9.5 SINGULARITIES IN GENERAL RELATIVITY

9.5.1 Geodesic completeness

How can we define a singularity in General Relativity? This is not a trivial issue, since singularities (either curvature or coordinate ones) *do not belong* to the spacetime manifold. As discussed in Sec. 9.3, looking at the singular points of the metric components $g_{\mu\nu}$ can be misleading: changing coordinates some of them could be mapped to regular points, some others could be mapped to infinity.

We clarify the last remark with an example. Let us consider a two-dimensional manifold with metric

$$ds^2 = \frac{1}{(x^2+y^2)^2}\left(dx^2+dy^2\right) . \tag{9.61}$$

Since the metric components are singular at $(x,y)=(0,0)$, one may think that this point (or, more precisely, this limit) is a singularity. However, the manifold with metric 9.61 is just the flat Euclidean space in a particular coordinate frame; indeed, changing to the frame (x',y') with

$$x' = \frac{x}{x^2+y^2} , \tag{9.62}$$

$$y' = \frac{y}{x^2+y^2} , \tag{9.63}$$

the metric 9.61 becomes $ds^2=(dx')^2+(dy')^2$ (we leave the proof as an exercise). In the coordinates (x',y'), the point $(x,y)=(0,0)$ is the limit $(x',y')\to(\infty,\infty)$ (the so-called "point at infinity"). Obviously, the point at infinity does not belong to the manifold (remind that a manifold is an open set, see Chapter 2), but this does not mean that the Euclidean space is singular. We can conclude that, although $(x,y)=(0,0)$ does not belong to the manifold, it should not be considered a singularity because it is just the point at infinity in disguise.

This example suggests a way to characterize the singularities: they can be seen as a sort of "hole", or "edge" of the spacetime, which does not belong to the manifold, but which *can actually be reached by a physical object*. If, instead, the spacetime is ill-defined in a certain limit ($(x,y)\to(0,0)$, i.e. $(x',y')\to(\infty,\infty)$ in the example above) which cannot be reached by physical objects, such limit should not be considered as a singularity.

We now introduce a formal, coordinate-invariant definition of singularities based on the following property: a spacetime is **geodesically complete** if every timelike and null geodesic can be extended to arbitrarily large values of the affine parameter. If the spacetime admits at least one incomplete (i.e., which cannot be extended) timelike or null geodesic, the spacetime is **geodesically incomplete**[4]. We say that the spacetime has a **singularity** if it is geodesically incomplete.

Coming back to the manifold with metric 9.61, no geodesic going towards $(x,y)=(0,0)$ reach that limit for a finite value of the affine parameter. For instance, the geodesic $(x'(\lambda),y'(\lambda))=(\lambda,0)$, in the coordinates (x,y) has the form $(x(\lambda),y(\lambda))=(\lambda^{-1},0)$, and $x\to 0$ as $\lambda\to\infty$.

We remark that this definition applies both to curvature and coordinate singularity. In the following we shall discuss the difference between these two classes of singularities, and how a coordinate singularity can be removed.

[4] We only consider timelike and null geodesics because, unlike spacelike geodesics which cannot be associated to the motion of physical objects, they describe the motion of massive and massless particles, i.e. of observers or signals.

9.5.2 How to remove a coordinate singularity

Some singularities can be removed with the following procedure. Let $\mathcal{M}$ be a spacetime manifold with a (timelike or null) geodesic which cannot be extended beyond a finite value of the affine parameter. Let $g_{\mu\nu}$ be the components of the metric tensor in a given coordinate frame $\{x^\mu\}$, defined in a domain $U \subset \mathbb{R}^4$. We remark that, by definition, the metric components $g_{\mu\nu}$ and the components of the inverse metric, $g^{\mu\nu}$, are regular in U (the latter requirement is equivalent to $g = \det(g_{\mu\nu}) \neq 0$ in U). To remove the singularity we follow these steps:

- We choose a new coordinate frame $\{x^{\alpha'}\}$. The domain $V \subset \mathbb{R}^4$ of the new coordinates is the image of U through the coordinate transformation; we choose the transformation $x^\mu \to x^{\alpha'}$ such that the metric components $g_{\alpha'\beta'}$ are regular and invertible in a *larger* domain in $\mathbb{R}^4$, $V' \supset V$.
- Let $\mathcal{M}'$ be the manifold described by the coordinates $\{x^{\alpha'}\}$ defined in the larger domain V', endowed with the metric tensor $g_{\alpha'\beta'}$. We have

$$\mathcal{M}' \supset \mathcal{M}\,. \tag{9.64}$$

  We say that the spacetime has been *extended* if we consider the larger manifold $\mathcal{M}'$ as the spacetime manifold.

The singularity corresponding to the incomplete geodesic has been *removed* if, in the new spacetime, that geodesic can be extended to arbirarily large values of the affine parameter.

As anticipated above, only some singularities can be removed with this procedure. They are called **coordinate singularities**. Those which cannot be removed are true spacetime singularities, and are called **curvature singularities**. As discussed in Sec. 9.3, curvature invariants are regular on coordinate singularities, while they can diverge approachng curvature singularities.

If several coordinate singularities are present the above procedure can be repeated, further extending the spacetime manifold. Once all coordinate singularities are removed, we have the *maximal extension* of the spacetime: in this case, all (timelike or null) geodesics which cannot be extended to arbitrarily large values of the affine parameter, correspond to true curvature singularities.

In the following we shall discuss a simple example of spacetime with a coordinate singularity: the Rindler spacetime, which presents interesting similarities with the Schwarzschild geometry. Subsequently, we shall discuss how to remove the $r = 2m$ singularity of the Schwarzschild spacetime.

Box 9-C

Some remarks on coordinate singularities

A metric space $\mathcal{M}$ (i.e. a manifold endowed with a metric tensor, see Section 2.5) is called *extendible* if it coincides with a subset of another metric space $\mathcal{M}'$, and the metric of $\mathcal{M}'$, restricted to this subset, coincides with the metric of $\mathcal{M}$. It has been argued (see e.g. [43, 106, 52]) that the spacetime describing the Universe should be *inextendible*. This assumption means that our spacetime is a maximal extension, and all singularities are curvature singularities. Many authors assume inextendibility in modelling the spacetime, although strictly speaking there is no actual proof of this conjecture.

Thus, when we refer to curvature singularities as "true" spacetime singularities, and to coordinate singularities as mere artefacts of an inadequate choice of the coordinate frame, we are implicitly assuming that the spacetime is inextendible.

9.5.3 Extension of the Rindler spacetime

The metric of the Rindler spacetime in two spacetime dimensions is

$$ds^2 = -x^2 dt^2 + dx^2, \qquad -\infty < t < \infty, \qquad 0 < x < \infty\,. \tag{9.65}$$

The metric is singular at $x \to 0$. Indeed, the determinant g vanishes in this limit, and $g^{\mu\nu}$ diverges.

Let $x^\mu(\tau)$ be a timelike geodesic in this spacetime, with proper time τ (we remind that the proper time is an affine parameter for timelike geodesics, see Chapter 3) and four-velocity $u^\mu = \frac{dx^\mu}{d\tau}$. Since the metric is independent of time, it admits a timelike Killing vector with components, in the frame (t,x), $k^\mu = (1,0)$. According to Eq. 8.52,

$$k_\alpha u^\alpha = g_{\alpha\beta} k^\alpha u^\beta = -x^2 u^0 = const \equiv -E\,, \tag{9.66}$$

therefore

$$u^0 = \frac{dt}{d\tau} = \frac{E}{x^2}\,. \tag{9.67}$$

Since the norm of the vector tangent to a timelike geodesic parametrized with proper time is -1,

$$u^\mu u^\nu g_{\mu\nu} = -x^2 \left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dx}{d\tau}\right)^2 = -1\,, \tag{9.68}$$

it follows that

$$\left(\frac{dx}{d\tau}\right)^2 = \frac{E^2}{x^2} - 1\,, \tag{9.69}$$

hence

$$\frac{dx}{d\tau} = \pm\sqrt{\frac{E^2}{x^2} - 1}, \quad \to \quad \tau = \int^x \frac{x dx}{\sqrt{E^2 - x^2}} = -\sqrt{E^2 - x^2} + const\,. \tag{9.70}$$

Thus, a particle starting its motion at some point x reaches $x = 0$ in a finite interval of the affine parameter: the Rindler spacetime is geodesically incomplete. However, as can easily be checked, the curvature scalars do not diverge at $x = 0$, therefore this could be

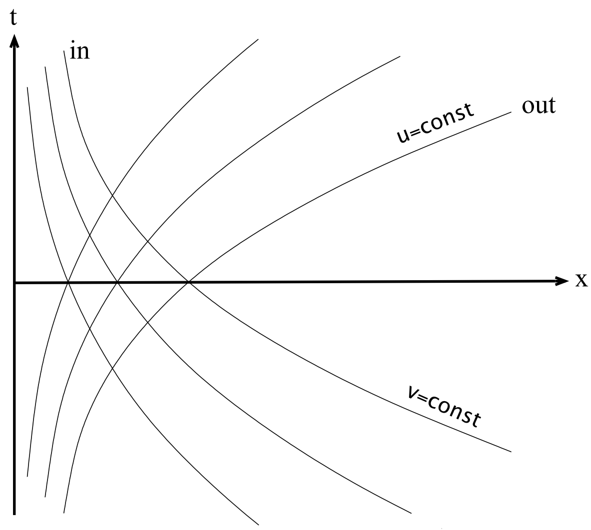

Figure 9.6: The Rindler spacetime in the coordinates (t, x). The logarithmic curves are, respectively, outgoing ($u = const$) and ingoing ($v = const$) null geodesics.

a mere coordinate singularity, which might be removed with a coordinate transformation. Unfortunately, a systematic approach to the problem of finding the coordinates which allow to extend the spacetime does not exist. We shall describe a procedure which is based on the behaviour of null geodesics, and which – as can be seen *a posteriori* – in some cases allows to find the appropriate transformation to remove a coordinate singularity.

Let $x^\mu(\lambda)$ be a null geodesic, with affine parameter λ and tangent vector

$$u^\mu = \frac{dx^\mu}{d\lambda} \,. \tag{9.71}$$

Since the geodesic is null,

$$g_{\mu\nu} u^\mu u^\nu = -x^2 \left(\frac{dt}{d\lambda}\right)^2 + \left(\frac{dx}{d\lambda}\right)^2 = 0 \,, \tag{9.72}$$

hence

$$\left(\frac{dt}{dx}\right)^2 = \frac{1}{x^2} \,. \tag{9.73}$$

The solution to this equation

$$t = \pm \log x + const, \tag{9.74}$$

shows that there are two families of null geodesics belonging to the $+$ and $-$ sign, respectively. As can be seen from Fig. 9.6, the "+" sign identifies the outgoing geodesics, for which time increases as x increases, whereas the "−" sign identifies the ingoing geodesics, for which time increases as x decreases. Accordingly, we can define the *null outgoing and ingoing coordinates*

$$u = t - \log x \qquad \text{and} \qquad v = t + \log x \,. \tag{9.75}$$

They are constant along any outgoing or ingoing geodesic, respectively. Since $dudv = dt^2 - x^{-2}dx^2$ and $e^{v-u} = x^2$, the metric 9.65 becomes

$$ds^2 = -e^{v-u}dudv. \tag{9.76}$$

The coordinates u and v are defined in the range $(-\infty, +\infty)$, and cover the original region $x > 0$, $-\infty < t < +\infty$. Since the singular point $x = 0$ (with t finite) is mapped to the point at infinity $(u, v) \to (+\infty, -\infty)$, the coordinate frame (u, v) does not allow to extend the spacetime, i.e. to apply the procedure described in Sec. 9.5.2.

We shall now define a new coordinate system (U, V), such that along any null geodesic, one coordinate is an affine parameter and the other is constant. In this new frame it will be possible to extend the spacetime and remove the coordinate singularity $x = 0$.

Let us consider an *outgoing* null geodesic, $u = const$, and the timelike Killing vector field $\vec{k}$ admitted by Rindler's metric. From Eqs. 9.66 and 9.71 it follows that

$$k_\alpha u^\alpha = g_{\alpha\beta}k^\alpha u^\beta = -x^2u^0 = const \equiv -E, \qquad \to \qquad d\lambda = \frac{x^2}{E}dt\,. \tag{9.77}$$

Since along a $u = const$ geodesic $dt = \frac{1}{2}d(u + v) = \frac{1}{2}dv$, we get

$$d\lambda = \frac{x^2}{2E}dv = \frac{e^{v-u}}{2E}dv = Ce^v dv \tag{9.78}$$

where $C = e^{-u}/(2E)$ is constant. In the same way, if we consider an *ingoing* null geodesic $v = const$,

$$d\lambda = \frac{x^2}{2E}du = \frac{e^{v-u}}{2E}du = C'e^{-u}du \tag{9.79}$$

with $C' = e^v/(2E)$ constant. If we define

$$\begin{aligned} U(u) &= -e^{-u} \\ V(v) &= e^v\,, \end{aligned} \tag{9.80}$$

from Eqs. 9.78 and 9.79 it follows that along a null outgoing geodesic $d\lambda = CdV$, and along a null ingoing geodesic $d\lambda = C'dU$. This means that on null outgoing geodesics

$$\lambda = CV + const\,, \tag{9.81}$$

and on null ingoing geodesics

$$\lambda = C'U + const\,. \tag{9.82}$$

Eqs. 9.81, 9.82 show that V, U are linear functions of λ on null outgoing and ingoing geodesics, respectively. Thus, since linear transformations map affine parameters into affine parameters, V is an affine parameter for the outgoing geodesics $U = const$, and U is an affine parameter for the ingoing geodesics $V = const$, i.e. (U, V) are the coordinates we were looking for.

Since $dU = e^{-u}du$ and $dV = e^v dv$, in the new coordinates the line element 9.76 simply becomes

$$ds^2 = -dUdV\,. \tag{9.83}$$

This metric is clearly free of singularities.

At this point it is useful to remind that the Rindler metric 9.65 was defined in the region $0 < x < \infty$, $-\infty < t < \infty$ of the (t, x) coordinates; this region was mapped to $-\infty < (u, v) < +\infty$ in the (u, v) coordinates, that corresponds to the domain $U < 0$, $V > 0$ in the (U, V) plane (see Fig. **??**). However, the line element 9.83 is perfectly well-behaved

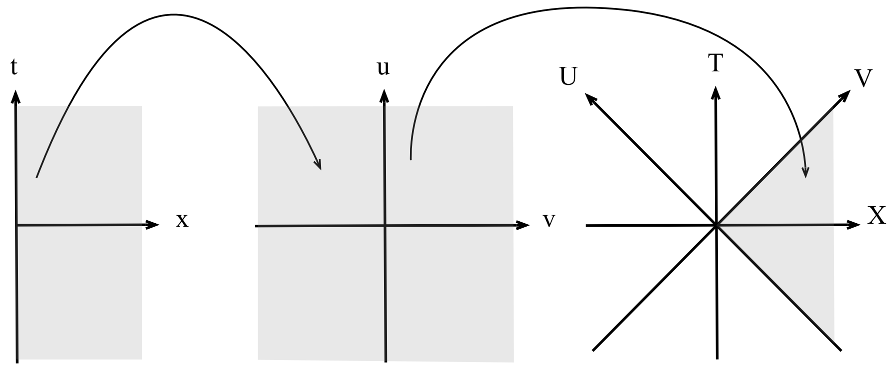

Figure 9.7: Different coordinate frames for Rindler's spacetime: (t, x) (with $x > 0$) is mapped to (u, v) (with (u, v) ranging between $(-\infty, +\infty)$), which is mapped to the region $U < 0$, $V > 0$ in the (U, V) plane. The (U, V) coordinates allow to extend the spacetime manifold.

in the entire (U, V) plane, $-\infty < U < +\infty\,, -\infty < V < +\infty$, therefore we can extend the original (Rindler) spacetime manifold $\mathcal{M}$, to the entire (U, V) space, and by defining the new coordinates (T, X) through the transformation

$$U = T + X\,, \qquad V = T - X\,, \tag{9.84}$$

the line element 9.83 becomes

$$ds^2 = -dT^2 + dX^2\,, \tag{9.85}$$

which is the metric of (two-dimensional) Minkowski's spacetime in Cartesian coordinates, defined in the domain $-\infty < T < +\infty\,, -\infty < X < +\infty$, i.e. in the Minkowski manifold $\mathcal{M}' \supset \mathcal{M}$ corresponding to the entire $T - X$ plane. Indeed, the Rindler metric is just a boosted version of Minkowski's metric.

With this procedure, we have eliminated the coordinate singularity in $x = 0$ of Rindler's spacetime and we have extended the spacetime to a larger manifold.

The relation between the initial coordinates (t, x) and the final coordinates (T, X) in the region $U < 0$, $V > 0$ (we leave it as an exercise) is

$$\begin{aligned} x &= (X^2 - T^2)^{\frac{1}{2}} \\ t &= \tanh^{-1}\left(\frac{T}{X}\right) = \frac{1}{2}\log\left(\frac{X+T}{X-T}\right). \end{aligned} \tag{9.86}$$

The singularity $x = 0$ corresponds to the lines $T = \pm X$. From the second of Eqs. 9.86

$$\begin{aligned} T &= -X \qquad &\text{corresponds to} \qquad &t \to -\infty \\ T &= X \qquad &\text{corresponds to} \qquad &t \to +\infty\,. \end{aligned} \tag{9.87}$$

The curves $x = const$ are the hyperbolae $X^2 - T^2 = const$, while the curves $t = const$ correspond to the straight lines

$$\frac{X+T}{X-T} = const \qquad \to \qquad T = const\,X\,. \tag{9.88}$$

An illustration of the original and extended spacetimes $\mathcal{M}$, $\mathcal{M}'$ is given in Figure 9.8. The Rindler space corresponds to the shaded region in the figure.

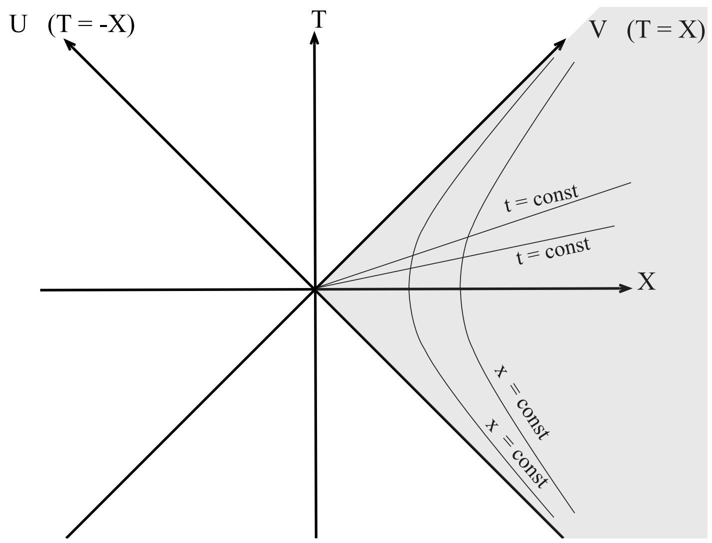

Figure 9.8: Rindler spacetime in the (U, V) coordinates. The curves $t = const$ and $x = const$, i.e. the coordinate lines in the initial (t, x) frame, are hyperbolae and straight lines, respectively, in the (U, V) frame.

9.5.4 Extension of the Schwarzschild spacetime

Let us now consider the Scharzschild spacetime. In the coordinates (t, r, θ, φ) the metric is given by Eq. 9.35, i.e.

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - \dfrac{2m}{r}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{9.89}$$

Strictly speaking, since the metric 9.89 is not defined at $r = 0$ and $r = 2m$, it describes the union of two disconnected manifolds, $\mathcal{M}_1$ with $0 < r < 2M$ (the black hole interior) and $\mathcal{M}_2$ with $r > 2M$ (the black hole exterior). A timelike geodesic, i.e. the worldline of a point particle (or of an observer), falling toward the black hole cannot be extended across $r = 2M$, since this hypersurface does not belong to $\mathcal{M}_1 \cup \mathcal{M}_2$, and the geodesic terminates at a finite value of the affine parameter as[5] $r \to 2m$. However, since $r = 2m$ is not a curvature singularity, it can be removed with the procedure outline in Sec. 9.5.2.

Let us consider a null geodesic $x^\mu(\lambda)$ in the Schwarzschild spacetime 9.89, with $\theta = const$, $\varphi = const$. The tangent vector

$$u^\mu = \frac{dx^\mu}{d\lambda} = \left(\frac{dt}{d\lambda}, \frac{dr}{d\lambda}, 0, 0\right) \tag{9.90}$$

[5] Actually, $r = 2m$ is not the unique coordinate singularity of the Schwazschild spacetime. Other singular points of the metric 9.89 are $\theta = 0, \pi$, $\varphi = 0, 2\pi$. These coordinate singularities are also present in Minkowski space in polar coordinates, and can easily be removed with a space rotation (see Box 2-C). These coordinate singularities are therefore "trivial", and will not be discussed here. However, as we shall see in Chapter 18, in other spacetimes the coordinate singularities $\theta = 0, \pi$ can acquire a subtler meaning and have to be studied in detail.

is a null vector, thus

$$g_{\mu\nu}u^\mu u^\nu = -\left(1-\frac{2m}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{2m}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 = 0\,. \tag{9.91}$$

Hence

$$\left(\frac{dr}{dt}\right)^2 = \left(1-\frac{2m}{r}\right)^2 \quad \rightarrow \quad \frac{dt}{dr} = \pm\frac{r}{r-2m}\,, \tag{9.92}$$

the solution of which is

$$t = \pm r_* + const \tag{9.93}$$

where

$$r_* = r + 2m\log\left(\frac{r}{2m}-1\right) \quad \text{if} \quad r > 2m \tag{9.94}$$

and

$$r_* = r + 2m\log\left(-\frac{r}{2m}+1\right) \quad \text{if} \quad 0 < r < 2m\,. \tag{9.95}$$

The coordinate r_* is called "tortoise" coordinate [6]. As $r \to +\infty$, $r_* \sim r$, while as $r \to 2m$, $r_* \to -\infty$. In other words, this change of the radial variable "pushes" the horizon to $-\infty$. Thus, as in Rindler's spacetime, in the Schwarzschild spacetime there exist two congruences of null geodesics (with θ, φ constant) corresponding, respectively, to the $+$ and $-$ sign in Eq. 9.93, and it is natural to define the null coordinates

$$u \equiv t - r_*\,, \qquad v \equiv t + r_*\,. \tag{9.96}$$

Null outgoing geodesics correspond to $u = const$, and null ingoing geodesics to $v = const$. Note that there are *two* (u, v) maps, both defined in $-\infty < u < +\infty$, $-\infty < v < +\infty$: one, with the definition 9.94, corresponding to the the manifold $\mathcal{M}_2$ $(r > 2m)$; the other, with the definition 9.95, corresponding to the the manifold $\mathcal{M}_1$ $(0 < r < 2m)$.

Let us consider the manifold $\mathcal{M}_2$ (the black hole exterior) $r > 2m$. Since

$$\frac{dr_*}{dr} = \frac{1}{1-\frac{2m}{r}}\,, \tag{9.97}$$

the metric in the coordinates (u, v, θ, φ) is

$$\begin{aligned} ds^2 &= -\left(1-\frac{2m}{r}\right)(dt^2 - dr_*^2) + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \\ &= -\left(1-\frac{2m}{r}\right)dudv + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,. \end{aligned} \tag{9.98}$$

Note that now r should not be considered as a coordinate: it is a function of the coordinates u and v, i.e. $r(u, v)$. The metric 9.98 is still singular at $r = 2m$. As in the Rindler case, we consider a new coordinate frame (U, V, θ, φ) such that – at least near the horizon – V is an affine parameter of the outgoing null geodesics, while U is an the affine parameter of ingoing null geodesics.

On null geodesics,

$$k_\alpha u^\alpha = g_{\alpha\beta}k^\alpha u^\beta = -\left(1-\frac{2m}{r}\right)\frac{dt}{d\lambda} = const \equiv -E\,, \tag{9.99}$$

[6] Like the famous Zeno's tortoise, the coordinate r_* "never" reaches the horizon $r = 2m$, but approaches it logarithmically.

where $\vec{k}$ is the timelike Killing vector field admitted by the Schwarzschild spacetime whose components, in the (t, r, θ, φ) frame, are $k^\alpha = (1, 0, 0, 0)$. Moreover, on $\mathcal{M}_2$ from Eq. 9.94 we find

$$\frac{r_* - r}{2m} = \ln\left(\frac{r}{2m} - 1\right) \qquad \rightarrow \qquad 1 - \frac{2m}{r} = \frac{2m}{r} e^{\frac{r_* - r}{2m}} = \frac{2m}{r} e^{-\frac{r}{2m}} e^{\frac{v-u}{4m}} . \tag{9.100}$$

Therefore, on null outgoing geodesics, where $u = const$ and $dt = \frac{dv}{2}$,

$$d\lambda = \left(1 - \frac{2m}{r}\right)\frac{dv}{2E} = \frac{2m}{r} e^{-\frac{r}{2m}} e^{-\frac{u}{4m}} e^{\frac{v}{4m}} dv . \tag{9.101}$$

Similarly, on null ingoing geodesics, $v = const$ and $dt = \frac{du}{2}$, therefore

$$d\lambda = \left(1 - \frac{2m}{r}\right)\frac{du}{2E} = \frac{2m}{r} e^{-\frac{r}{2m}} e^{\frac{v}{4m}} e^{-\frac{u}{4m}} du . \tag{9.102}$$

We now define the coordinates

$$U \equiv -e^{-\frac{u}{4m}} , \qquad V \equiv e^{\frac{v}{4m}} . \tag{9.103}$$

Near the horizon, as $r \to 2m$, from Eq. 9.101 it follows that, on the null *outgoing* geodesics $U = const$, the affine parameter is given by $d\lambda \propto e^{\frac{v}{4m}} dv$, and consequently $d\lambda = C dV$ with C constant, i.e. V is an affine parameter for outgoing geodesics as in the Rindler case (see Eq. 9.81).

Similarly, from Eq. 9.102 it follows that, on the null *ingoing* geodesics $V = const$, the affine parameter is given by $d\lambda \propto e^{-\frac{u}{4m}} du$, and consequently $d\lambda = C' dU$ with C' constant, i.e. U is an affine parameter for ingoing geodesics, as in the Rindler case (see Eq. 9.82).

Thus, (U, V) are the coordinates we were looking for, but we should keep in mind that in the Schwarzschild case (U, V) are affine parameters along the null ingoing and outgoing null geodesics *only near the horizon.*

In the coordinate frame (U, V, θ, φ), called the **Kruskal coordinates**, the metric is

$$ds^2 = -\frac{32m^3}{r} e^{-\frac{r}{2m}} dU dV + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{9.104}$$

as can easily be shown by replacing Eqs. 9.100 and 9.103 in Eq. 9.98. The metric 9.104 is no longer singular on $r = 2m$.

Note that the coordinates (U, V) (Eq. 9.103) are defined in the quadrant $U < 0$, $V > 0$. Thus, the spacetime exterior to the black hole, i.e. the manifold $\mathcal{M}_2$, $r > 2m$ in the coordinates (t, r, θ, φ), has been mapped to the region $U < 0$, $V > 0$ in the Kruskal coordinates.

Since

$$UV = -e^{\frac{v-u}{4m}} = -e^{\frac{r_*}{2m}} = \left(1 - \frac{r}{2m}\right) e^{\frac{r}{2m}} , \tag{9.105}$$

the limit $r \to 2m$ corresponds to $U \to 0$ or $V \to 0$.

Let us now consider the manifold $\mathcal{M}_1$ $(0 < r < 2m)$. The null coordinates are defined, as before, as $u = t - r_*$, $v = t + r_*$, but r_* is now given by Eq. 9.95. The tortoise coordinate is always negative, it tends to zero as $r \to 0$, and to $-\infty$ as $r \to 2M$. Since Eq. 9.97 still holds, the metric in the coordinates (u, v) is given by Eq. 9.98,

$$ds^2 = -\left(1 - \frac{2m}{r}\right) du dv + r^2(d\theta^2 + \sin^2\theta d\varphi^2) . \tag{9.106}$$

Since Eq. 9.95 gives

$$1 - \frac{2m}{r} = -\frac{2m}{r} e^{-\frac{r}{2m}} e^{\frac{v-u}{4m}} . \tag{9.107}$$

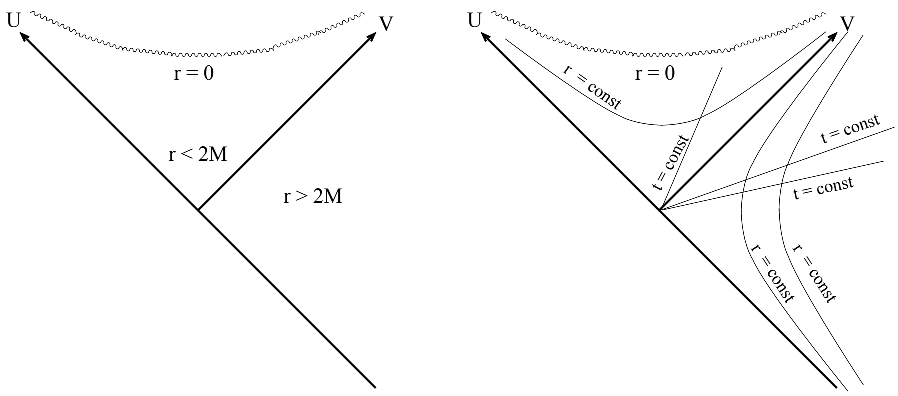

Figure 9.9: Interior and exterior of a Schwarzschild black hole in Kruskal coordinates. In the right panel we show the curves at t and r constant.

defining

$$U \equiv +e^{-\frac{u}{4m}}\,, \qquad V \equiv e^{\frac{v}{4m}} \tag{9.108}$$

we find the same expression for the metric in Kruskal coordinates given in Eq. 9.104. The coordinates (U, V) defined in Eq. 9.108 are defined in the domain $U > 0$, $V > 0$. They are still affine parameters, in the near-horizon limit, of null outgoing and ingoing geodesics, respectively (Eqs. 9.101 and 9.102 still hold, with the opposite sign). Note that the singularity $r = 0$ corresponds to $r_* = 0$, and thus $u = v$ (in $\mathcal{M}_1$) and $UV = e^{\frac{v-u}{4m}} = 1$.

Summarizing, the Kruskal coordinates (U, V, θ, φ) describe both the manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ (see Fig. 9.9). In these coordinates, the black hole exterior $r > 2m$ is mapped to the region $(U < 0,\ V > 0)$, while the interior $0 < r < 2m$ is mapped to $(U > 0,\ V > 0$ with $UV < 1)$; the coordinate singularity $r = 2m$ (and $t = +\infty$) corresponds to the semiaxis $(U = 0,\ V > 0)$; the curvature singularity $r = 0$ corresponds to the upper branch of the hyperbole $UV = 1$.

In the coordinate frame (U, V, θ, φ) the manifold $\mathcal{M}_1 \cup \mathcal{M}_2$ can be extended across the semiaxis $(U = 0,\ V > 0)$ separating $\mathcal{M}_1$ and $\mathcal{M}_2$, since the line metric 9.104 is not singular there. Thus, we consider a new manifold,

$$\mathcal{M} \supset \mathcal{M}_1 \cup \mathcal{M}_2\,, \tag{9.109}$$

defined by

$$V > 0\,, \qquad UV < 1\,. \tag{9.110}$$

Generally, when studying phenomena which occur near a Schwarzschild black hole such as the capture of particles, we implicitly consider the extended manifold $\mathcal{M}$: for instance, as we shall do in the next chapter, we assume that an object falling inside the black hole crosses the horizon $r = 2m$ which, therefore, has to belong to the manifold. The discussion in Sec. 9.4 about the $r = const$ hypersurfaces also assumes that the manifold is $\mathcal{M}$, since $r = 2m$ has been considered as a part of the manifold. However, as explained above in this section, the coordinates (t, r, θ, φ) do not cover the manifold $\mathcal{M}$. Strictly speaking, when we want to describe the black hole horizon, we should use a coordinate system in which the $r = 2m$ singularity is removed.

It is customary to define (as in Rindler's spacetime, see Eqs. 9.84) the coordinates T, X

as

$$T = \frac{U+V}{2} \qquad V = \frac{U-V}{2}\,. \tag{9.111}$$

In terms of these coordinates, the metric 9.104 becomes

$$ds^2 = -\frac{32m^3}{r}e^{-\frac{r}{2m}}(-dT^2 + dX^2) + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{9.112}$$

The relation between the coordinate frames (t, r, θ, φ) and (T, X, θ, φ) is similar to the expressions obtained in Rindler's spacetime (Eq. 9.86). Indeed,

$$X^2 - T^2 = -UV = \pm e^{\frac{v-u}{4m}} = \pm e^{\frac{r_*}{2m}} = \pm e^{\frac{r}{2m}}\left(\frac{r}{2m} - 1\right) \tag{9.113}$$

where the upper/lower sign refers to the first $(U < 0)$ and second $(U > 0)$ quadrants, respectively. We also note that

$$t = 2m\log e^{\frac{u+v}{4m}} = 2m\log\left(\pm\frac{V}{U}\right) = 2m\log\left|\frac{T+X}{T-X}\right|\,. \tag{9.114}$$

While in the case of Rindler's spacetime the coordinates (t, x) are defined in $U < 0$, $V > 0$, in Schwarzschild's spacetime the coordinates (t, r) are defined in $U \neq 0$, $V > 0$ (i.e., the exterior *and* the interior of the black hole), therefore the coordinate transformations are also defined in a larger domain.

The curves $t = const$ and $r = const$ are shown in the right panel of Fig. 9.9. Eq. 9.114 shows that the $t = const$ curves are straight lines in the $U - V$ (and $T - X$) plane; Eq. 9.113 that the $r = const$ curves are hyperbolae.

The manifold $\mathcal{M}$ can still be extended: timelike and null geodesics from $V = 0$ cannot be continued to large negative values of the affine parameter, unless we extend the manifold to $V \leq 0$. By including the region $-\infty < U < +\infty$, $-\infty < V < +\infty$ with $UV < 1$, we obtain the maximal extension of the Schwarzschild spacetime, shown in Fig. 9.10. The dashed line represents the worldline of an observer falling into the black hole, and the wave-like curves represent the curvature singularity $r = 0$.

In the Kruskal coordinates 9.104 the null worldlines with θ, φ constant are straight lines at 45^o, i.e. $U = const.$ or $V = const.$; this can easily be seen from the metric 9.104: any worldline with tangent vector either $(1, 0, 0, 0)$ or $(0, 1, 0, 0)$ is null. Therefore, the light cones can be drawn as in Minkowski's spacetime, and the description of causal connections among events is easy and intuitive (see Fig. 9.10). In particular, we see that signals from region I can be sent only to region II; furthermore, there is a copy of region I, i.e. region IV, which is causally disconnected from I, but can receive signals only from region III and send signals to region II only. Region III is often called *white hole*, since all signals starting in this region have to escape to regions I or IV across the horizon.

The incomplete (timelike and null) geodesics of the maximally extended manifold correspond to the true singularity $r = 0$, i.e., in Kruskal coordinates $UV = 1$. As we shall show in the next chapter, these geodesics reach the singularity at a finite value of the affine parameter, and cannot be extended through it; for instance, an observer that falls inside the black hole reaches the singularity in a finite amount of proper time.

The Kruskal coordinates, besides providing the maximal extension of the Schwarzschild spacetime, can be useful to clarify an important feature of the horizon. We have shown in Section 9.4 that, since $r = 2M$ is a null hypersurface, it can be crossed in one direction only; but which is this direction: inwards or outwards? The $r = 2M$ hypersurface *in the future of the events outside the black hole*, i.e. in the future of region I, is the semiaxis $U = 0$, $V > 0$, and can only be crossed inwards (see e.g. the worldline shown in Fig. 9.10). The

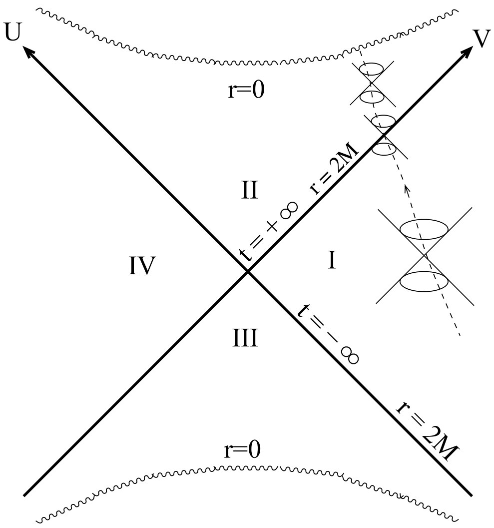

Figure 9.10: Maximal extension of Schwarzschild spacetime in Kruskal coordinates. The dashed line represents the worldline of an observer falling into the black hole. Different regions are marked with I, II, III, and IV.

$r = 2M$ hypersurface in the past of region I is a *different* hypersurface, the semiaxis $U = 0$, $V < 0$, which can only be crossed outwards. Similarly, the black hole interior in the future of region I is region II, where the spacelike $r = const$ hypersurfaces can only be crossed inwards, while the black hole interior in the past of region I is region III, where the spacelile $r = const$ hypersurfaces can only be crossed outwards.

It should be stressed that the maximal extension of the Schwarzschild spacetime has no meaning if we consider a black hole as an astrophysical object, formed in the gravitational collapse of a star. Indeed it describes an *eternal black hole*, whereas the stellar collapse occurs at a finite value of t. In particular, region III cannot exist for an astrophysical black hole, because the semiaxis ($U < 0$, $V = 0$) corresponds to $t = -\infty$ when the black hole was not formed yet. The Kruskal coordinates are not appropriate to describe the stellar collapse, in which the horizon and the singularity appear at finite time. In Section 9.5.5 we shall show how to describe this process in a different coordinate system. Here we only note that we do not have to worry about the meaning of regions III and IV since they do not exist in astrophysical black holes, and we can leave the discussion on the existence of other universes (such as regions III and IV of the construction above) to science-fiction writers.

The final fate of an observer who reaches the singularity is unknown and this poses a problem for the predictability of the theory and for its self-consistence. On the other hand, such problem is not severe from an operational point of view, because no signal from the observer reaching the singularity can be sent outside the black hole: the consistency of the theory, in a certain sense, is preserved by the existence of the horizon. Roger Penrose has conjectured that there is a fundamental principle, the *cosmic censorship hypothesis*, stating that all singularities in the Universe (with the exception of a possible initial singularity) are concealed behind a horizon and that *naked singularities* cannot exist in nature. There is no definitive proof of this conjecture, but there are indications supporting it, at least under some reasonable assumptions about the matter fields that can produce a singularity, e.g., during gravitational collapse.

The presence of a curvature singularities, althogh concealed within the event horizon of black holes, strongly suggests that Einstein's theory is not the last word on gravity. It is customary to consider General Relativity as an effective theory which is valid at curvature scales much smaller than Planckian curvature (curvature $\sim 1/l_{\rm P}^2$, where $l_{\rm P}$ is the Planck length, see Box 9-B). Near the singularity, the curvature is so large that higher-order curvature corrections to General Relativity might become dominant. There are several proposal for such corrections but, at the moment, none of them is supported by observations. Even if General Relativity will eventually have to be modified at the Planck scale, the corrections are expected to be negligible for the astrophysical objects and for the gravitational-wave sources discussed in the rest of this book.

9.5.5 Eddington-Finkelstein coordinates

The Kruskal coordinates allow to define a maximal extension of the Schwarzschild spacetime covered by a unique coordinate choice. However, if we are only interested in removing the singularity $r = 2m$ in regions I and II, there is a simpler, and most practical choice: the Eddington-Finkelstein coordinates [7]

$$(v, r, \theta, \varphi) \qquad -\infty < v < +\infty \qquad 0 < r < +\infty\,. \tag{9.115}$$

Here, as before, $v = t + r_*$, and the tortoise coordinate is defined in Eqs. 9.94, 9.95, which can be unified in

$$r_* \equiv r + 2m \ln\left|\frac{r}{2m} - 1\right|\,. \tag{9.116}$$

Let us consider the Schwarzschild metric written as in Eq. 9.98

$$ds^2 = -\left(1 - \frac{2m}{r}\right)(dt^2 - dr_*^2) + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{9.117}$$

It describes both the $\mathcal{M}_1$ and $\mathcal{M}_2$ manifolds. Being $dr_* = dv - dt$, from Eq. 9.97 we find

$$dt^2 - dr_*^2 = dv^2 - 2dvdr_* = dv^2 - 2\frac{dr_*}{dr}dvdr = dv^2 - 2\frac{dvdr}{1 - \frac{2m}{r}}\,, \tag{9.118}$$

therefore the metric in the Eddington-Finkelstein coordinates is

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\,. \tag{9.119}$$

This metric covers both the interior and the exterior of the black hole, i.e. the regions I and II of the Kruskal construction, and is regular and invertible on the horizon $r = 2m$ (the determinant does not vanish due to the off-diagonal components). Note that on the horizon v is finite because $t \to +\infty$ and $r_* \to -\infty$, while u diverges. All the computations and derivations involving the interior and the exterior of the black hole, such as the study of the $r = const$ surfaces of Sec. 9.4, can be rigorously performed in the Eddington-Finkelstein coordinates (note that $g^{rr} = 1 - 2m/r$ in these coordinates, too).

[7] Strictly speaking, (v, r, θ, φ) are the *ingoing* Eddington-Finkelstein coordinates, while (u, r, θ, φ) are called *outgoing* Eddington-Finkelstein coordinates. We shall omit this specification because we only consider the ingoing coordinates, which allow to remove the coordinate singularity between the regions I and II.

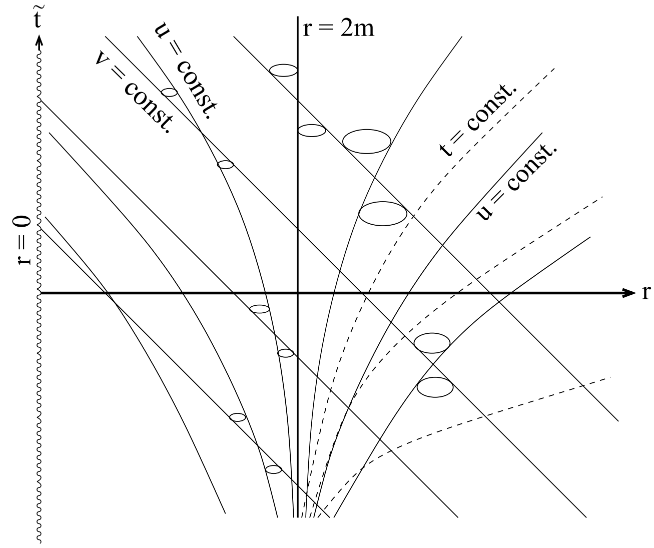

Figure 9.11: Finkelstein diagram of a Schwarzschild black hole.

The Finkelstein diagram

A useful way to visualize the Schwarzschild spacetime is the Finkelstein diagram, in which the axes are $(\tilde{t}, r)$ where

$$\tilde{t} \equiv v - r = t + 2m \ln \left| \frac{r}{2m} - 1 \right| . \tag{9.120}$$

In this diagram the null lines $v = const$, corresponding to ingoing massless particles, are straight lines at 45^o; the null lines $u = const$, corresponding to outgoing massless particles, are hyperbolic curves. These two sets of curves define the light cones centered in any spacetime point, and allow to establish the causal relations among different events. While the light cones in the (t, r) plane collapse to lines at the horizon, the light cones in the Finkelstein diagram remain regular.

Since $\tilde{t} \simeq t$ for $r \ll 2m$ and $r \gg 2m$, the coordinate $\tilde{t}$ coincides with t far away from the horizon, but they are very different near to the horizon (where, as we shall discuss in Chapter 10, the operational definition of the coordinate t is problematic). We also note that the coordinate $\tilde{t}$ cannot be considered a "time" inside the horizon, because the vector $\partial/\partial\tilde{t}$ is spacelike.

In Fig. 9.11 the coordinate lines $r = const$ are vertical straight lines, whereas $t = const$ are hyperbolic (dashed) curves; the $\tilde{t}$-axis represents the singularity, and for this reason it is drawn wave-like.

As mentioned above, astrophysical black holes are the result of a gravitational collapse (see Chapter 16) occurring at some time and producing a singularity for some $t = t_0$. A qualitative view of the spacetime of a realistic black hole is shown in the Finkelstein diagram in Fig. 9.12, where the shaded area represents the interior of the star. The $r = 0$ axis is a curvature singularity for $\tilde{t} \geq \tilde{t}_0$, i.e. after the singularity forms; for $\tilde{t} < \tilde{t}_0$, the $r = 0$ axis is simply the (trivial) coordinate singularity at the origin of polar coordinates. The horizon, represented by the dashed line, is also formed during the collapse.

It is important to stress that, although we have discussed the entire Schwarzschild so-

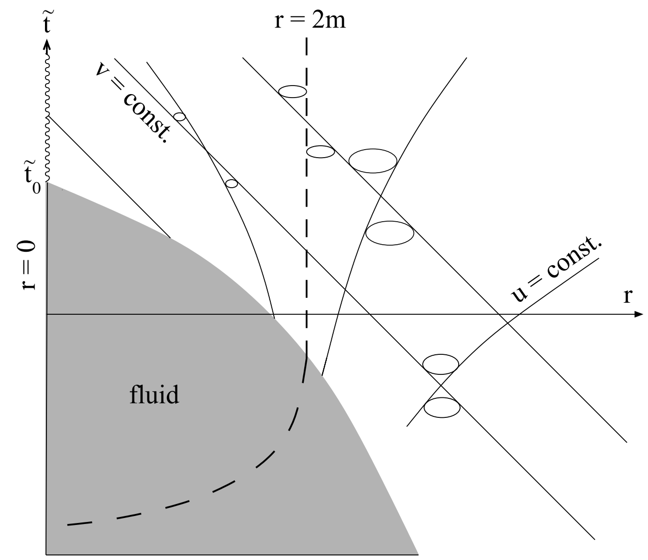

Figure 9.12: Finkelstein diagram of a stellar collapse originating a Schwarzschild black hole. The shaded area represents the fluid interior of the star. The curvature singularity (wave-like line) is formed at $\tilde{t} = \tilde{t}_0$. The horizon is represented by the dashed line.

lution, only the $r > 2m$ region is directly relevant for astrophysical observations: relativity imposes that no signal can come from the interior of a black hole horizon.

Finally, it is worth mentioning that a useful way to represent the causal structure of a spacetime is through the so-called Penrose-Carter diagrams. We do not discuss this interesting topic in this book, and we refer the interested reader to more specialized work, e.g. [85].

9.6 THE BIRKHOFF THEOREM

In Sec. 9.2 we derived the Schwarzschild metric as the solution to Einstein's equations in vacuum, under the assumption of staticity and spherically symmetry. This solution represents the gravitational field external to a non-rotating, spherically symmetric body, the structure of which is time-independent. However, the Schwarzschild solution is more general, since, as shown by George Birkhoff in 1923, it is the only *spherically symmetric, asymptotically flat solution to Einstein's field equations in vacuum.* Thus, to prove Birkhoff's theorem we need to relax the assumption that the metric admits a timelike, hypersurface-orthogonal Killing vector field. We shall now generalize the results of Sec. 9.1, where we showed how to choose the coordinates by imposing the spherical symmetry, assuming that the metric depends on time. As in Sec. 9.1 we fill the three-dimensional space with two-spheres, with two-metric (see Eq. 9.8)

$$ds^2_{(2)} = a^2(x^0, x^1)(d\theta^2 + \sin^2\theta d\varphi^2)\,, \tag{9.121}$$

where $a^2(x^0, x^1)$ is an unspecified function. Contrary to what we did in Sec. 9.1, we shall now retain the dependence on x^0. The basis vectors $\vec{e}_{(\theta)}$ and $\vec{e}_{(\varphi)}$ are tangent, respectively, to the coordinate lines ($\varphi = const, \theta = const$), which we choose on the two-spheres. Then, we align the poles of all spheres as explained in Sec. 9.1; in addition we choose the basis