \int_{\partial V} \zeta^{00k} n^k r^2 d\Omega \,. \tag{17.93}$$

As discussed in Section 17.1, the three-momentum of the volume element d^3x located at a point of coordinates x^i is $\mathcal{P}^i d^3x$, and the angular momentum of the same element is $dJ^i = (\mathbf{x} \times \mathcal{P})^i \, d^3x = -\epsilon_{ijk} \mathcal{P}^j x^k d^3x$. In the present case $\mathcal{P}^j$ is [4]

$$\mathcal{P}^j = (-g)(T^{0j} + t^{0j}) = (-g)(T^{j0} + t^{j0}) = \frac{\partial}{\partial x^l} \zeta^{j0l} \,, \tag{17.94}$$

therefore the total angular momentum which generalizes Eq. 17.30 and includes the contribution of the gravitational field is

$$\begin{aligned} J^i_{\rm tot} &= -\epsilon_{ijk} \int_V d^3x \frac{\partial \zeta^{j0l}}{\partial x^l} x^k = -\epsilon_{ijk} \int_V d^3x \left[\frac{\partial(\zeta^{j0l} x^k)}{\partial x^l} - \zeta^{j0l} \frac{\partial x^k}{\partial x^l} \right] \\ &= -\epsilon_{ijk} \int_V d^3x \left[\frac{\partial(\zeta^{j0l} x^k)}{\partial x^l} - \zeta^{j0k} \right] . \end{aligned} \tag{17.95}$$

where the latter equality holds in stationary spacetimes, and V is assumed to be larger than the source volume. By replacing Eq. 17.87 in Eq. 17.95 we find

$$J^i_{\rm tot} = -\epsilon_{ijk} \int_V d^3x \frac{\partial}{\partial x^l} \left(\zeta^{j0l} x^k - \lambda^{j0kl} \right) \,, \tag{17.96}$$

and, by Gauss' theorem, the components of the total angular momentum of the source are

$$J^i_{\rm tot} = -\epsilon_{ijk} \int_{\partial V} \left(\zeta^{j0l} x^k - \lambda^{j0kl} \right) n^l r^2 d\Omega \,. \tag{17.97}$$

[4] Note that $\zeta^{0jl}{}_{,l} = \zeta^{j0l}{}_{,l}$ but $\zeta^{0jl} \neq \zeta^{j0l}$; we write Eq. 17.94 in terms of ζ^{j0l} because it turns out that in this way the subsequent computations are simpler.

Box 17-B

Summary: total mass-energy and angular momentum of a stationary, isolated body using the stress-energy pseudo-tensor

The total mass and angular momentum of a stationary, isolated source, which includes the contribution of the gravitational field it generates, can be written in terms of the stress-energy pseudo-tensor as follows:

$$M_{\rm tot} = \int_V d^3x(-g)(T^{0\mu} + t^{0\mu}) = \int_{\partial V} \zeta^{00i} n^i r^2 d\Omega \tag{17.98}$$

$$J^i_{\rm tot} = \epsilon_{ijk} \int_V (-g)(T^{0k} + t^{0k}) x^j d^3x = -\epsilon_{ijk} \int_{\partial V} \left(\zeta^{j0l} x^k - \lambda^{j0kl}\right) n^l r^2 d\Omega\,, \tag{17.99}$$

where $\zeta^{\alpha\mu\nu}$, $\lambda^{\mu\nu\alpha\beta}$ are given in Eqs. 17.87, 17.88, respectively, and V is much larger than the source size, so that the surface ∂V is located in the far-field region.

Let us now explicitly compute Eqs. 17.98 and 17.99 assuming that the surface enclosing the source is located in the far-field region; in this case the metric to be used to evaluate the surface integrals is $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is given by Eqs. 17.40:

$$\begin{aligned} h_{00} &= \frac{2M}{r} + O\left(\frac{1}{r^2}\right) \\ h_{0i} &= \frac{2}{r^3}\epsilon_{ijk}x^j J^k + O\left(\frac{1}{r^3}\right) \\ h_{ij} &= \frac{2M}{r}\delta_{ij} + O\left(\frac{1}{r^2}\right). \end{aligned} \tag{17.100}$$

We first recall that (see Section 6.1) that

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O(h^2)\,, \tag{17.101}$$

where the indices of $h_{\mu\nu}$ have been raised with Minkowski's metric. The determinant of $g_{\mu\nu}$ is

$$g = (-1 + h_{00})(1 + h_{ii}) = -(1 + \frac{4M}{r}) + O\left(\frac{1}{r^2}\right). \tag{17.102}$$

Note that in this expression we have neglected the term J/r^3 with respect to M/r, since we are in the far-field limit.

The quantity ζ^{00i} which appears in Eq. 17.98, can be evaluated using Eqs. 17.87, 17.88 and the metric 17.100; neglecting terms $O\left(1/r^2\right)$, we find

$$\begin{aligned} \zeta^{00i}n^i &= \frac{1}{16\pi} n^i \frac{\partial}{\partial x^j}\left[(-g)\left(g^{00}g^{ij} - g^{0i}g^{0j}\right)\right] = \frac{1}{16\pi} n^i \frac{\partial}{\partial x^j}\left[(-g)g^{00}g^{ij}\right] + O(h^2) \\ &= \frac{1}{16\pi} n^i \frac{\partial}{\partial x^j}\left[\left(1 + \frac{4M}{r}\right)\left(-1 - \frac{2M}{r}\right)\left(1 - \frac{2M}{r}\right)\delta_{ij}\right] + O\left(\frac{1}{r^2}\right) \\ &= -\frac{1}{16\pi} n^i \frac{\partial}{\partial x^j}\frac{4M}{r}\delta_{ij} + O\left(\frac{1}{r^2}\right). \end{aligned} \tag{17.103}$$

Since $\frac{\partial}{\partial x^j}\frac{1}{r} = -\frac{n^j}{r^2}$

$$\zeta^{00i}n^i = \frac{1}{4\pi}\frac{M}{r^2} n^i n^i = \frac{1}{4\pi}\frac{M}{r^2}\,, \tag{17.104}$$

and finally

$$M_{\rm tot} = P^0 = \int_{\partial V} \zeta^{00i} n^i r^2 d\Omega = M\,. \tag{17.105}$$

Thus, *the constant M appearing in the metric of the far-field limit is the total mass-energy of the source.*

Let us now prove that the components of the angular momentum defined in Eq. 17.99 coincide with the constants J^i which appear in the metric of far-field limit 17.100. Using Eq. 17.87, and the relation $x^k = rn^k$, Eq. 17.99 becomes

$$\begin{aligned} J^i_{\rm tot} &= -\epsilon_{ijk} \int_{\partial V} \left(\zeta^{j0l} x^k - \lambda^{j0kl}\right) dS_l \\ &= -\epsilon_{ijk} \int_{\partial V} \left(n^k n^l r^3 \frac{\partial}{\partial x^m} \lambda^{j0lm} - n^l r^2 \lambda^{j0kl} \right) d\Omega\,. \end{aligned} \tag{17.106}$$

Using Eq. 17.88, 17.102, and

$$g^{0j} = -h^{0j} = h_{0j} = \frac{2}{r^3} \epsilon_{jrs} x^r J^s + O\left(\frac{1}{r^3}\right), \tag{17.107}$$

neglecting terms $O(h^2)$ we find

$$\begin{aligned} \lambda^{j0lm} &= \frac{1}{16\pi}(h_{0j}\delta_{lm} - h_{0m}\delta_{jl}) + O\left(\frac{1}{r^3}\right) \\ &= \frac{1}{8\pi r^3}(\epsilon_{jrs}\delta_{lm} - \epsilon_{mrs}\delta_{jl}) x^r J^s + O\left(\frac{1}{r^3}\right). \end{aligned} \tag{17.108}$$

Let us compute the first term in Eq. 17.106. Neglecting higher-order terms in $1/r$ we obtain

$$-\epsilon_{ijk} n^k n^l r^3 \frac{\partial}{\partial x^m} \lambda^{j0lm} = -\frac{1}{8\pi} \epsilon_{ijk} n^k n^l r^3 \frac{\partial}{\partial x^m} \left(\frac{\epsilon_{jrs}\delta_{lm} - \epsilon_{mrs}\delta_{jl}}{r^3} x^r \right) J^s\,; \tag{17.109}$$

since $\delta_{jl}\epsilon_{ijk} n^k n^l = \epsilon_{ijk} n^k n^j = 0$ and

$$n^m \frac{\partial}{\partial x^m}\left(\frac{x^r}{r^3}\right) = \frac{\delta_{rm} - 3n^m n^r}{r^3} n^m = -2\frac{n^r}{r^3}\,, \tag{17.110}$$

using the property $\epsilon_{ijk}\epsilon_{jrs} = \delta_{kr}\delta_{is} - \delta_{ks}\delta_{ir}$ (see Eq. 17.44), we find

$$\begin{aligned} & -\epsilon_{ijk} n^k n^l r^3 \frac{\partial}{\partial x^m} \lambda^{0jlm} = -\frac{1}{8\pi} \epsilon_{ijk}\epsilon_{jrs} J^s n^k n^m r^3 \frac{\partial}{\partial x^m}\left(\frac{x^r}{r^3}\right) \\ &= \frac{1}{4\pi}\epsilon_{ijk}\epsilon_{jrs} J^s n^k n^r = \frac{1}{4\pi}(\delta_{kr}\delta_{is} - \delta_{ks}\delta_{ir}) J^s n^k n^r \\ &= \frac{1}{4\pi}(J^i - J^k n^k n^i)\,. \end{aligned} \tag{17.111}$$

For the second term in Eq. 17.106 we find (using the property $\epsilon_{ijk}\delta_{jk} = 0$)

$$\begin{aligned} \epsilon_{ijk} n^l r^2 \lambda^{0jkl} &= \frac{n^l}{8\pi} \epsilon_{ijk}(\epsilon_{jrs}\delta_{kl} - \epsilon_{lrs}\delta_{jk}) n^r J^s \\ = \frac{1}{8\pi}\epsilon_{ijk}\epsilon_{jrs} n^k n^r J^s &= \frac{1}{8\pi}(\delta_{kr}\delta_{is} - \delta_{ks}\delta_{ir}) n^k n^r J^s = \frac{1}{8\pi}(J^i - J^k n^k n^i)\,. \end{aligned} \tag{17.112}$$

Replacing in Eq. 17.106,

$$\begin{aligned} J^i_{\rm tot} &= -\epsilon_{ijk}\int_{\partial V}\left(n^k n^l r^3 \frac{\partial}{\partial x^m}\lambda^{0jlm} - n^l r^2 \lambda^{j0kl}\right) d\Omega \\ &= \frac{3}{2}\frac{1}{4\pi}\int_{\partial V}(J^i - J^k n^k n^i) d\Omega = \frac{3}{2}J^k \frac{1}{4\pi}\int_{\partial V}(\delta_{ik} - n^i n^k) d\Omega = J^i \end{aligned} \tag{17.113}$$

where we have used $\int_{\partial V} d\Omega n^i n^k = (4\pi/3)\delta_{ik}$ (see Eq. 13.145).

We can conclude that the integration constants M and J appearing in the metric of the far-field limit of a stationary, isolated source given in Eq. 17.3, can be correctly interpreted as the mass-energy and the angular momentum of the system. For a weakly gravitating source, the contribution of the gravitational field is negligible; if the gravitational field of the source is strong, it contributes to these quantities through the stress-energy pseudotensor $t^{\mu\nu}$.

17.4 PRECESSION OF A GYROSCOPE IN A GRAVITATIONAL FIELD

A *gyroscope* is a body with an intrinsic angular momentum which is not subjected to external torques; this means that non-gravitational forces, if present, act on its center of mass. We can model a gyroscope in General Relativity as a point particle with four-velocity $\vec{u}$, and with an *intrinsic spin vector* $\vec{S}$.

In a LIF the laws of Special Relativity hold and, if the body moves on a geodesic, absence of external torques implies [5]

$$\frac{dS^\mu}{d\tau} = 0\,, \tag{17.114}$$

which in a LIF can be written as $u^\alpha S^\mu{}_{,\alpha} = u^\alpha S^\mu{}_{;\alpha} = 0$. By definition, the intrinsic spin vector in Special Relativity – like the orbital angular momentum of a moving particle – is orthogonal to the four-velocity of the body: $S_\mu u^\mu = 0$ (see Eq. 10.38) which, in a comoving frame, reduces to $S^0 = 0$. Since these are tensor equations, they hold in any coordinate frame:

$$u^\alpha S^\mu{}_{;\alpha} = 0 \tag{17.115}$$

$$S_\mu u^\mu = 0\,. \tag{17.116}$$

If the gyroscope does not move on a geodesic, i.e. non-gravitational forces act on its center of mass, there is a non-vanishing four-acceleration $a^\mu = du^\mu/d\tau = u^\nu u^\mu{}_{;\nu}$. It can be shown that the four-acceleration induces a rotation of the spin vector in the plane containing the four-velocity and the four-acceleration:

$$u^\alpha S^\mu{}_{;\alpha} = -u^\mu a^\alpha S_\alpha\,. \tag{17.117}$$

This phenomenom, also present in Special Relativity, is called *Thomas precession* and will not be discussed in this book.

17.4.1 Gyroscope in the gravitational field of a rotating body: the Lense-Thirring precession

Let us consider a gyroscope in the gravitational field generated by an isolated, stationary object with non-vanishing angular momentum J^i. Be $\{x^\mu\}$ the coordinate frame in which $g_{\mu\nu,0} = 0$ and, in the far-field limit, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{\mu\nu}$ given by Eqs. 17.100. We shall

[5] Eq. 17.114 is the special relativistic generalization of the conservation of the angular momentum in Newtonian physics $\mathbf{L}$ in the absence of an external torque, $\mathbf{M}$, i.e. in general $d\mathbf{L}/dt = \mathbf{M}$.

show that the spin of the gyroscope precesses, due to the coupling with the angular momentum of the central body. This effect, called **Lense-Thirring precession**, is a remarkable prediction of General Relativity: the orbital motion is affected by the angular momentum of the central body.

Let us first consider, for simplicity, a gyroscope which does not move in space, i.e. such that $u^i = 0$. The space components of Eq. 17.115 reduce to [6]

$$u^\mu S^i{}_{;\mu} = u^0 S^i{}_{;0} = u^0 S^i{}_{,0} + \Gamma^i_{0j} S^i = 0 \tag{17.118}$$

and therefore, neglecting $O(h^2)$ terms,

$$\frac{dS^i}{d\tau} = u^0 S^i{}_{,0} = -\Gamma^i_{0j} S^j = -\frac{1}{2}(h_{0i,j} - h_{0j,i})S^j = -2B_{ij}S^j \tag{17.119}$$

where we have defined

$$B_{ij} = -B_{ji} = \frac{1}{4}(h_{0i,j} - h_{0j,i})\,. \tag{17.120}$$

As shown in Box 17-A, if we define $\omega^k = \epsilon^{kij} B_{ij}$, it follows that $B_{ij} = \frac{1}{2}\epsilon_{ijk}\omega^k$. Hence, Eq. 17.119 can be written as

$$\frac{dS^i}{d\tau} = -\epsilon_{ijk}\omega^k S^j = \epsilon_{ijk}\omega^j S^k\,, \qquad \text{i.e.} \qquad \frac{d\mathbf{S}}{d\tau} = \boldsymbol{\omega} \times \mathbf{S} : \tag{17.121}$$

the three-vector $\mathbf{S} = \{S^i\}$ rotates in space with angular velocity $\boldsymbol{\omega} = \{\omega^i\}$, which can be found using Eq. 17.120,

$$\omega^k = \frac{1}{2}\epsilon^{kij} h_{0i,j}\,. \tag{17.122}$$

Since the gyroscope is in the far-field region of a stationary, isolated object, h_{0i} is given by Eqs. 17.100:

$$h_{0i} = \frac{2}{r^3}\epsilon_{ijk} x^j J^k\,. \tag{17.123}$$

The angular velocity of the spin precession is then

$$\omega^k = \frac{1}{2}\epsilon^{ijk} h_{0i,j} = \epsilon^{ijk}\epsilon_{ilm}\left(\frac{x^l J^m}{r^3}\right)_{,j} = \frac{J^m}{r^3}\left(\delta^j_l \delta^k_m - \delta^j_m \delta^k_l\right)\left(\delta^l_j - \frac{3x^l x^j}{r^2}\right)\,, \tag{17.124}$$

where we have used the relation $r^n{}_{,i} = n x^i r^{n-2}$ and the property 17.44, $\epsilon^{ijk}\epsilon_{klm} = \delta^i_l \delta^j_m - \delta^i_m \delta^j_l$. Then, since $\delta^j_l(\delta^l_j - 3x^l x^j/r^2) = 0$,

$$\omega^k = -\frac{J^j}{r^3}\left(\delta^k_j - \frac{3x^k x^j}{r^2}\right) = \frac{1}{r^3}\left(-J^k + 3\frac{J^j x^j x^k}{r^2}\right)\,, \tag{17.125}$$

i.e.

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{\rm LT} \equiv \frac{1}{r^3}\left(-\mathbf{J} + 3\frac{\mathbf{J}\cdot\mathbf{x}}{r^2}\mathbf{x}\right)\,. \tag{17.126}$$

The precession frequency $\boldsymbol{\omega}_{\rm LT}$ defined in Eq. 17.126 is called *Lense-Thirring frequency*, and this phenomenom is called **Lense-Thirring effect**.

The Lense-Thirring effect is an example of *dragging of inertial frames* (which will be also discussed in Sec. 18.5): as the central body rotates, it drags the inertial frames in the

[6] Strictly speaking, a static body in a gravitational field has a non-vanishing acceleration, and thus the general formula 17.117 should be used: $u^\alpha S^\mu{}_{;\alpha} = -u^\mu a^\alpha S_\alpha$. However, since we are only interested in the equations for the space components S^i and $u^i = 0$, the acceleration term disappears and Eq. 17.118 is correct.

surrounding spacetime, because the angular momentum of the central body couples with the spin of the gyroscope. Thus, although the gyroscope keeps its direction fixed with respect to the LIF, it rotates with respect to the asymptotically flat frame. In practice, the gyroscope rotates with respect to the direction of far-away stars (which – neglecting for simplicity the effects of cosmology – have "fixed" directions in the asymptotically flat frame).

17.4.2 Moving gyroscope: geodesic precession

In the previous section we considered a gyroscope at rest, i.e. with $u^i = 0$. If the gyroscope is in motion, its spin undergoes a further precession (distinct from the Lense-Thirring one) due to the coupling between the orbital angular momentum of the gyroscope and its spin. This phenomenom is called **geodesic precession** or *de Sitter precession*, and is typically much larger than the Lense-Thirring precession.

In order to study the gyroscope motion it is convenient to consider the comoving frame, i.e. the local Fermi frame $\{x^{*\mu}\}$ adapted to the source (see Sec. 3.9). For simplicity we shall assume that the gyroscope moves along a geodesic.

We remind that the Fermi frame $\{\vec{e}^{\,*}_{(\mu)}\}$ is an orthonormal basis ($\vec{e}^{\,*}_{(\mu)} \cdot \vec{e}^{\,*}_{(\nu)} = \eta_{\mu\nu}$) such that $\vec{e}^{\,*}_{(0)} = \vec{u}$ is the four-velocity of the body (in this case, the gyroscope), and $\{\vec{e}^{\,*}_{(i)}\}$ are three spacelike vectors orthogonal to $\vec{u}$. With this definition the spatial orientation of the vectors $\{\vec{e}^{\,*}_{(i)}\}$ can be arbitrarily chosen; we choose the spacelike basis vectors to have the same space orientation as the coordinate basis vectors associated to the coordinates (r, θ, φ) of the asymptotically flat metric.

The motion of a gyroscope moving on a general geodesic orbit is quite involved. We shall here consider the simple case of a *circular, equatorial* orbit around a spherically symmetric object (thus neglecting its angular momentum). Obviously, in this case the Lense-Thirring precession will not be present. As discussed in Chapter 9, the spacetime outside the central body is described by the Schwarzschild metric,

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{17.127}$$

where we remind that M is the mass of the central body in geometrized units. We shall first determine how the spin of the gyroscope evolves in the frame $\{x^\mu\} = (t, r, \theta, \varphi)$, and then derive the corresponding equations in the local Fermi frame.

Since the gyroscope is a (massive) body in circular equatorial motion around the central object[7], $u^\mu = u^t(1, 0, 0, \omega)$ where (see Section 10.5)

$$\omega = \frac{d\varphi}{dt} = \frac{u^\varphi}{u^t} = \sqrt{\frac{M}{r^3}} \tag{17.128}$$

is the Keplerian orbital frequency of the body. Moreover, since $u^\mu u_\mu = -1$ and $\theta = \pi/2$,

$$g_{\mu\nu} u^\mu u^\nu \;=\; (u^t)^2 \left[-\left(1 - \frac{2M}{r}\right) + \omega^2 r^2\right] = -(u^t)^2 \left(1 - \frac{3M}{r}\right) = -1\,,$$

therefore

$$u^t = \left(1 - \frac{3M}{r}\right)^{-1/2}\,. \tag{17.129}$$

Eq. 17.116, $S_\mu u^\mu = 0$, yields

$$g_{\mu\nu} S^\mu u^\nu = -\left(1 - \frac{2M}{r}\right) S^t u^t + r^2 S^\varphi \omega u^t = 0\,, \tag{17.130}$$

[7] We assume that the mass of the gyroscope is much smaller than that of the central body.

hence

$$S^t = r^2 \left(1 - \frac{2M}{r}\right)^{-1} \omega S^\varphi \,. \tag{17.131}$$

The evolution equation of the spin vector is given by Eq. 17.115, $u^\alpha S^\mu{}_{;\alpha} = 0$, i.e.

$$\frac{dS^i}{d\tau} = u^\alpha S^i{}_{,\alpha} = -\Gamma^i_{\alpha\beta} u^\alpha S^\beta \,. \tag{17.132}$$

The non-vanishing Christoffel symbols of the Schwarzschild metric are given in Box 9-B; on the equatorial plane they become

$$\begin{array}{lll} \Gamma^t_{tr} = \Gamma^t_{rt} = \frac{M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1} & \Gamma^r_{tt} = \frac{M}{r^2}\left(1 - \frac{2M}{r}\right) & \Gamma^r_{rr} = -\frac{M}{r^2}\left(1 - \frac{2M}{r}\right)^{-1} \\ \Gamma^r_{\theta\theta} = -(r - 2M) & \Gamma^r_{\varphi\varphi} = -(r - 2M) & \Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r} \,. \end{array} \tag{17.133}$$

Using these expressions, the θ-component of Eq. 17.132 gives

$$\frac{dS^\theta}{d\tau} = -\Gamma^\theta_{\alpha\beta} u^\alpha S^\beta = -\Gamma^\theta_{r\theta}(u^r S^\theta + u^\theta S^r) = 0 \,, \tag{17.134}$$

since $u^r = u^\theta = 0$. Thus, S^θ is constant; let us assume for simplicity that $S^\theta = 0$, i.e. that the spin of the gyroscope lies in the equatorial orbit.

For the r-component of Eq. 17.132 we find (using $u^\varphi = \omega u^t$ and Eq. 17.131)

$$\begin{aligned} \frac{dS^r}{d\tau} &= -\Gamma^r_{\alpha\beta} u^\alpha S^\beta = -\Gamma^r_{tt} u^t S^t - \Gamma^r_{rr} u^r S^r - \Gamma^r_{\theta\theta} u^\theta S^\theta - \Gamma^r_{\varphi\varphi} u^\varphi S^\varphi & (17.135) \\ &= -u^t \left(\Gamma^r_{tt} S^t + \Gamma^r_{\varphi\varphi} \omega S^\varphi\right) = -\omega u^t S^\varphi \left[\Gamma^r_{tt} r^2 \left(1 - \frac{2M}{r}\right)^{-1} + \Gamma^r_{\varphi\varphi}\right] & (17.136) \\ &= -\omega u^t S^\varphi \left[(M - (r - 2M)\right] = \omega u^t (r - 3M) S^\varphi \,. \end{aligned}$$

For the φ-component we find

$$\frac{dS^\varphi}{d\tau} = -\Gamma^\varphi_{\alpha\beta} u^\alpha S^\beta = -\Gamma^\varphi_{\varphi r} u^\varphi S^r = -\omega \frac{u^t}{r} S^r \,. \tag{17.137}$$

Finally, in terms of the global time coordinate t, we find the system of equations

$$\begin{aligned} \frac{dS^r}{dt} &= \frac{1}{u^t}\frac{dS^r}{d\tau} = \omega r \left(1 - \frac{3M}{r}\right) S^\varphi \\ \frac{dS^\varphi}{dt} &= \frac{1}{u^t}\frac{dS^\varphi}{d\tau} = -\omega \frac{1}{r} S^r \,. \end{aligned} \tag{17.138}$$

The Fermi frame with spacelike vectors oriented as those of the coordinate basis associated to (r, θ, φ), i.e. $\{\vec{e}_{(\mu)}\} = \left\{\frac{\partial}{\partial x^\mu}\right\}$, in the equatorial plane is

$$\begin{aligned} \vec{e}^{\,*}_{(0)} &= \vec{u} = u^t \left(\vec{e}_{(0)} + \omega \vec{e}_{(3)}\right) \\ \vec{e}^{\,*}_{(1)} &= \left(1 - \frac{2M}{r}\right)^{1/2} \vec{e}_{(1)} \\ \vec{e}^{\,*}_{(2)} &= \frac{1}{r}\vec{e}_{(2)} \\ \vec{e}^{\,*}_{(3)} &= u^t \left[\omega r \left(1 - \frac{2M}{r}\right)^{-1/2} \vec{e}_{(0)} + \left(1 - \frac{2M}{r}\right)^{1/2} \frac{1}{r}\vec{e}_{(3)}\right] \,, \end{aligned} \tag{17.139}$$

Note that the spacelike vector $\vec{e}^{\,*}_{(3)}$ is a linear combination of $\vec{e}_{(3)}$ and $\vec{e}_{(0)}$ (the latter is needed to have $\vec{e}^{\,*}_{(3)} \cdot \vec{u} = 0$), and it is orthogonal to the other space vectors $\vec{e}_{(1)}$ and $\vec{e}_{(2)}$. It is simple to check, using the property $\vec{e}_{(\mu)} \cdot \vec{e}_{(\nu)} = g_{\mu\nu}$ and Eq. 17.129, that $\vec{e}^{\,*}_{(\mu)} \cdot \vec{e}^{\,*}_{(\nu)} = \eta_{\mu\nu}$ (we leave the proof as an exercise).

The spin vector is

$$\vec{S} = S^\mu \vec{e}_{(\mu)} = S^{*\mu} \vec{e}^{\,*}_{(\mu)} \tag{17.140}$$

therefore, recalling that $\vec{e}_{(1)} \cdot \vec{e}^{\,*}_{(3)} = 0$ and using Eq. 17.131, we find that the components of the spin of the gyroscope in the Fermi frame are

$$\begin{aligned}
S^{*t} &= \vec{S} \cdot \vec{e}^{\,*}_{(0)} = \vec{S} \cdot \vec{u} = 0 \\
S^{*r} &= \vec{S} \cdot \vec{e}^{\,*}_{(1)} = S^r \vec{e}_{(1)} \cdot \vec{e}^{\,*}_{(1)} = \left(1 - \frac{2M}{r}\right)^{-1/2} S^r \\
S^{*\theta} &= \vec{S} \cdot \vec{e}^{\,*}_{(2)} = 0 \\
S^{*\varphi} &= \vec{S} \cdot \vec{e}^{\,*}_{(3)} = (S^t \vec{e}_{(0)} + S^r \vec{e}_{(1)} + S^\varphi \vec{e}_{(3)}) \cdot \vec{e}^{\,*}_{(3)} \\
&= \left[r^2 \left(1 - \frac{2M}{r}\right)^{-1} \omega S^\varphi \vec{e}_{(0)} + S^\varphi \vec{e}_{(3)} \right] \\
&\quad \cdot u^t \left[\omega r \left(1 - \frac{2M}{r}\right)^{-1/2} \vec{e}_{(0)} + \left(1 - \frac{2M}{r}\right)^{1/2} \frac{1}{r} \vec{e}_{(3)} \right] \\
&= u^t \left[-\omega^2 r^3 \left(1 - \frac{2M}{r}\right)^{-1/2} + r \left(1 - \frac{2M}{r}\right)^{1/2} \right] S^\varphi \\
&= u^t r \left(1 - \frac{2M}{r}\right)^{-1/2} \left(1 - \frac{2M}{r} - \omega^2 r^2\right) S^\varphi \\
&= r \left(1 - \frac{2M}{r}\right)^{-1/2} \left(1 - \frac{3M}{r}\right)^{1/2} S^\varphi .
\end{aligned} \tag{17.141}$$

Then, multiplying Eqs. 17.138 by $(1 - 2M/r)^{-1/2}$ and $r(1 - 3M/r)^{1/2}(1 - 2M/r)^{-1/2}$, respectively, we obtain

$$\begin{aligned}
\frac{dS^{*r}}{dt} &= \omega r \left(1 - \frac{3M}{r}\right) \left(1 - \frac{2M}{r}\right)^{-1/2} S^\varphi = \omega \left(1 - \frac{3M}{r}\right)^{1/2} S^{*\varphi} \\
\frac{dS^{*\varphi}}{dt} &= -\omega \left(1 - \frac{3M}{r}\right)^{1/2} S^{*r} .
\end{aligned} \tag{17.142}$$

Defining

$$\omega' = \left(1 - \frac{3M}{r}\right)^{1/2} \omega \,, \tag{17.143}$$

Eqs. 17.142 become

$$\begin{aligned}
\frac{dS^{*r}}{dt} &= \omega' S^{*\varphi} \\
\frac{dS^{*\varphi}}{dt} &= -\omega' S^{*r} .
\end{aligned} \tag{17.144}$$

These equations show that the spin vector rotates with frequency ω' in the comoving frame

$\{\vec{e}^{\,*}_{(r)}, \vec{e}^{\,*}_{(\varphi)}\}$. If, for instance, at $t = 0$ the spin points towards the radial direction, $S^{*r}(0) = S_0$, $S^{*\varphi} = 0$, then $S^{*r}(t) = S_0 \cos\omega' t$ and $S^{*\varphi}(t) = -S_0 \sin\omega' t$.

We now determine the spin of the gyroscope with respect to the asymptotically flat directions, i.e. the directions which, in the far-field limit, correspond to the Cartesian coordinate lines of the Minkowski metric $\eta_{\mu\nu}$. These are often called "fixed stars directions" (neglecting the effect of the cosmological expansion).

To this aim, reminding that we have the freedom to rotate the spatial vectors of the Fermi frame, we change the basis vectors $\{\vec{e}^{\,*}_{(r)}, \vec{e}^{\,*}_{(\varphi)}\}$ to $\{\vec{e}^{\,*}_{(x)}, \vec{e}^{\,*}_{(y)}\}$, parallel to the Cartesian coordinate axes in the far-field limit, defined as follows:

$$\begin{aligned} \vec{e}^{\,*}_{(x)} &= \cos\varphi \vec{e}^{\,*}_{(r)} - \sin\varphi \vec{e}^{\,*}_{(\varphi)} \\ \vec{e}^{\,*}_{(y)} &= \sin\varphi \vec{e}^{\,*}_{(r)} + \cos\varphi \vec{e}^{\,*}_{(\varphi)} \, . \end{aligned} \tag{17.145}$$

These expressions can be obtained from the rotation $x = r\cos\varphi$, $y = r\sin\varphi$, by inverting Eq. 3.132. We stress that the Cartesian vectors $\vec{e}^{\,*}_{(x)}$, $\vec{e}^{\,*}_{(y)}$ are still vectors of the local Fermi frame of the gyroscope, with a different space orientation.

The spin components along the Cartesian directions are

$$\begin{aligned} S^{*x} &= \vec{S} \cdot \vec{e}^{\,*}_{(x)} = (S^{*r}\vec{e}^{\,*}_{(r)} + S^{*\varphi}\vec{e}^{\,*}_{(\varphi)}) \cdot (\cos\varphi \vec{e}^{\,*}_{(r)} - \sin\varphi \vec{e}^{\,*}_{(\varphi)}) \\ &= \cos\varphi S^{*r} - \sin\varphi S^{*\varphi} \\ S^{*y} &= \vec{S} \cdot \vec{e}^{\,*}_{(y)} = (S^{*r}\vec{e}^{\,*}_{(r)} + S^{*\varphi}\vec{e}^{\,*}_{(\varphi)}) \cdot (\sin\varphi \vec{e}^{\,*}_{(r)} + \cos\varphi \vec{e}^{\,*}_{(\varphi)}) \\ &= \sin\varphi S^{*r} + \cos\varphi S^{*\varphi} \, , \end{aligned} \tag{17.146}$$

and since $S^{*r}(t) = S_0 \cos\omega' t$, $S^{*\varphi} = -S_0 \sin\omega' t$ and, along the circular orbit, $\varphi = \omega t$,

$$\begin{aligned} S^{*x} &= S_0(\cos\omega t \cos\omega' t + \sin\omega t \cos\omega' t) = S_0 \cos(\omega - \omega')t \\ S^{*y} &= S_0(\sin\omega t \cos\omega' t - \cos\omega t \sin\omega' t) = -S_0 \sin(\omega - \omega')r \, . \end{aligned} \tag{17.147}$$

In the Newtonian limit $\omega = \omega'$ and thus (S^{*x}, S^{*y}) are constant: as it is well known, a moving gyroscope keeps its direction fixed wih respect to far-away stars. In General Relativity, instead, the gyroscope precesses with respect to the "fixed stars directions", with the **geodesic precession** frequency

$$\omega_{\rm GP} = \omega - \omega' = \left[1 - \left(1 - \frac{3M}{r}\right)^{1/2}\right]\omega \simeq \frac{3M}{2r}\omega = \frac{3M}{2r}\sqrt{\frac{M}{r^3}} \, , \tag{17.148}$$

where we have neglected higher-order terms in $1/r$ while using Eq. 17.143.

When the central body rotates, both the geodesic precession and the Lense-Thirring precession are present. It can be shown that (in the weak-field limit) the spin, measured in the local frame with the orientation of the asymptotically Cartesian directions, satisfies the equation

$$\frac{d\mathbf{S}^*}{dt} = (\boldsymbol{\omega}_{\rm GP} + \boldsymbol{\omega}_{\rm LT}) \times \mathbf{S}^* \tag{17.149}$$

where $\boldsymbol{\omega}_{\rm GP}$ is the geodesic precession frequency (which reduces, in the case of circular equatorial motion, to Eq. 17.148) and $\boldsymbol{\omega}_{\rm LT}$ is the Lense-Thirring frequency given in Eq. 17.126.

As mentioned above, the geodesic precession frequency is typically much larger than the Lense-Thirring frequency. Let us consider, for instance, a gyroscope in a spacecraft moving on a circular, equatorial orbit 600 km above the surface of Earth (i.e. about 7000 km from the center of Earth). In this case the Earth angular momentum is orthogonal to the position vector **x** of the gyroscope, hence Eqs. 17.126, 17.148 give (in physical units, see Box 9-A)

$$|\omega_{\rm LT}| = \frac{GJ}{c^2 r^3} \quad (17.150)$$

$$|\omega_{\rm GP}| = \frac{3GM}{2c^2 r}\sqrt{\frac{GM}{r^3}} \quad (17.151)$$

where $M = 5.9 \times 10^{27}$ g, $r = 7.0 \times 10^8$ cm, and, with the rough approximation of a uniform-density Earth, $J \sim \frac{2}{5}MR^2\Omega_\oplus$. Since the rotation frequency of the Earth is $\Omega_\oplus = 2\pi/86400 = 7.3 \times 10^{-5}$ s^{-1} and $G = 6.7 \times 10^{-8}$ cm^3 g^{-1}s^{-2}, $c = 3.0 \times 10^{10}$ cm/si (see Table 0.1), we find $J \sim 7 \times 10^{40}$ g cm^2 s^{-1}; hence $|\omega_{\rm LT}| \sim 1.5 \times 10^{-14}$ rad/s ~ 0.1 arcsec/year, and $|\omega_{\rm GP}| \sim 10^{-12}$ rad/s ~ 6 arcsec/year.

The configuration discussed above maximizes the Lense-Thirring effect. If the angular momentum of the cental object is not orthogonal to the orbital plane (as in the case of an orbit along a meridian circle of the Earth, see Section 17.4.3) the Lense-Thirring frequency is even smaller, as shown by Eq. 17.126, while the geodesic precession frequency is the same.

17.4.3 Measurement of geodesic and Lense-Thirring frequencies: Gravity Probe B

The idea of an experiment to measure the geodesic and Lense-Thirring precessions dates back to 1960, when Shiff [97] noted that a gyroscope on a spacecraft orbiting around Earth would undergo geodesic and Lense-Thirring precessions with respect to the directions of "fixed stars" at infinity (see also [92] for a similar, independent proposal). An estimate of the order of magnitude of these effects showed that they would be "difficult, but not impossible, to observe".

The realization of this experiment – called Gravity Probe B (GPB) – has been extremely difficult. The satellite was launched in 2004 and the mission ended in 2005; after that, six more years were needed to analyze the data and understand all sources of error. In 2011 the final results were published [42]: the geodesic precession was measured with an accuracy of $\simeq 0.3\%$, and the more elusive Lense-Thirring precession was measured with an accuracy of $\simeq 20\%$. The observed values are compatible with the predictions of General Relativity, and can be considered a further *kinematical test of General Relativity*, which adds to those discussed in Chapter 11.

As discussed above the geodesic precession frequency is much larger than the Lense-Thirring frequency. Thus if $\boldsymbol{\omega}_{\rm GP}$ is parallel to $\boldsymbol{\omega}_{\rm LT}$, the Lense-Thirring frequency is "buried" in the geodesic precession frequency, increasing the difficulty of the measurement. To overcome this problem the orbit of GPB has been chosen to be orthogonal to the equatorial plane (see Fig. 17.2), and the spin of the gyroscope has been oriented orthogonally to both the angular momenta of the Earth and of the orbit. Thus, in the Cartesian frame $(Oxyz)$ in Fig. 17.2, the angular momentum of the Earth is $J^i = (0, 0, J)$, the orbit lies in the $x - z$ plane and the spin of the gyroscope is $S^i = (S, 0, 0)$. With this configuration, the geodesic precession frequency - which is always ortoghonal to the orbital plane - is parallel to the y-axis, and determines a precession of the spin in the $x - z$ plane, i.e. in the North-South direction. The Lense-Thirring frequency 17.126, instead, lies in the orbital plane $x - z$ (because both the Earth angular momenum **J** and the radial direction **n** belong to that plane), and since the spin **S** is parallel to the x-axis, the Lense-Thirring precession 17.126 – $\delta\mathbf{S} \sim \boldsymbol{\omega}_{\rm LT} \times \mathbf{S} = (0, \omega^z_{\rm LT}S, 0)$ – is parallel to the y-axis, i.e. in the West-East direction. It was thus possible to independently measure the two effects.

Taking into account the actual value of the Earth moment of inertia (known from geophysical modelling of the Earth) $I_\oplus = 8.0 \times 10^{44}$ km m^2, Eq. 17.151 yields $|\omega_{\rm GP}| = 6.6$ arcsec/year. To compute the Lense-Thirring frequency 17.126 we note that $\mathbf{n} =$

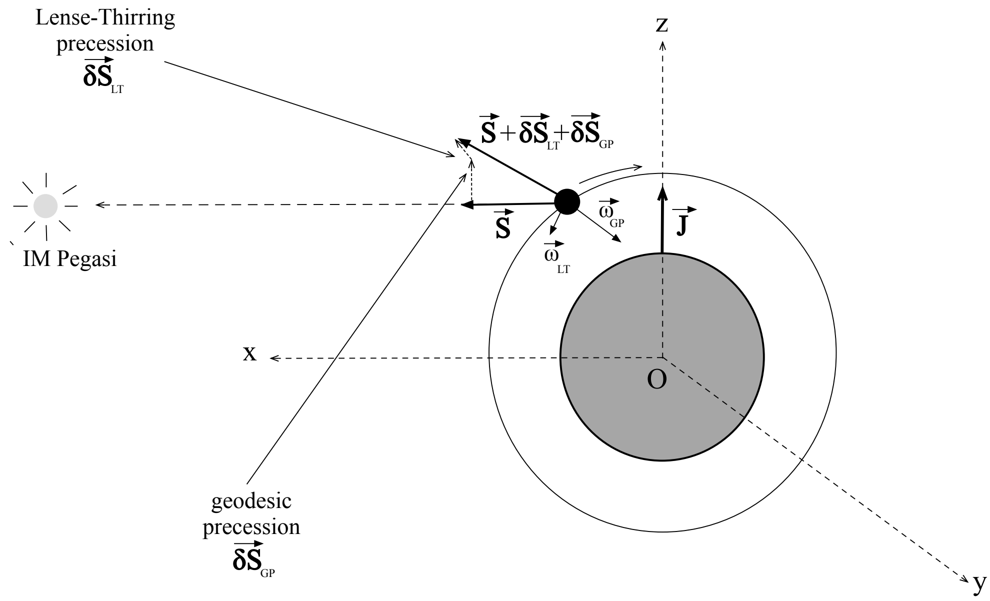

Figure 17.2: Schematic description of the GPB experiment. The spin of the satellite has a geodesic precession from South to North, and a Lense-Thirring precession from East to West.

$(\cos\Omega_\oplus t, 0, \sin\Omega_\oplus t)$ and $\mathbf{J} = (0, 0, J)$, therefore

$$-\mathbf{J} + 3(\mathbf{J}\cdot\mathbf{n})\mathbf{n} = (3J\sin\Omega_\oplus t\cos\Omega_\oplus t, 0, -J(1-3\sin^2\Omega_\oplus t)) \tag{17.152}$$

hence, averaging the z-component of the Lense-Thirring frequency on an orbit with period $T = 2\pi/\Omega_\oplus$,

$$\langle\omega^z_{\rm LT}\rangle = \frac{1}{T}\int_0^T J\left(3\sin^2\frac{2\pi}{T}t - 1\right)dt = \frac{J}{2\pi}\int_0^{2\pi}\left(3\sin^2\alpha - 1\right)d\alpha = \frac{1}{2}J\,, \tag{17.153}$$

which, in physical units, gives

$$|\omega^z_{\rm LT}| = \frac{GI\Omega_\oplus}{2c^2r^3} = 0.04\,\text{arcsec/year}\,. \tag{17.154}$$

GPB contained four gyroscopes, which were electrostatically suspended rotating spheres. Each sphere contained a superconducting loop producing a magnetic dipole moment parallel to the spin direction. A magnetometer measured the direction of the magnetic moment, and then of the spin, of each gyroscope. These directions were compared with the direction of the far-away star IM Pegasi, determined by a telescope on the spacecraft. The results obtained by combining the observations of the four gyroscopes are shown in Table 17.1, and compared with the theoretical predictions of General Relativity.

	$\omega_{\rm LT}$ (milli-arcsec/year)	$\omega_{\rm GP}$ (milli-arcsec/year)
General Relativity prediction	-39.2	-6606.1
GPB	-37.2 ± 7.2	-6601.8 ± 18.3

Table 17.1: Precession frequency measured by Gravity Probe B along the North-South direction (Lense-Thirring frequency) and along the West-East direction (geodesic precession frequency), compared with the theoretical values predicted by General Relativity.

Remarkably, when the results of this experiment were released the geodesic and Lense-Thirring effect of the Earth had already been measured by analyzing the motion of a pair of satellites, whose position was constantly monitored by sending laser impulses from the Earth to the satellites. This experiment, called LAGEOS, required the subtraction of the non-spherical component of the Earth gravitational field, which had previously been determined by a similar experiment, GRACE. In 2004 the LAGEOS experiment measured the geodesic precession frequency with an accuracy of $\simeq 0.7\%$, and the Lense-Thirring frequency with an accuracy of $\simeq 10\%$. The accuracy of these measurements has significantly been improved with a recent analysis of the LAGEOS data [34, 76].

We also note that while the Lense-Thirring effect was firstly observed by the GPB and LAGEOS experiments, the geodesic precession had been measured decades earlier studying the motion of binary pulsars such as PSR 1913+16 (see e.g. [110] and references therein).