Chapter 13

Gravitational Waves

One of the most interesting predictions of the theory of General Relativity is the existence of gravitational waves. The idea that a perturbation of the gravitational field should propagate as a wave is, in some sense, intuitive. For example electromagnetic waves were introduced when the Coulomb theory of electrostatics was replaced by the theory of electrodynamics, and it was shown that they transport through space the information about the evolution of charged systems. In a similar way when a mass-energy distribution changes in time, the information about this change should propagate in the form of waves. However, gravitational waves have a distinctive feature: due to the twofold nature of $g_{\mu\nu}$, which is the metric tensor and the gravitational potential, gravitational waves are **metric waves**. Thus when they propagate the geometry, and consequently the proper distance between spacetime points, change in time.

Gravitational waves can be studied by following two different approaches, one based on perturbative methods, the second on the solution of the non linear Einstein equations.

The perturbative approach

Be $g^0_{\mu\nu}$ a known exact solution of Einstein's equations; it can be, for instance, the metric of flat spacetime $\eta_{\mu\nu}$, or the metric generated by a Schwarzschild black hole. Let us consider a small perturbation of $g^0_{\mu\nu}$ caused by some source described by a stress-energy tensor $T^{\mu\nu}_{pert}$. We shall write the metric tensor of the perturbed spacetime, $g_{\mu\nu}$, as follows

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}, \tag{13.1}$$

where $h_{\mu\nu}$ is the small perturbation

$$|h_{\mu\nu}| << |g^0_{\mu\nu}|.$$

It is clear that this assumption is ambiguous, because we should specify in which reference frame this is true; however we shall assume that this frame does exists.

The inverse metric can be written as

$$g^{\mu\nu} = g^{0\,\mu\nu} - h^{\mu\nu} + O(h^2)\,, \tag{13.2}$$

where the indices of $h^{\mu\nu}$ have been raised with the *unperturbed* metric

$$h^{\mu\nu} \equiv g^{0\,\mu\alpha} g^{0\,\nu\beta} h_{\alpha\beta}\,. \tag{13.3}$$

Indeed, with this definition,

$$\left(g^{0\,\mu\nu} - h^{\mu\nu})(g^0_{\nu\alpha} + h_{\nu\alpha}\right) = \delta^\mu_\alpha + O(h^2)\,. \tag{13.4}$$

In order to find the equations that describe $h_{\mu\nu}$, we shall write Einstein's equations for the metric (13.1) in the form

$$R_{\mu\nu} = \frac{8\pi G}{c^4}\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)\,, \tag{13.5}$$

where $T_{\mu\nu}$ is the sum of two terms, one associate to the source that generates the background geometry $g^0_{\mu\nu}$, say $T^0_{\mu\nu}$, and one associate to the source of the perturbation $\delta T_{\mu\nu}$, which is of order h. We remind that the Ricci tensor $R_{\mu\nu}$ is

$$R_{\mu\nu} = \frac{\partial}{\partial x^\alpha}\Gamma^\alpha{}_{\mu\nu} - \frac{\partial}{\partial x^\nu}\Gamma^\alpha{}_{\mu\alpha} + \Gamma^\alpha{}_{\sigma\alpha}\Gamma^\sigma{}_{\mu\nu} - \Gamma^\alpha{}_{\sigma\nu}\Gamma^\sigma{}_{\mu\alpha}, \tag{13.6}$$

and that the affine connections $\Gamma^\gamma_{\beta\mu}$ are

$$\Gamma^\gamma_{\beta\mu} = \frac{1}{2}g^{\gamma\alpha}\left[g_{\alpha\beta,\mu} + g_{\alpha\mu,\beta} - g_{\beta\mu,\alpha}\right]. \tag{13.7}$$

The $\Gamma^\gamma_{\beta\mu}$ computed for the perturbed metric (13.1) are

$$\begin{aligned}
\Gamma^\gamma_{\beta\mu}\left(g_{\mu\nu}\right) &= \frac{1}{2}\left[g^{0\alpha\gamma} - h^{\alpha\gamma}\right]\left[\left(g^0_{\alpha\beta,\mu} + g^0_{\alpha\mu,\beta} - g^0_{\beta\mu,\alpha}\right) + \left(h_{\alpha\beta,\mu} + h_{\alpha\mu,\beta} - h_{\beta\mu,\alpha}\right)\right] \\
&= \frac{1}{2}g^{0\alpha\gamma}\left[g^0_{\alpha\beta,\mu} + g^0_{\alpha\mu,\beta} - g^0_{\beta\mu,\alpha}\right] + \frac{1}{2}g^{0\alpha\gamma}\left[h_{\alpha\beta,\mu} + h_{\alpha\mu,\beta} - h_{\beta\mu,\alpha}\right] \\
&- \frac{1}{2}h^{\alpha\gamma}\left[g^0_{\alpha\beta,\mu} + g^0_{\alpha\mu,\beta} - g^0_{\beta\mu,\alpha}\right] + O(h^2) \\
&\equiv \Gamma^\gamma_{\beta\mu}\left(g^0\right) + \delta\Gamma^\gamma_{\beta\mu}\left(h\right) + O(h^2),
\end{aligned} \tag{13.8}$$

where $\delta\Gamma^\gamma_{\beta\mu}\left(h\right)$ are of first order in $h_{\mu\nu}$

$$\delta\Gamma^\gamma_{\beta\mu}\left(h\right) = \frac{1}{2}g^{0\alpha\gamma}\left[h_{\alpha\beta,\mu} + h_{\alpha\mu,\beta} - h_{\beta\mu,\alpha}\right] - \frac{1}{2}h^{\alpha\gamma}\left[g^0_{\alpha\beta,\mu} + g^0_{\alpha\mu,\beta} - g^0_{\beta\mu,\alpha}\right]. \tag{13.9}$$

When we substitute eq. (13.9) in the Ricci tensor we get

$$\begin{aligned}
R_{\mu\nu}\left(g_{\mu\nu}\right) &= R^0_{\mu\nu}\left(g^0\right) \\
&+ \frac{\partial}{\partial x^\alpha}\delta\Gamma^\alpha{}_{\mu\nu}\left(h\right) - \frac{\partial}{\partial x^\nu}\delta\Gamma^\alpha{}_{\mu\alpha}\left(h\right) \\
&+ \Gamma^\alpha{}_{\sigma\alpha}\left(g^0\right)\delta\Gamma^\sigma{}_{\mu\nu}\left(h\right) + \delta\Gamma^\alpha{}_{\sigma\alpha}\left(h\right)\Gamma^\sigma{}_{\mu\nu}\left(g^0\right) \\
&- \Gamma^\alpha{}_{\sigma\nu}\left(g^0\right)\delta\Gamma^\sigma{}_{\mu\alpha}\left(h\right) - \delta\Gamma^\alpha{}_{\sigma\nu}\left(h\right)\Gamma^\sigma{}_{\mu\alpha}\left(g^0\right) + O(h^2) \equiv R^0_{\mu\nu}\left(g^0\right) + \delta R_{\mu\nu}\left(h\right) + O(h^2)
\end{aligned} \tag{13.10}$$

We now have to work out the right-hand side of the Einstein equations (13.5), i.e. $\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)$, and separate the terms which are of order h. Since $T_{\mu\nu} = T^0_{\mu\nu} + \delta T_{\mu\nu}$

$$\begin{aligned}
T &= g^{\mu\nu}T_{\mu\nu} = \left(g^{0\mu\nu} - h^{\mu\nu}\right)\left(T^0_{\mu\nu} + \delta T_{\mu\nu}\right) \\
&= g^{0\mu\nu}T^0_{\mu\nu} - h^{\mu\nu}T^0_{\mu\nu} - g^{0\mu\nu}\delta T_{\mu\nu} + O(h^2) \equiv T^0 + \delta T + O(h^2).
\end{aligned} \tag{13.11}$$

Consequently

$$\begin{aligned}\left(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T\right) &= T^0_{\mu\nu}+\delta T_{\mu\nu}-\frac{1}{2}\left(g^0_{\mu\nu}+h_{\mu\nu}\right)\left(T^0+\delta T\right)+O(h^2) \\ &= \left(T^0_{\mu\nu}-\frac{1}{2}g^0_{\mu\nu}T^0\right)+\left[\delta T_{\mu\nu}-\frac{1}{2}\left(g^0_{\mu\nu}\delta T+h_{\mu\nu}T^0\right)\right]+O(h^2).\end{aligned} \tag{13.12}$$

Combining eqs. (13.10) and (13.11), and reminding that $g^0_{\mu\nu}$ is, by assumption, the exact solution of Einstein's equations in vacuum $R_{\mu\nu}\left(g^0\right)=\frac{8\pi G}{c^4}\left(T^0_{\mu\nu}-\frac{1}{2}g^0_{\mu\nu}T^0\right)$, Einstein's equations for the perturbations $h_{\mu\nu}$ reduce to

$$\begin{aligned}&\frac{\partial}{\partial x^\alpha}\Gamma^\alpha{}_{\mu\nu}\left(h\right)-\frac{\partial}{\partial x^\nu}\Gamma^\alpha{}_{\mu\alpha}\left(h\right) \\ +\ &\Gamma^\alpha{}_{\sigma\alpha}\left(g^0\right)\Gamma^\sigma{}_{\mu\nu}\left(h\right)+\Gamma^\alpha{}_{\sigma\alpha}\left(h\right)\Gamma^\sigma{}_{\mu\nu}\left(g^0\right) \\ -\ &\Gamma^\alpha{}_{\sigma\nu}\left(g^0\right)\Gamma^\sigma{}_{\mu\alpha}\left(h\right)-\Gamma^\alpha{}_{\sigma\nu}\left(h\right)\Gamma^\sigma{}_{\mu\alpha}\left(g^0\right)=\frac{8\pi G}{c^4}\left[\delta T_{\mu\nu}-\frac{1}{2}\left(g^0_{\mu\nu}\delta T+h_{\mu\nu}T^0\right)\right]+O(h^2),\end{aligned} \tag{13.13}$$

that are linear in $h_{\mu\nu}$. Their solution describes the propagation of gravitational waves in the considered background.[1] This approximation works sufficiently well in a variety of physical situations because gravitational waves are very weak. This point will be better understood in the next chapter, when we will discuss the generation of gravitational waves.

The "exact" approach

The second approach to the study of gravitational waves seeks for exact solutions of Einstein's equations which describe both the source and the emitted wave, but no solution of this kind has been found so far. Of course the non-linearity of the equations makes the problem very difficult; however, it may be noted that also in electrodynamics an exact solution of Maxwell's equations appropriate to describe the electromagnetic field produced by a current which decreases in an electric oscillator due to the emission of electromagnetic waves has never been found, although Maxwell's equations are linear.

Exact solutions of Einstein's equations describing gravitational waves can be found only if one imposes some particular symmetry as for example plane, spherical, or cylindrical symmetry. The interaction of plane waves can also be described in terms of exact solutions, and due to the non-linearity of the equations of gravity it is very different from the interaction of electromagnetic waves.

In the following we shall use the perturbative approach to show that a weak perturbation of the flat spacetime satisfies the wave equation.

[1] Notice that the right-hand side of eq.(13.13) is a particular case of the Palatini identity.

13.1 A perturbation of the flat spacetime propagates as a wave

Let us consider the flat spacetime described by the metric tensor $\eta_{\mu\nu}$ and a small perturbation $h_{\mu\nu}$, such that the resulting metric can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad |h_{\mu\nu}| << 1. \tag{13.14}$$

The affine connections (13.8) computed for the metric (13.14) give

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2}\eta^{\lambda\rho}\left[\frac{\partial}{\partial x^{\mu}}h_{\rho\nu} + \frac{\partial}{\partial x^{\nu}}h_{\rho\mu} - \frac{\partial}{\partial x^{\rho}}h_{\mu\nu}\right] + O(h^2). \tag{13.15}$$

Since the metric $g^0_{\mu\nu} \equiv \eta_{\mu\nu}$ is constant, $\Gamma^{\lambda}{}_{\mu\nu}(g^0) = 0$ and the right-hand side of eq. (13.13) simply reduces to

$$\begin{aligned} \frac{\partial\Gamma^{\alpha}{}_{\mu\nu}}{\partial x^{\alpha}} &- \frac{\partial\Gamma^{\alpha}{}_{\mu\alpha}}{\partial x^{\nu}} + O(h^2) \\ &= \frac{1}{2}\left\{-\Box_F h_{\mu\nu} + \left[\frac{\partial^2}{\partial x^{\lambda}\partial x^{\mu}}h^{\lambda}_{\nu} + \frac{\partial^2}{\partial x^{\lambda}\partial x^{\nu}}h^{\lambda}_{\mu} - \frac{\partial^2}{\partial x^{\mu}\partial x^{\nu}}h^{\lambda}_{\lambda}\right]\right\} + O(h^2). \end{aligned} \tag{13.16}$$

The operator $\Box_F$ is the D'Alambertian in flat spacetime

$$\Box_F = \eta^{\alpha\beta}\frac{\partial}{\partial x^{\alpha}}\frac{\partial}{\partial x^{\beta}} = -\frac{\partial^2}{c^2\partial t^2} + \nabla^2. \tag{13.17}$$

Einstein's equations (13.5) for $h_{\mu\nu}$ finally become

$$\left\{\Box_F h_{\mu\nu} - \left[\frac{\partial^2}{\partial x^{\lambda}\partial x^{\mu}}h^{\lambda}_{\nu} + \frac{\partial^2}{\partial x^{\lambda}\partial x^{\nu}}h^{\lambda}_{\mu} - \frac{\partial^2}{\partial x^{\mu}\partial x^{\nu}}h^{\lambda}_{\lambda}\right]\right\} = -\frac{16\pi G}{c^4}\left(\delta T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\delta T\right). \tag{13.18}$$

As already discussed in chapter 8, the solution of eqs. (13.18) is not uniquely determined. If we make a coordinate transformation, the transformed metric tensor is still a solution: it describes the same physical situation seen from a different frame. But since we are working in the weak field limit, we are entitled to make only those transformations which preserve the condition $|h'_{\mu\nu}| << 1$ (note that in this Section we denote the transformed tensor as $h'_{\mu\nu}$ rather than as $h_{\mu\nu\prime}$, since this simplifies the discussion of infinitesimal coordinate transformations).

If we make an infinitesimal coordinate transformation

$$x^{\mu\prime} = x^{\mu} + \epsilon^{\mu}(x), \tag{13.19}$$

(the prime refers to the coordinate x^{μ}, not to the index μ) where ϵ^{μ} is an arbitrary vector such that $\frac{\partial\epsilon^{\mu}}{\partial x^{\nu}}$ is of the same order of $h_{\mu\nu}$, then

$$\frac{\partial x^{\alpha\prime}}{\partial x^{\mu}} = \delta^{\alpha}_{\mu} + \frac{\partial\epsilon^{\alpha}}{\partial x^{\mu}}\,. \tag{13.20}$$

Since

$$\begin{aligned} g_{\mu\nu} &= g'_{\alpha\beta}\frac{\partial x^{\alpha\prime}}{\partial x^{\mu}}\frac{\partial x^{\beta\prime}}{\partial x^{\nu}} = \left(\eta_{\alpha\beta}+h'_{\alpha\beta}\right)\left(\delta^{\alpha}_{\mu}+\frac{\partial \epsilon^{\alpha}}{\partial x^{\mu}}\right)\left(\delta^{\beta}_{\nu}+\frac{\partial \epsilon^{\beta}}{\partial x^{\nu}}\right) \\ &= \eta_{\mu\nu}+h'_{\mu\nu}+\epsilon_{\nu,\mu}+\epsilon_{\mu,\nu}+O(h^2)\,, \end{aligned} \tag{13.21}$$

and $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$, then (up to $O(h^2)$)

$$h'_{\mu\nu} = h_{\mu\nu}-\frac{\partial \epsilon_{\mu}}{\partial x^{\nu}}-\frac{\partial \epsilon_{\nu}}{\partial x^{\mu}}. \tag{13.22}$$

In order to simplify eq. (13.18) it appears convenient to choose a coordinate system in which the harmonic gauge condition is satisfied, i.e.

$$g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu}=0. \tag{13.23}$$

Let us see why. This condition is equivalent to say that, up to terms that are first order in $h_{\mu\nu}$, the following equation is satisfied [2]

$$\frac{\partial}{\partial x^{\mu}}h^{\mu}{}_{\nu}=\frac{1}{2}\frac{\partial}{\partial x^{\nu}}h^{\mu}{}_{\mu}. \tag{13.24}$$

Using this condition the term in square brackets in eq. (13.18) vanishes, and Einstein's equations reduce to a simple wave equation supplemented by the condition (13.24)

$$\begin{cases} \Box_F h_{\mu\nu} = -\frac{16\pi G}{c^4}\left(\delta T_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}\delta T\right) \\ \frac{\partial}{\partial x^{\mu}}h^{\mu}{}_{\nu}=\frac{1}{2}\frac{\partial}{\partial x^{\nu}}h^{\mu}{}_{\mu}, \end{cases} \tag{13.25}$$

(to hereafter, we omit the superscript 'pert' to indicate the stress-energy tensor associated to the source of the perturbation). If we introduce the tensor

$$\bar{h}_{\mu\nu}\equiv h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h\,, \tag{13.26}$$

where $h=\eta^{\mu\nu}h_{\mu\nu}\equiv h^{\mu}{}_{\mu}$, eqs. (13.25) become

$$\begin{cases} \Box_F \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}\delta T_{\mu\nu} \\ \frac{\partial}{\partial x^{\mu}}\bar{h}^{\mu}{}_{\nu}=0\,, \end{cases} \tag{13.27}$$

and outside the source where $\delta T_{\mu\nu}=0$

$$\begin{cases} \Box_F \bar{h}_{\mu\nu} = 0 \\ \frac{\partial}{\partial x^{\mu}}\bar{h}^{\mu}{}_{\nu}=0\,. \end{cases} \tag{13.28}$$

---

[2]

$$g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu}=\frac{1}{2}\eta^{\mu\nu}\eta^{\lambda\kappa}\left\{\frac{\partial h_{\kappa\mu}}{\partial x^{\nu}}+\frac{\partial h_{\kappa\nu}}{\partial x^{\mu}}-\frac{\partial h_{\mu\nu}}{\partial x^{\kappa}}\right\}=\frac{1}{2}\eta^{\lambda\kappa}\left\{h^{\nu}{}_{\kappa,\nu}+h^{\mu}{}_{\kappa,\mu}-h^{\nu}{}_{\nu,\kappa}\right\}$$

Since the first two terms are equal we find

$$g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu}=\eta^{\lambda\kappa}\left\{h^{\mu}{}_{\kappa,\mu}-\frac{1}{2}h^{\nu}{}_{\nu,\kappa}\right\}$$

q.e.d.

Thus, we have shown that a perturbation of a flat spacetime propagates as a wave travelling at the speed of light, and that Einstein's theory of gravity predicts the existence of gravitational waves.

As in electrodynamics, the solution of eqs. (13.27) can be written in terms of retarded potentials

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = \frac{4G}{c^4}\int \frac{T_{\mu\nu}(t-\frac{|\mathbf{x}\text{-}\mathbf{x}'|}{c},\mathbf{x}')}{|\mathbf{x}\text{-}\mathbf{x}'|}\, d^3x', \tag{13.29}$$

and the integral extends over the past light-cone of the event $(t,\mathbf{x})$. In eq. (13.29) we have removed the 'δ' in front of the stress energy tensor which, to hereafter, will be considered as a quantity of order h. Equation (13.29) describes the gravitational waves generated by the source $T_{\mu\nu}$.

We may now ask how eqs. (13.28) and (13.27) should be modified if, instead of considering the perturbation of a flat spacetime, we would consider the perturbation of a curved background. For example, suppose $g^0_{\mu\nu}$ is the Schwarzshild solution for a non rotating black hole. In this case, it is possible to show that, by a suitable choice of the gauge, the Einstein equations written for certain combinations of the components of the metric tensor, can be reduced to a form similar to eqs. (13.27). However, since the background spacetime is now curved, the propagation of the waves will be modified with respect to the flat case. The curvature will act as a potential barrier by which waves are scattered and the final equation will have the form

$$\Box_F\Phi - V(x^\mu)\Phi = -\frac{16\pi G}{c^4}T \tag{13.30}$$

where Φ is the appropriate combination of metric functions, T is a combination of the stress-energy tensor components, $\Box_F$ is the d'Alambertian of the flat spacetime and V is the potential barrier generated by the spacetime curvature. In other words, the perturbations of a sperically symmetric, stationary gravitational field would be described by a Schroedinger-like equation! A complete account on the theory of perturbations of black holes can be found in the book *The Mathematical Theory of Black Holes* by S. Chandrasekhar, Oxford: Claredon Press, (1984).

13.2 How to choose the harmonic gauge

We shall now show that if the harmonic-gauge condition is not satisfied in a reference frame, we can always find a new frame where it is, by making an infinitesimal coordinate transformation

$$x^{\lambda'} = x^\lambda + \epsilon^\lambda, \tag{13.31}$$

provided

$$\Box_F\epsilon_\rho = \frac{\partial h^\beta_\rho}{\partial x^\beta} - \frac{1}{2}\frac{\partial h^\beta_\beta}{\partial x^\rho}. \tag{13.32}$$

Indeed, when we change the coordinate system $\Gamma^\lambda = g^{\mu\nu}\Gamma^\lambda{}_{\mu\nu}$ transforms according to equation (9.63), i.e.

$$\Gamma^{\lambda'} = \frac{\partial x^{\lambda'}}{\partial x^\rho}\Gamma^\rho - g^{\rho\sigma}\frac{\partial^2 x^{\lambda'}}{\partial x^\rho \partial x^\sigma}, \tag{13.33}$$

where, from eq. (13.31)

$$\frac{\partial x^{\lambda'}}{\partial x^\rho} = \delta^\lambda_\rho + \frac{\partial \epsilon^\lambda}{\partial x^\rho}.$$

If $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ (see footnote after eq. (13.23))

$$\Gamma^\rho = \eta^{\rho\kappa}\left\{h^\mu{}_{\kappa,\mu} - \frac{1}{2}h^\nu{}_{\nu,\kappa}\right\}; \tag{13.34}$$

moreover

$$\begin{aligned} g^{\rho\sigma}\frac{\partial^2 x^{\lambda'}}{\partial x^\rho \partial x^\sigma} &= g^{\rho\sigma}\left[\frac{\partial}{\partial x^\rho}\left(\frac{\partial x^\lambda}{\partial x^\sigma} + \frac{\partial \epsilon^\lambda}{\partial x^\sigma}\right)\right] = \\ & g^{\rho\sigma}\left[\frac{\partial}{\partial x^\rho}\left(\delta^\lambda_\sigma + \frac{\partial \epsilon^\lambda}{\partial x^\sigma}\right)\right] \simeq \eta^{\rho\sigma}\left[\frac{\partial^2 \epsilon^\lambda}{\partial x^\rho \partial x^\sigma}\right] = \Box_F \epsilon^\lambda, \end{aligned} \tag{13.35}$$

therefore in the new gauge the condition $\Gamma^{\lambda\prime} = 0$ becomes

$$\Gamma^{\lambda'} = \left[\delta^\lambda_\rho + \frac{\partial \epsilon^\lambda}{\partial x^\rho}\right]\eta^{\rho\kappa}\left[\frac{\partial h^\mu_\kappa}{\partial x^\mu} - \frac{1}{2}\frac{\partial h^\nu{}_\nu}{\partial x^\kappa}\right] - \Box_F \epsilon^\lambda = 0. \tag{13.36}$$

If we neglect second order terms in h eq.(13.36) becomes

$$\Gamma^{\lambda'} = \eta^{\lambda\kappa}\left[\frac{\partial h^\mu_\kappa}{\partial x^\mu} - \frac{1}{2}\frac{\partial h^\nu{}_\nu}{\partial x^\kappa}\right] - \Box_F \epsilon^\lambda = 0.$$

Contracting with $\eta_{\lambda\alpha}$ and remembering that $\eta_{\lambda\alpha}\eta^{\lambda\kappa} = \delta^\kappa_\alpha$ we finally find

$$\Box_F \epsilon_\alpha = \left(\frac{\partial h^\mu{}_\alpha}{\partial x^\mu} - \frac{1}{2}\frac{\partial h^\nu{}_\nu}{\partial x^\alpha}\right).$$

This equation can in principle be solved to find the components of ϵ_α, which identify the coordinate system in which the harmonic gauge condition is satisfied.

13.3 Plane gravitational waves

The simplest solution of the wave equation in vacuum (13.28) is a monocromatic plane wave

$$\bar{h}_{\mu\nu} = \Re\left\{A_{\mu\nu}e^{ik_\alpha x^\alpha}\right\}, \tag{13.37}$$

where $A_{\mu\nu}$ is the polarization tensor, i.e. the wave amplitude and $\vec{k}$ is the wave vector. By direct substitution of (13.37) into the first equation we find

$$\begin{aligned} \Box_F \bar{h}_{\mu\nu} = \eta^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta}\left(e^{ik_\gamma x^\gamma}\right) = \eta^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\left[ik_\gamma \frac{\partial x^\gamma}{\partial x^\beta}e^{ik_\gamma x^\gamma}\right] = \\ \eta^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\left[ik_\gamma \delta^\gamma{}_\beta e^{ik_\gamma x^\gamma}\right] = \eta^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\left[ik_\beta e^{ik_\gamma x^\gamma}\right] = \\ = -\eta^{\alpha\beta}k_\alpha k_\beta\ e^{ik_\gamma x^\gamma} = 0, \qquad \rightarrow \qquad \eta^{\alpha\beta}k_\alpha k_\beta = 0, \end{aligned} \tag{13.38}$$

thus, (13.37) is a solution of (13.28) if $\vec{k}$ is a null vector. In addition the harmonic gauge condition requires that

$$\frac{\partial}{\partial x^\mu}\bar{h}^\mu{}_\nu = 0\,, \tag{13.39}$$

which can be written as

$$\eta^{\mu\alpha}\frac{\partial}{\partial x^\mu}\bar{h}_{\alpha\nu} = 0\,. \tag{13.40}$$

Using eq. (13.37) it gives

$$\eta^{\mu\alpha}\frac{\partial}{\partial x^\mu}A_{\alpha\nu}e^{ik_\gamma x^\gamma} = 0 \quad \rightarrow \quad \eta^{\mu\alpha}A_{\alpha\nu}k_\mu = 0 \quad \rightarrow \quad k_\mu A^\mu{}_\nu = 0\,. \tag{13.41}$$

This further condition expresses the orthogonality of the wave vector and of the polarization tensor.

Since $\bar{h}_{\mu\nu}$ is constant on those surfaces where

$$k_\alpha x^\alpha = const, \tag{13.42}$$

these are the equations of the **wavefront**. It is conventional to refer to k^0 as $\frac{\omega}{c}$, where ω is the frequency of the waves. Consequently

$$\vec{k} = (\frac{\omega}{c}, \mathbf{k}). \tag{13.43}$$

Since $\vec{k}$ is a null vector

$$-(k_0)^2 + (k_x)^2 + (k_y)^2 + (k_z)^2 = 0, \qquad \text{i.e.} \tag{13.44}$$

$$\omega = ck_0 = c\sqrt{(k_x)^2 + (k_y)^2 + (k_z)^2}, \tag{13.45}$$

where (k_x, k_y, k_z) are the components of the unit 3-vector $\mathbf{k}$.

13.4 The TT-gauge

We now want to see how many of the ten components of $h_{\mu\nu}$ have a real physical meaning, i.e. what are the degrees of freedom of a gravitational plane wave. Let us consider a wave propagating in flat spacetime along the $x^1 = x$-direction. Since $h_{\mu\nu}$ is independent of y and z, eqs. (13.28) become (as before we raise and lower indices with $\eta_{\mu\nu}$)

$$\left(-\frac{\partial^2}{c^2\partial t^2} + \frac{\partial^2}{\partial x^2}\right)\bar{h}^\mu{}_\nu = 0, \tag{13.46}$$

i.e. $\bar{h}^\mu{}_\nu$ is an arbitrary function of $t \pm \frac{x}{c}$, and

$$\frac{\partial}{\partial x^\mu}\bar{h}^\mu{}_\nu = 0\,. \tag{13.47}$$

Let us consider, for example, a progressive wave $\bar{h}^{\mu}{}_{\nu} = \bar{h}^{\mu}{}_{\nu}[\chi(t,x)]$, where $\chi(t,x) = t - \frac{x}{c}$. Being

$$\begin{cases} \frac{\partial}{\partial t}\bar{h}^{\mu}{}_{\nu} = \frac{\partial \bar{h}^{\mu}{}_{\nu}}{\partial \chi}\frac{\partial \chi}{\partial t} = \frac{\partial \bar{h}^{\mu}{}_{\nu}}{\partial \chi}, \\ \frac{\partial}{\partial x}\bar{h}^{\mu}{}_{\nu} = \frac{\partial \bar{h}^{\mu}{}_{\nu}}{\partial \chi}\frac{\partial \chi}{\partial x} = -\frac{1}{c}\frac{\partial \bar{h}^{\mu}{}_{\nu}}{\partial \chi}, \end{cases} \tag{13.48}$$

eq. (13.47) gives

$$\frac{\partial}{\partial x^{\mu}}\bar{h}^{\mu}{}_{\nu} = \frac{1}{c}\frac{\partial \bar{h}^{t}{}_{\nu}}{\partial t} + \frac{\partial \bar{h}^{x}{}_{\nu}}{\partial x} = \frac{1}{c}\frac{\partial}{\partial \chi}\left[\bar{h}^{t}{}_{\nu} - \bar{h}^{x}{}_{\nu}\right] = 0. \tag{13.49}$$

This equation can be integrated, and the constants of integration can be set equal to zero because we are interested only in the time-dependent part of the solution. The result is

$$\begin{aligned} &\bar{h}^{t}{}_{t} = \bar{h}^{x}{}_{t}, \qquad && \bar{h}^{t}{}_{y} = \bar{h}^{x}{}_{y}, \\ &\bar{h}^{t}{}_{x} = \bar{h}^{x}{}_{x}, \qquad && \bar{h}^{t}{}_{z} = \bar{h}^{x}{}_{z}. \end{aligned} \tag{13.50}$$

We now observe that the harmonic gauge condition does not determine the gauge uniquely. Indeed, if we make an infinitesimal coordinate transformation

$$x^{\mu\prime} = x^{\mu} + \epsilon^{\mu}, \tag{13.51}$$

from eq. (13.33) we find that, if in the old frame $\Gamma^{\rho} = 0$, in the new frame $\Gamma^{\lambda\prime} = 0$, provided

$$\eta^{\rho\sigma}\frac{\partial^2 x^{\lambda\prime}}{\partial x^{\rho}\partial x^{\sigma}} = 0, \tag{13.52}$$

namely, if ϵ^{μ} satisfies the wave equation

$$\Box_F \epsilon^{\mu} = 0. \tag{13.53}$$

If we have a solution of the wave equation,

$$\Box_F \bar{h}_{\mu\nu} = 0 \tag{13.54}$$

and we perform a gauge transformation, the perturbations in the new gauge

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_{\mu}\epsilon_{\nu} - \partial_{\nu}\epsilon_{\mu} \tag{13.55}$$

give

$$\bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \partial_{\mu}\epsilon_{\nu} - \partial_{\nu}\epsilon_{\mu} + \eta_{\mu\nu}\partial^{\alpha}\epsilon_{\alpha} \tag{13.56}$$

and, due to (13.53), the new tensor is solution of the wave equation,

$$\Box_F \bar{h}'_{\mu\nu} = 0\,. \tag{13.57}$$

It can be shown that the converse is also true: it is always possible to find a vector ϵ_{μ} satisfying (13.53) to set to zero four components of $\bar{h}_{\mu\nu}$ solution of (13.54).

Thus, we can use the four functions ϵ^{μ} to set to zero the following four quantities

$$\bar{h}^{t}{}_{x} = \bar{h}^{t}{}_{y} = \bar{h}^{t}{}_{z} = \bar{h}^{y}{}_{y} + \bar{h}^{z}{}_{z} = 0. \tag{13.58}$$

From eq. (13.50) it then follows that

$$\bar{h}^x{}_x = \bar{h}^x{}_y = \bar{h}^x{}_z = \bar{h}^t{}_t = 0. \tag{13.59}$$

The remaining non-vanishing components are $\bar{h}^z{}_y$ and $\bar{h}^y{}_y - \bar{h}^z{}_z$. These components cannot be set equal to zero, because we have exhausted our gauge freedom.

From eqs. (13.58) and (13.59) it follows that

$$\bar{h}^\mu{}_\mu = \bar{h}^t{}_t + \bar{h}^x{}_x + \bar{h}^y{}_y + \bar{h}^z{}_z = 0, \tag{13.60}$$

and since

$$\bar{h}^\mu{}_\mu = h^\mu{}_\mu - 2h^\mu{}_\mu = -h^\mu{}_\mu, \tag{13.61}$$

it follows that

$$h^\mu{}_\mu = 0, \qquad \rightarrow \qquad \bar{h}^\mu{}_\nu \equiv h^\mu{}_\nu, \tag{13.62}$$

i.e. in this gauge $h_{\mu\nu}$ and $\bar{h}_{\mu\nu}$ coincide and are traceless. Thus, a plane gravitational wave propagating along the x-axis is characterized by two functions h_{xy} and $h_{yy} = -h_{zz}$, while the remaining components can be set to zero by choosing the gauge as we have shown:

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_{yy} & h_{yz} \\ 0 & 0 & h_{yz} & -h_{yy} \end{pmatrix}. \tag{13.63}$$

In conclusion, a gravitational wave has only two physical degrees of freedom which correspond to the two possible polarization states. The gauge in which this is clearly manifested is called the TT-gauge, where 'TT-' indicates that the components of the metric tensor $h_{\mu\nu}$ are different from zero only on the plane orthogonal to the direction of propagation (transverse), and that $h_{\mu\nu}$ is traceless.

13.5 How does a gravitational wave affect the motion of a single particle

Consider a particle at rest in flat spacetime before the passage of the wave. We set an inertial frame attached to this particle, and take the x-axis coincident with the direction of propagation of an incoming TT-gravitational wave. The particle will follow a geodesic of the curved spacetime generated by the wave

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \equiv \frac{dU^\alpha}{d\tau} + \Gamma^\alpha{}_{\mu\nu}U^\mu U^\nu = 0. \tag{13.64}$$

At $t = 0$ the particle is at rest ($U^\alpha = (1, 0, 0, 0)$) and the acceleration impressed by the wave will be

$$\left(\frac{dU^\alpha}{d\tau}\right)_{(t=0)} = -\Gamma^\alpha{}_{00} = -\frac{1}{2}\eta^{\alpha\beta}\left[h_{\beta 0,0} + h_{0\beta,0} - h_{00,\beta}\right], \tag{13.65}$$

but since we are in the TT-gauge it follows that

$$\left(\frac{dU^\alpha}{d\tau}\right)_{(t=0)} = 0. \tag{13.66}$$

Thus, U^α remains constant also at later times, which means that the particle is not accelerated neither at $t = 0$ nor later! It remains at a **constant coordinate position**, regardeless of the wave. We conclude that **the study of the motion of a single particle is not sufficient to detect a gravitational wave.**

13.6 Geodesic deviation induced by a gravitational wave

We shall now study the relative motion of particles induced by a gravitational wave. Consider two neighbouring particles A and B, with coordinates x^μ_A, x^μ_B. We shall assume that the two particles are initially at rest, and that a plane-fronted gravitational wave reaches them at some time $t = 0$, propagating along the x-axis. We shall also assume that we are in the TT-gauge, so that the only non-vanishing components of the wave are those on the (y, z)-plane. In this frame, the metric is

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (\eta_{\mu\nu} + h^{TT}_{\mu\nu})dx^\mu dx^\nu \,. \tag{13.67}$$

Since $g_{00} = \eta_{00} = -1$, we can assume that both particles have proper time $\tau = ct$. Since the two particles are initially at rest, they will remain at a constant coordinate position even later, when the wave arrives, and their *coordinate* separation

$$\delta x^\mu = x^\mu_B - x^\mu_A \tag{13.68}$$

remains constant. However, since the metric changes, the proper distance between them will change. For example if the particles are on the y-axis,

$$\Delta l = \int ds = \int_{y_A}^{y_B} |g_{yy}|^{\frac{1}{2}} dy = \int_{y_A}^{y_B} |1 + h^{TT}{}_{yy}(t - x/c)|^{\frac{1}{2}} dy \neq \text{constant}. \tag{13.69}$$

We now want to study the effect of the wave by using the equation of geodesic deviation. To this purpose, it is convenient to change coordinate system and use a locally inertial frame $\{x^{\alpha\prime}\}$ centered on the geodesic of one of the two particles, say the particle A; in the neighborhood of A the metric is

$$ds^2 = \eta_{\alpha\prime\beta\prime}dx^{\alpha\prime}dx^{\beta\prime} + O(|\delta x|^2) \,. \tag{13.70}$$

i.e. it differs from Minkowski's metric by terms of order $|\delta x|^2$. It may be reminded that, as discussed in Chapter 1, it is always possible to define such a frame.

In this frame the particle A has space coordinates $x^{i\prime}_A = 0$ $(i = 1, 2, 3)$, and

$$t_A = \tau/c \,, \qquad \frac{dx^{\mu'}}{d\tau}_{|A} = (1, 0, 0, 0) \,, \qquad g_{\mu'\nu'\,|A} = \eta_{\mu'\nu'} \,, \qquad g_{\mu'\nu',\alpha'\,|A} = 0 \;\; (\text{i.e.}\, \Gamma^{\alpha'}_{\mu'\nu'\,|A} = 0) \,, \tag{13.71}$$

where the subscript $|A$ means that the quantity is computed along the geodesic of the particle A. Moreover, the space components of the vector $\delta x^{\mu'}$ which separates A and B are the coordinates of the particle B:

$$x^{i'}_B = \delta x^{i'} \,. \tag{13.72}$$

To simplify the notation, in the following we will rename the coordinates of this locally inertial frame attached to A as $\{x^\mu\}$, and we will drop all the primes.

The separation vector δx^μ satisfies the equation of geodesic deviation (see Chapter 7):

$$\frac{D^2 \delta x^\mu}{d\tau^2} = R^\mu_{\alpha\beta\gamma} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \delta x^\gamma \,. \tag{13.73}$$

If we evaluate this equation along the geodesic of the particle A, using eqs. (13.71) (removing the primes) we find

$$\frac{1}{c^2} \frac{d^2 \delta x^i}{dt^2} = R^i_{00j} \delta x^j \,. \tag{13.74}$$

If the gravitational wave is due to a perturbation of the flat metric, as discussed in this chapter, the metric can be written as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and the Riemann tensor

$$\begin{aligned} R_{\alpha\kappa\lambda\mu} &= \frac{1}{2} \left(\frac{\partial^2 g_{\alpha\mu}}{\partial x^\kappa \partial x^\lambda} + \frac{\partial^2 g_{\kappa\lambda}}{\partial x^\alpha \partial x^\mu} - \frac{\partial^2 g_{\alpha\lambda}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 g_{\kappa\mu}}{\partial x^\alpha \partial x^\lambda} \right) + \\ &+ g_{\nu\sigma} \left(\Gamma^\nu{}_{\kappa\lambda} \Gamma^\sigma{}_{\alpha\mu} - \Gamma^\nu{}_{\kappa\mu} \Gamma^\sigma{}_{\alpha\lambda} \right) , \end{aligned} \tag{13.75}$$

after neglecting terms which are second order in $h_{\mu\nu}$, becomes

$$R_{\alpha\kappa\lambda\mu} = \frac{1}{2} \left(\frac{\partial^2 h_{\alpha\mu}}{\partial x^\kappa \partial x^\lambda} + \frac{\partial^2 h_{\kappa\lambda}}{\partial x^\alpha \partial x^\mu} - \frac{\partial^2 h_{\alpha\lambda}}{\partial x^\kappa \partial x^\mu} - \frac{\partial^2 h_{\kappa\mu}}{\partial x^\alpha \partial x^\lambda} \right) + O(h^2); \tag{13.76}$$

consequently

$$R_{i00m} = \frac{1}{2} \left(\frac{\partial^2 h_{im}}{\partial x^0 \partial x^0} + \frac{\partial^2 h_{00}}{\partial x^i \partial x^m} - \frac{\partial^2 h_{i0}}{\partial x^0 \partial x^m} - \frac{\partial^2 h_{0m}}{\partial x^i \partial x^0} \right) = \frac{1}{2} h^{TT}_{im,00}, \tag{13.77}$$

because in the TT-gauge $h_{i0} = h_{00} = 0$. i and m can assume only the values 2 and 3, i.e. they refer to the y and z components. It follows that

$$R^\lambda{}_{00m} = \eta^{\lambda i} R_{i00m} = \frac{1}{2} \, \eta^{\lambda i} \, \frac{\partial^2 h^{TT}{}_{im}}{c^2 \partial t^2}, \tag{13.78}$$

and the equation of geodesic deviation (13.74) becomes

$$\frac{d^2}{dt^2} \delta x^\lambda = \frac{1}{2} \, \eta^{\lambda i} \, \frac{\partial^2 h^{TT}{}_{im}}{\partial t^2} \delta x^m. \tag{13.79}$$

For $t \leq 0$ the two particles are at rest relative to each other, and consequently

$$\delta x^j = \delta x^j_0, \qquad \text{with} \qquad \delta x^j_0 = \quad \text{const}, \qquad t \leq 0. \tag{13.80}$$

Since $h_{\mu\nu}$ is a small perturbation, when the wave arrives the relative position of the particles will change only by infinitesimal quantities, and therefore we put

$$\delta x^j(t) = \delta x_0^j + \delta x_1^j(t), \qquad t > 0, \tag{13.81}$$

where $\delta x_1^j(t)$ has to be considered as a small perturbation with respect to the initial position δx_0^j . Substituting (13.81) in (13.79), remembering that δx_0^j is a constant and retaining only terms of order $O(h)$, eq. (13.79) becomes

$$\frac{d^2}{dt^2}\delta x_1^j = \frac{1}{2}\,\eta^{ji}\frac{\partial^2 {h^{TT}}_{ik}}{\partial t^2}\delta x_0^k. \tag{13.82}$$

This equation can be integrated and the solution is

$$\delta x^j = \delta x_0^j + \frac{1}{2}\,\eta^{ji}\;{h^{TT}}_{ik}\;\delta x_0^k, \tag{13.83}$$

which clearly shows the tranverse nature of the gravitational wave; indeed, using the fact that if the wave propagates along x only the components $h_{22} = -h_{33}, h_{23} = h_{32}$ are different from zero, from eqs. (13.83) we find

$$\begin{aligned}
\delta x^1 &= \delta x_0^1 + \frac{1}{2}\,\eta^{11}{h^{TT}}_{1k}\;\delta x_0^k = \delta x_0^1 \\
\delta x^2 &= \delta x_0^2 + \frac{1}{2}\,\eta^{22}{h^{TT}}_{2k}\;\delta x_0^k = \delta x_0^2 + \frac{1}{2}\left({h^{TT}}_{22}\;\delta x_0^2 + {h^{TT}}_{23}\;\delta x_0^3\right) \\
\delta x^3 &= \delta x_0^3 + \frac{1}{2}\,\eta^{33}{h^{TT}}_{3k}\;\delta x_0^k = \delta x_0^3 + \frac{1}{2}\left({h^{TT}}_{32}\;\delta x_0^2 + {h^{TT}}_{33}\;\delta x_0^3\right).
\end{aligned} \tag{13.84}$$

Thus, the particles will be accelerated only in the plane orthogonal to the direction of propagation.

Let us now study the effect of the polarization of the wave. Consider a plane wave whose nonvanishing components are (we omit in the following the superscript TT)

$$\begin{aligned}
h_{yy} &= -h_{zz} = 2\Re\left\{A_+ e^{i\omega(t-\frac{x}{c})}\right\}, \\
h_{yz} &= h_{zy} = 2\Re\left\{A_\times e^{i\omega(t-\frac{x}{c})}\right\}.
\end{aligned} \tag{13.85}$$

Consider two particles located, as indicated in figure (13.1) at $(0, y_0, 0)$ and $(0, 0, z_0)$. Let us consider the polarization '+' first, i.e. let us assume

$$A_+ \neq 0 \qquad \text{and} \qquad A_\times = 0. \tag{13.86}$$

Assuming A_+ real eqs. (13.85) give

$$h_{yy} = -h_{zz} = 2A_+ \cos\omega(t - \frac{x}{c}), \qquad h_{yz} = h_{zy} = 0. \tag{13.87}$$

If at $t = 0$ $\omega(t - \frac{x}{c}) = \frac{\pi}{2}$, eqs. (13.84) written for the two particles for $t > 0$ give

$$\begin{aligned}
&1) \qquad z = 0, \quad y = y_0 + \frac{1}{2}h_{yy}\;y_0 = y_0\;[1 + A_+\cos\omega(t - \frac{x}{c})], \\
&2) \qquad y = 0, \quad z = z_0 + \frac{1}{2}h_{zz}\;z_0 = z_0[1 - A_+\cos\omega(t - \frac{x}{c})].
\end{aligned} \tag{13.88}$$

After a quarter of a period ($\cos\omega(t-\frac{x}{c})=-1$)

$$\begin{array}{lll} 1) & z=0, & y=y_0[1-A_+], \\ 2) & y=0, & z=z_0[1+A_+]. \end{array} \tag{13.89}$$

After half a period ($\cos\omega(t-\frac{x}{c})=0$)

$$\begin{array}{lll} 1) & z=0, & y=y_0, \\ 2) & y=0, & z=z_0. \end{array} \tag{13.90}$$

After three quarters of a period ($\cos\omega(t-\frac{x}{c})=1$)

$$\begin{array}{lll} 1) & z=0, & y=y_0[1+A_+], \\ 2) & y=0, & z=z_0[1-A_+]. \end{array} \tag{13.91}$$

Similarly, if we consider a small ring of particles centered at the origin, the effect produced by a gravitational wave with polarization '+' is shown in figure (13.2).

Let us now see what happens if $A_\times \neq 0$ and $A_+ = 0$:

$$h_{yy}=h_{zz}=0, \qquad h_{yz}=h_{zy}=2A_\times\cos\omega(t-\frac{x}{c}). \tag{13.92}$$

Comparing with eqs. (13.84) we see that a generic particle initially at $P=(y_0,z_0)$, when $t>0$ will move according to the equations

$$\begin{aligned} y &= y_0+\frac{1}{2}h_{yz}\ z_0 = y_0+z_0A_\times\cos\omega(t-\frac{x}{c}), \\ z &= z_0+\frac{1}{2}h_{zy}\ y_0 = z_0+y_0A_\times\cos\omega(t-\frac{x}{c}). \end{aligned} \tag{13.93}$$

Let us consider four particles disposed as indicated in figure (13.3)

$$\begin{array}{lll} 1) & y=r, & z=r, \\ 2) & y=-r, & z=r, \\ 3) & y=-r, & z=-r, \\ 4) & y=r, & z=-r. \end{array} \tag{13.94}$$

As before, we shall assume that the initial time $t=0$ corresponds to $\omega(t-\frac{x}{c})=\frac{\pi}{2}$. After a quarter of a period $(\cos\omega(t-\frac{x}{c})=-1)$, the particles will have the following positions

$$\begin{array}{lll} 1) & y=r[1-A_\times], & z=r[1-A_\times], \\ 2) & y=r[-1-A_\times], & z=r[1+A_\times], \\ 3) & y=r[-1+A_\times], & z=r[-1+A_\times], \\ 4) & y=r[1+A_\times], & z=r[-1-A_\times]. \end{array} \tag{13.95}$$

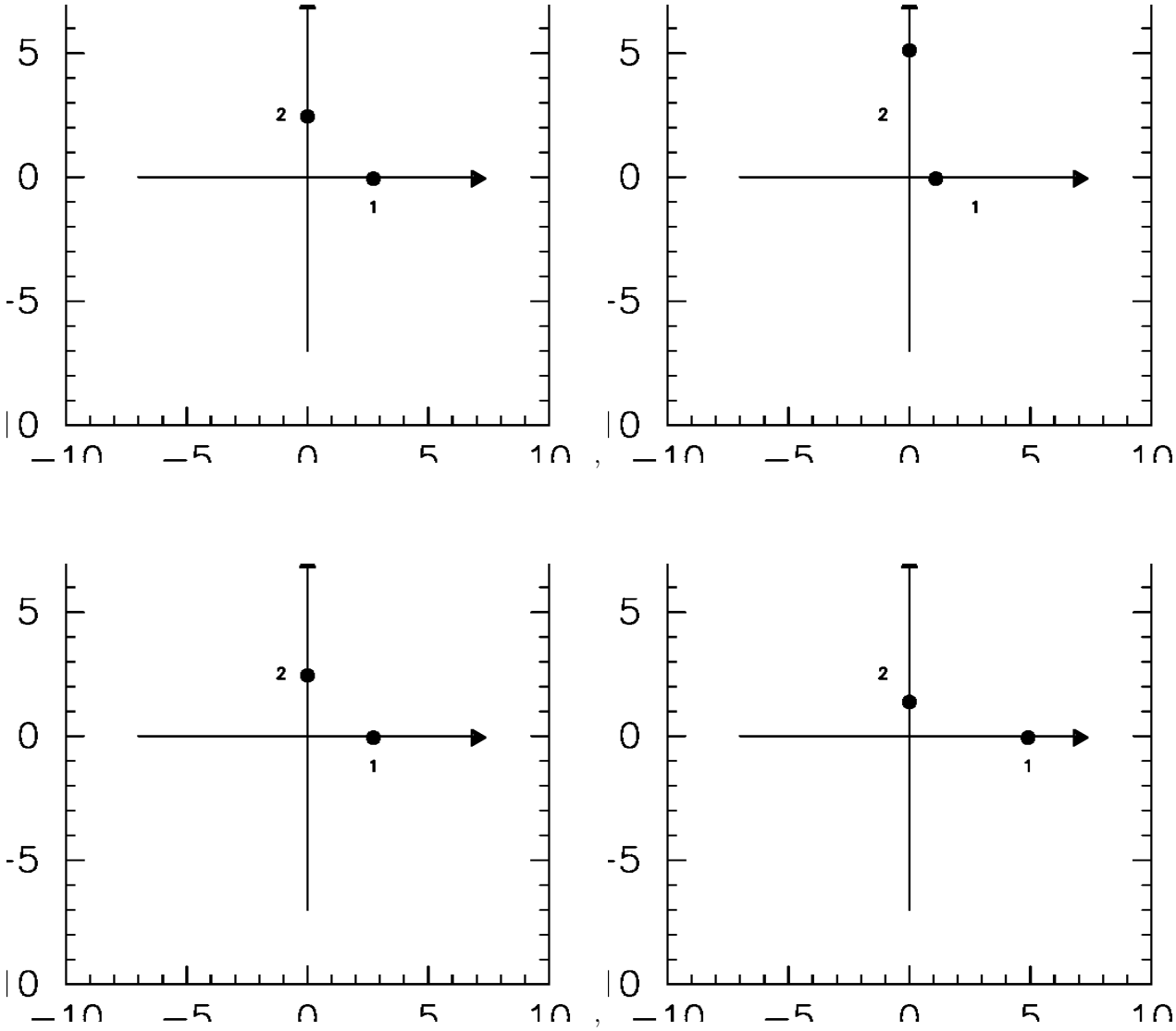

Figure 13.1:

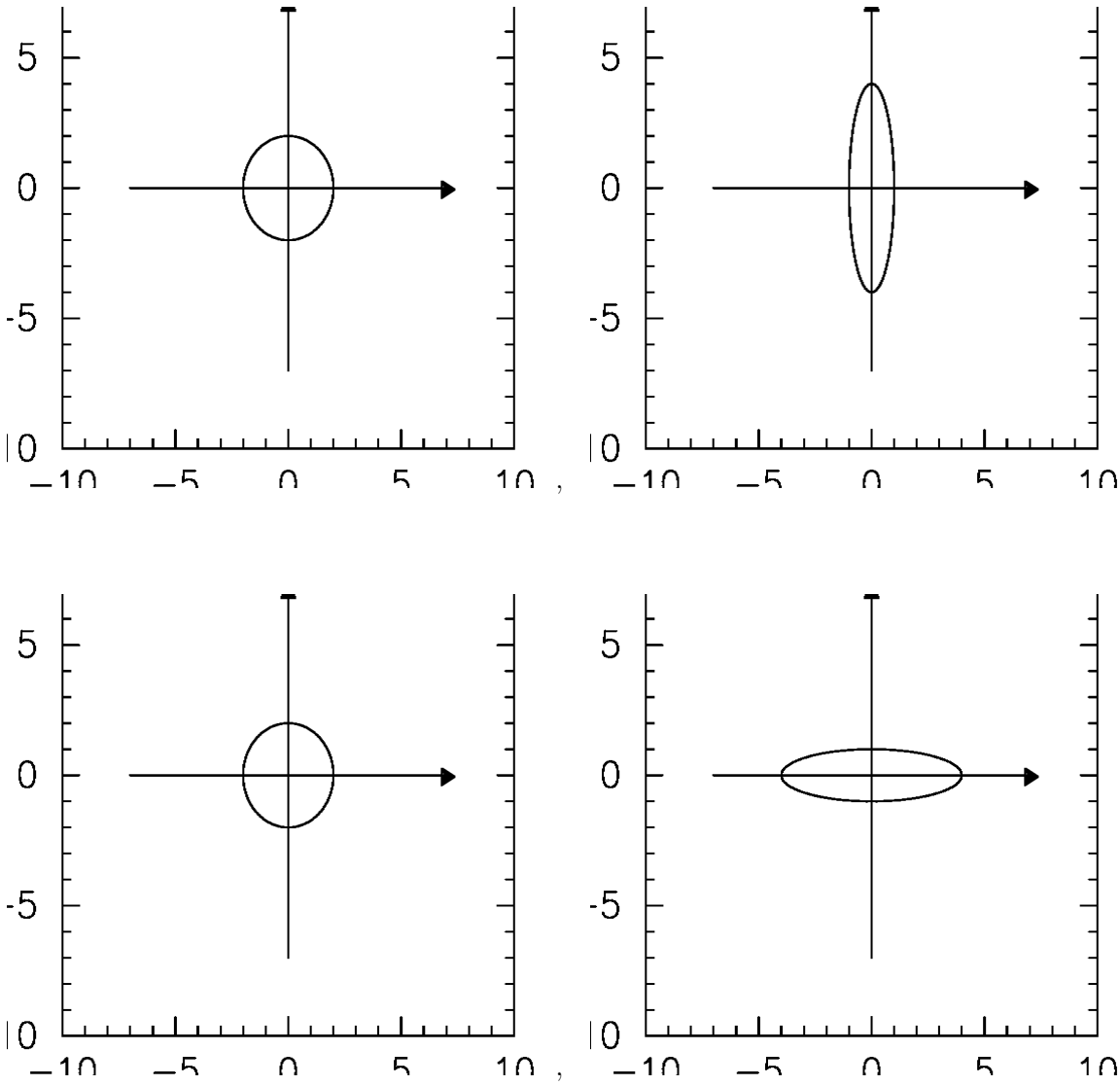

Figure 13.2:

After half a period $\cos\omega(t-\frac{x}{c})=0$, and the particles go back to the initial positions. After three quarters of a period, when $\cos\omega(t-\frac{x}{c})=1$

$$\begin{array}{lll} 1) & y=r[1+A_\times], & z=r[1+A_\times], \\ 2) & y=r[-1+A_\times], & z=r[1-A_\times], \\ 3) & y=r[-1-A_\times], & z=r[-1-A_\times], \\ 4) & y=r[1-A_\times], & z=r[-1+A_\times]. \end{array} \tag{13.96}$$

The motion of the particles is indicated in figure (13.3).

It follows that a small ring of particles centered at the origin, will again become an ellipse, but rotated at 45^0 (see figure (13.4)) with respect to the case previously analysed. In conclusion, we can define A_+ and $A_\times$ as the **polarization amplitudes** of the wave. The wave will be linearly polarized when only one of the two amplitudes is different from zero.

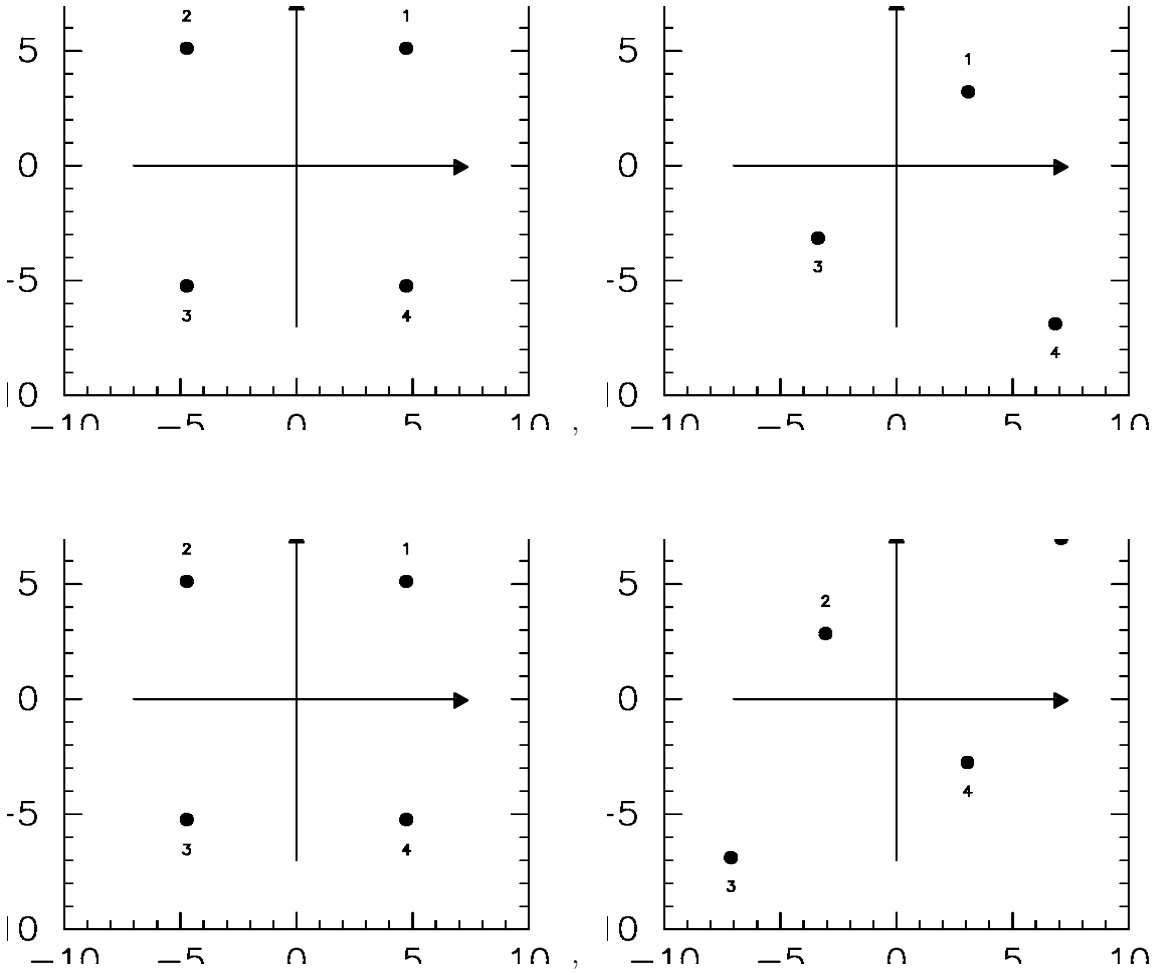

Figure 13.3:

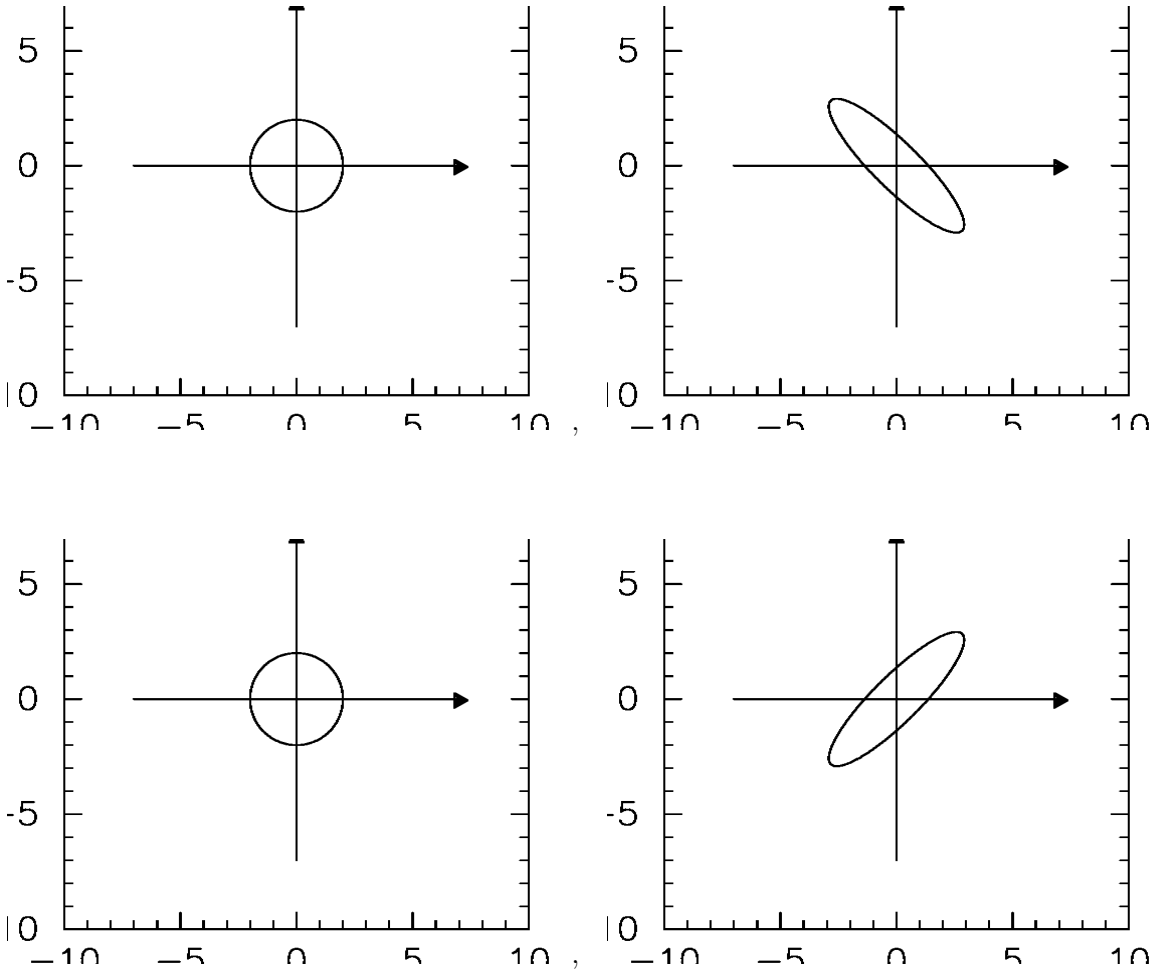

Figure 13.4: