which proves Eq. 18.107. The Kerr metric in Kerr-Schild coordinates is then

$$\begin{aligned} ds^2 &= -d\bar{t}^2 + dx^2 + dy^2 + dz^2 \\ &+ \frac{2Mr^3}{r^4 + a^2 z^2}\left[d\bar{t} + \frac{r(xdx + ydy) - a(xdy - ydx)}{r^2 + a^2} + \frac{zdz}{r}\right]^2 . \end{aligned} \tag{18.111}$$

The above metric depends on a function $r(x, y, z)$, which is defined implicitly by

$$r^4 - (x^2 + y^2 + z^2 - a^2)r^2 - a^2 z^2 = 0 . \tag{18.112}$$

Indeed, by combining Eqs. 18.93 we find $r^2 - (x^2 + y^2 + z^2 - a^2) = a^2 \cos^2\theta = z^2 a^2/r^2$, which is equivalent to Eq. 18.112.

Note that the metric 18.111 has the form

$$g_{\mu\nu} = \eta_{\mu\nu} + H l_\mu l_\nu \tag{18.113}$$

with

$$H \equiv \frac{2Mr^3}{r^4 + a^2 z^2} \tag{18.114}$$

and, in Kerr-Schild coordinates,

$$l_\mu dx^\mu = -\left(d\bar{t} + \frac{r(xdx + ydy) - a(xdy - ydx)}{r^2 + a^2} + \frac{zdz}{r}\right) , \tag{18.115}$$

while in Kerr coordinates

$$l_\alpha dx^\alpha = -d\bar{t} - dr + a\sin^2\theta d\bar{\varphi} = -dv + a\sin^2\theta d\bar{\varphi} ; \tag{18.116}$$

thus l_μ is exactly the null vector given in Eq. 18.52, i.e. the generator of the principal null geodesics which have been used to define the Kerr coordinates.

18.8 THE INTERIOR OF AN ETERNAL KERR BLACK HOLE

While the Kerr metric describes the exterior – i.e., the region outside the outer horizon $r = r_+$ – of a stationary, astrophysical black hole formed in the gravitational collapse of a star, it cannot describe its interior, i.e. the region $r < r_+$ (see also Sec. 9.5.4). Strictly speaking, the Kerr metric (which includes the external and the internal regions) describes an *eternal* Kerr black hole. In this section we shall discuss some peculiar properties of the interior of an eternal Kerr black hole, in particular of the region close to $r = 0$. In reality, the interior of an astrophysical black hole is not empty but contains the matter fields that underwent the gravitational collapse. This might change the inner structure significantly and resolve some of the potential pathologies that we are going to discuss.[6]

18.8.1 Extensions of the Kerr metric

Let us consider the metric in the Kerr-Schild coordinates $\{\bar{t}, x, y, z\}$ (Eq. 18.111)

$$\begin{aligned} ds^2 &= -d\bar{t}^2 + dx^2 + dy^2 + dz^2 \\ &+ \frac{2Mr^3}{r^4 + a^2 z^2}\left[d\bar{t} + \frac{r(xdx + ydy) - a(xdy - ydx)}{r^2 + a^2} + \frac{zdz}{r}\right]^2 . \end{aligned} \tag{18.117}$$

[6] The curvature singularities that exist in the Schwarzschild and Kerr metric are not a prerogative of these solutions. Indeed, a remarkable series of results due to Penrose and Hawking – the so-called "singularity theorems" – show that in General Relativity curvature singularities emerge generically as the outcome of a gravitational collapse (see [52] for a rigorous discussion).

The function $r(x, y, z)$ is given by Eq. 18.112 which, for each values of x, y, z, has two real solutions (besides two unphysical complex conjugate solutions), one positive and one negative. We could be tempted to discard the $r < 0$ solution as unphysical, but in this way – although the coordinates $\{\bar{t}, x, y, z\}$ are continuous across the disk $r = 0$, $\theta \neq 0$, the coordinate singularity on the disk would not be removed, as we are going to show.

Let us consider, for simplicity, the $x = y = 0$ submanifold, i.e. the polar axis, where the metric 18.111 reduces to

$$ds^2 = -d\bar{t}^2 + dz^2 + \frac{2Mr^3}{r^4 + a^2 z^2}\left[d\bar{t} + \frac{z dz}{r}\right]^2 , \qquad (18.118)$$

and Eq. 18.112 reduces to $r^4 - (z^2 - a^2)r^2 - z^2 a^2 = 0$, whose solutions are $r(z) = \pm z$. If we require that the radial coordinate is positive, $r = |z|$ and Eq. 18.118 becomes

$$ds^2 = -d\bar{t}^2 + dz^2 + \frac{2M|z|}{z^2 + a^2}\left[d\bar{t} + \frac{z dz}{|z|}\right]^2 , \qquad (18.119)$$

which is continuous but not differentiable at $z = 0$. A computation of the Christoffel symbols shows that $\Gamma^z_{\bar{t}\bar{t}}$ is discontinuous across the disk and thus, for a timelike geodesic with tangent vector $\frac{dx^\mu}{d\lambda}$ crossing the disk, $\frac{d^2 z}{d\lambda^2}$ would also be discontinuous.

These problems arise because we have forced r to be positive, but there is no fundamental reason for this assumption. If we allow r to have negative values, when an observer crosses the disk the coordinate r changes sign. For instance, in the $x = y = 0$ submanifold, we can choose the solution $r = z$ of Eq. 18.112 along the entire axis. The metric in this submanifold is then

$$ds^2 = -d\bar{t}^2 + dz^2 + \frac{2Mz}{z^2 + a^2}\left[d\bar{t} + dz\right]^2 , \qquad (18.120)$$

which is regular at $z = 0$. Note that this choice also "cures" the discontinuity of $dr/d\lambda$ across the disk, and the discontinuity of $\theta(\lambda)$ as well, since (being $z = r\cos\theta$) $\theta = 0$ along the entire axis.

In order to extend the spacetime across the $r = 0$ disk we have to consider a manifold formed by at least two copies of the spacetime described by Eq. 18.111: one with $r > 0$, the other with $r < 0$. The $r < 0$ spacetime is also asymptotically flat, but it has no horizons. If the top of the disk of the $r > 0$ spacetime is identified with the bottom of the disk with $r < 0$ spacetime and vice versa (see Figure 18.4), the worldlines crossing the disk move from one copy to the other. In this way, the metric is regular across the disk, and the coordinate singularity is removed[7]. Note, however, that there is no reason to assume that two observers in the $r > 0$ spacetime, one crossing the disk from the top, the other from the bottom, reach the same $r < 0$ spacetime, as in Fig. 18.4. A larger spacetime would consists of different copies of the same manifold, such that the two observers crossing the disk from different sides end up in different $r < 0$ manifolds.

This is not the maximal extension of the Kerr metric. A detailed analysis of geodesic completeness – which is extremely involved and would go far beyond the scope of this book, see e.g. [52] – shows that if we require that the spacetime is inextendible, i.e. that all timelike or null geodesics either hit the ring singularity, or can be extended to arbitarily large values of the affine parameter (see Box 9-C and Sec. 9.5.4), we have to patch together an infinite number of spaces like those in Fig. 18.4. Here we only describe the maximal extension of the $x = y = 0$ submanifold, i.e. of the $\theta = 0$ axis. The structure of this spacetime is shown

[7]This extension is analogous to the extension of the complex plane to Riemann surfaces for the representation of multi-valued functions of a complex variable.

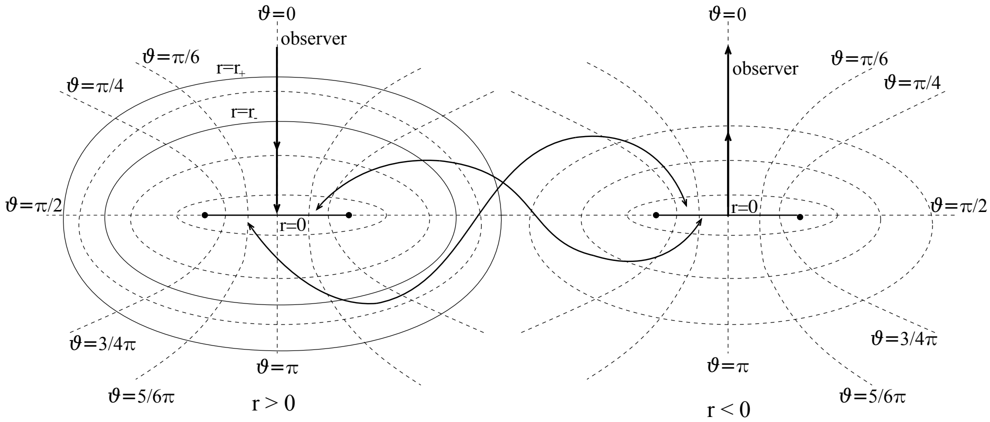

Figure 18.4: Two copies of the spacetime, one with $r > 0$, one with $r < 0$, are patched together, identifying the top of the $r > 0$ disc with the bottom of the $r < 0$ disk, and vice versa. The $r > 0$ spacetime contains the $r = r_\pm$ horizons. An observer enters in the disk from the top of the $r > 0$ space and emerges from the top of the $r < 0$ space.

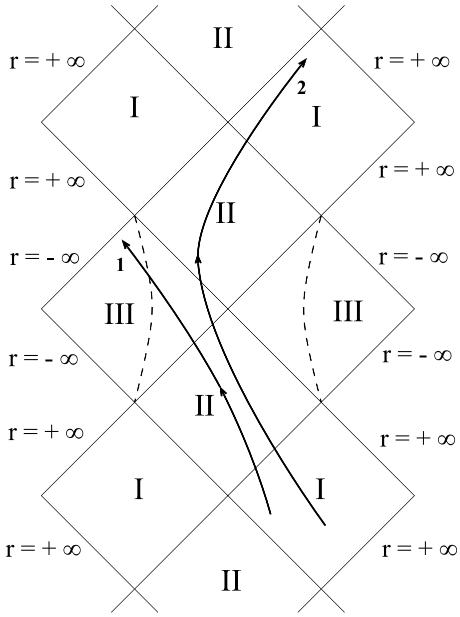

Figure 18.5: Schematic representation of the maximal extension of the Kerr metric, along the $\theta = 0$ axis. The dashed hyperbolic curves correspond to $r = 0$. We indicate with I the exterior of the black hole, with II the regions between the inner and outer horizons, with III the inner regions, which include the center of the $r = 0$ disk and the asymptotically flat region with $r < 0$. The solid lines represent two timelike observers escaping to different asymptotically flat spaces.

in Fig. 18.5, where the regions I,II,III correspond to:

$$\begin{array}{lcll} I & : & r_+ < r < +\infty & \text{(exterior of the black hole, asymptotically flat)} \\ II & : & r_- < r < r_+ & \text{(where the } r = const. \text{ surfaces are spacelike)} \\ III & : & -\infty < r < r_- & \text{(ring singularity and } r < 0 \text{ asymptotically flat space)}\,. \end{array}$$

The dashed hyperbolic curves correspond to $r = 0$. Note that, as in the Schwarzschild spacetime (see Sec. 9.5.4), the outer horizon $r = r_+$ in the future of region I is crossed inwards by timelike and null worldlines, which also cross inwards the inner horizon $r = r_-$ in the future of regions I and II.

Remarkably, in this spacetime an observer can travel through different asymptotically flat regions. This is different from the maximal extension of the Schwarzschild spacetime discussed in Sec. 9.5.4, where the observers falling in the black hole necessarily hit the singularity, and the two asymptotically flat regions (indicated as I and IV in Figure 9.10) are causally disconnected. In the maximally extended Kerr spacetime, instead, once an observer crosses the inner horizon and enters in region III, (s)he can either cross the disk $r = 0$, and escape to the asymptotically flat region $r < 0$ (trajectory 1 in Fig. 18.5), or move outwards[8] and cross the inner horizon again (trajectory 2 in Fig. 18.5). In this case the observer, after leaving region III, would enter into a different copy of region II, where (s)he can keep moving to increasing values of r, and finally enter into a different copy of region I.

This fascinating structure, however, is unlikely to be realized in actual astrophysical objects. As we noted at the beginning of this section, the Kerr metric and its extensions only describe *eternal black holes* (similarly to case of the Schwarzschild metric in Kruskal's coordinates discussed in Sec. 9.5.4). In an astrophysical black hole originating from a gravitational collapse, a region I cannot receive signals from a region II, because these signals would come from $t \to -\infty$, when the black hole was not born yet. This prevents the formation of the multiple copies shown in Figs 18.4 and 18.5.

Moreover, it has been shown that the inner horizon $r = r_-$ is unstable: a small perturbation produced by mass accretion would grow up [88], potentially leading to drastic changes in the structure of the $r \leq r_-$ region. However, the nature of this instability is still unclear (see e.g. [13] or, more recently, [35]).

18.8.2 Causality violations

Let us consider a curve γ on the equatorial plane, consisting in a ring just outside the curvature singularity ring, in the spacetime with $r < 0$:

$$\gamma : \quad \left\{ \bar{t} = const,\, r = const,\, \theta = \frac{\pi}{2},\, 0 \leq \bar{\varphi} \leq 2\pi \right\} \quad \text{with} \quad |r| \ll M,\, r < 0\,. \tag{18.121}$$

The curve γ belongs to region III of the black hole, and can be reached by an observer from positive, large values of r who crosses the two horizons, passes through the $r = 0$ ring, and turns around it up to the $z = 0$ plane, just outside the ring (see Fig. 18.6).

The tangent vector to the curve γ is the Killing vector $\vec{m}$, and its norm is

$$m^\mu m^\nu g_{\mu\nu} = g_{\bar{\varphi}\bar{\varphi}} = (r^2 + a^2)\sin^2\theta + \frac{2Mr}{\Sigma} a^2 \sin^4\theta = r^2 + a^2 + \frac{2Ma^2}{r}\,, \tag{18.122}$$

where we used Eq. 18.53 evaluated in $\theta = \pi/2$. Since $r < 0$ and $|r| \ll M$, the term $2Ma^2/r$ is negative and dominates the others, therefore $m^\mu m^\nu g_{\mu\nu} < 0$. The curve γ is then

[8]We remind that in region III the $r = const$ surfaces are timelike and can be crossed in both directions.

a timelike curve, and can be interpreted as the worldline of an observer; however, it is a closed curve and its existence may violate causality: the observer moving on a *closed timelike curve* (CTC)[9] would meet him/herself in its own past. However, it has been recently argued that this paradox might be resolved including thermodynamical considerations; a causality violation would require not only a particle, but a thermodynamical system – for instance, a clock keeping track of time – meeting itself in its own future (for details see [95]). There

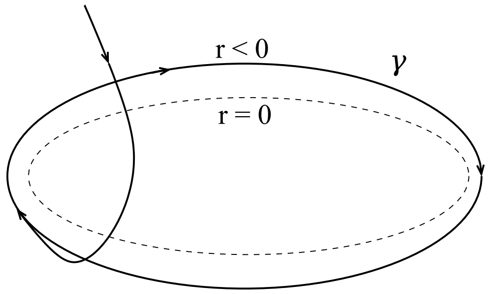

Figure 18.6: Closes timelike curve in Kerr spacetime.

are reasons to believe that in a rotating black hole born in a gravitational collapse, the structure of the ring singularity (and then the occurrence of CTCs) would be destroyed by the presence of the matter and/or by the instability of the inner horizon. However, there is no clear proof that this is the case.

A possible point of view to interpret these troublesome features of Kerr's spacetime is that causality violations, together with the existence of singularities (where some timelike or null geodesics end in a finite amount of proper time), are *inconsistencies* of the theory of General Relativity, which would disappear once a more fundamental theory unifying General Relativity with quantum field theory will take its place. Indeed, quantum gravity effects are expected to be significant near the singularities.

A different point of view is that these features are not problematic since, being hidden behind horizons, cannot be observed. This is a further motivation for the cosmic censorship conjecture discussed Sec. 9.5.4.

18.9 GENERAL BLACK HOLE SOLUTIONS

When a black hole forms in the gravitational collapse of a sufficiently massive star, the violent oscillations that follow the collapse are damped by gravitational wave emission and other dissipative processes. Thus we expect that, after some time related to the damping time of the black hole quasi-normal modes (see Chapter 15), the remnant of the collapse settles down to a stationary configuration. Thus, *stationary* black holes are the final outcome of gravitational collapse.

In addition to the Schwarzschild and Kerr solutions, there exist a stationary, axisymmetric solution of the Einstein-Maxwell equations[10] known as the **Kerr-Newman solu-**

[9]The occurrence of closed timelike curves was first found by Kurt Gödel in an exact solution of Einstein's equations, which is considered to be unphysical.

[10]The equations which couple the electromagnetic field to gravity can be derived form a variational principle, as shown in Chapter 7, by adding Maxwell's action to the Einstein-Hilbert action.

tion [80]; the metric, in the Boyer-Lindquist coordinates, is

$$ds^2 = -\frac{\Delta}{\Sigma}(dt - a\sin^2\theta d\varphi)^2 + \frac{\sin^2\theta}{\Sigma}[adt - (r^2+a^2)d\varphi]^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2\,, \tag{18.123}$$

where

$$\Delta(r) \equiv r^2 - 2Mr + a^2 + Q^2\,, \tag{18.124}$$
$$\Sigma(r,\theta) \equiv r^2 + a^2\cos^2\theta\,,$$

and Q is the *black hole electric charge*; here we have used electric units [11] such that $4\pi\epsilon_0 = 1$ (see also Box 18-B). It is easy to check that if $Q = 0$, the solution 18.123 reduces to the Kerr metric.

In the case of zero spin, $a = 0$ and the Kerr-Newman solution reduces to the **Reissner-Nordström** solution,

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2 d\Omega^2\,, \tag{18.125}$$

which describes a spherically-symmetric, electrically charged black hole in Einstein-Maxwell theory. However, as discussed in Box 18-B, the charge of astrophysical black holes is expected to be negligibly small and therefore astrophysical black holes are perfectly described by the Kerr solution.

There are some remarkable theorems [94] on stationary black holes, derived by S. Hawking, W. Israel, B. Carter and others, which prove the following:

- A stationary, asymptotically flat black hole must be axially symmetric (while, as we know from Birkoff's theorem, a static black hole is necessarily spherically symmetric).
- Any stationary, asymptotically flat black hole, with no electric charge, is described by the Kerr solution.
- Any stationary, asymptotically flat black hole is described by the Kerr-Newman solution, and it is therefore characterized by *only three parameters*: the mass M, the angular momentum aM, and the charge Q.

Besides the mass, angular momentum, and electric charge, all other features that the star possessed before collapsing, such as a particular structure of the magnetic field, departure from axisymmetry, matter currents, differential rotation, etc., disappear when the final black hole forms. This result, which goes under the name of *no-hair theorem*, has been nicely summarized with the sentence: "*A black hole has no hair*" [12].

[11] In geometrized ($G = 1$) and unrationalized Gaussian ($4\pi\epsilon_0 = 1$) units, the ratio Q/M is dimensionless.
[12] The quote is attributed to John Archibald Wheeler who, in turn, attributed it to Jacob Bekenstein.

Box 18-B

The charge of astrophysical black holes

Astrophysical black holes are considered to be electrically neutral for various reasons: quantum discharge effects, electron-positron pair production, and charge neutralization by astrophysical plasmas. Without entering into the details, these arguments rely — one way or another — on the huge charge-to-mass ratio of the electron. The simplest argument is purely kinematical. Let us consider a black hole with mass M and electric charge Q and a low-energy electron in radial motion with a small, initial radial velocity. For the electron to be absorbed by the black hole, the magnitude of the electric (Coulomb) force

$$F_C = \frac{Qe}{4\pi\epsilon_0 r^2} \tag{18.126}$$

(here ϵ_0 is the vacuum dieletric constant) must be smaller than the gravitational force (we use a Newtonian approximation for the sake of simplicity)

$$F_N = \frac{GMm_e}{r^2}\,. \tag{18.127}$$

The condition $F_C < F_N$ implies

$$eQ \leq 4\pi\epsilon_0 GMm_e\,. \tag{18.128}$$

Note that the quantity $\sqrt{4\pi\epsilon_0 G}M$ has the dimensions of a charge. It is convenient to use geometrized ($G = 1$) and unrationalized Gaussian ($4\pi\epsilon_0 = 1$) units, in which the charge-to-mass ratio is dimensionless. In these units, 1 C $= 1.16\times10^{13}$ g. Therefore, the charge of the electron is (see Table A) $e = 1.602\times10^{-19}$ C $\sim 2\times10^{-6}$ g $\sim 2\times10^{21}m_e$ and Eq. (18.128) can be written as

$$\frac{Q}{M} \leq \frac{m_e}{e} \sim 5\times10^{-22}\,. \tag{18.129}$$

Therefore, due to the tiny mass-to-charge ratio of the electron, the dimensionless parameter Q/M must be extremely small.

Similarly to the spin parameter of a Kerr black hole (which is limited by $|a|/M \leq 1$) also the charge of a Reissner-Nordström black hole is limited, $Q^2/M^2 \leq 1$. For $Q = M$ the black hole is said to be *extremal*, whereas for $Q > M$ there is no horizon and a naked singularity appears. In the $Q/M \ll 1$ limit one recovers the Schwarzschild solution and Eq. (18.129) implies that Q must be negligibly small in units of the black hole mass.

The above argument does not apply if the initial radial velocity of the electron is very large. However, more sophisticated arguments (e.g. charge neutralization by surrounding plasma) show that – even when the electrons have large velocities – the charge of astrophysical black holes is always incredibly small and can always be neglected.