Chapter 14

Gravitational waves generation

In this chapter we will introduce the quadrupole formalism which allows to estimate the gravitational energy and the waveforms emitted by an evolving physical system described by the stress-energy tensor $T^{\mu\nu}$. We shall solve eq. (13.27) under the following assumption: we shall assume that the region where the source is confined, namely

$$|x^i| < \epsilon, \qquad T_{\mu\nu} \neq 0, \tag{14.1}$$

is much smaller than the wavelenght of the emitted radiation, $\lambda_{GW} = \frac{2\pi c}{\omega}$. This implies that

$$\frac{2\pi c}{\omega} \gg \epsilon \quad \to \quad \epsilon\, \omega \ll c \quad \to \quad v_{typical} \ll c,$$

i.e. the velocities typical of the physical processes we are considering are much smaller than the speed of light; for this reason this is called *the slow-motion approximation*.
Let us consider the first equation in (13.27)

$$\Box_F \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}, \tag{14.2}$$

where

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h \qquad \text{and} \qquad \Box_F = \left[-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2 \right].$$

By Fourier-expanding both $\bar{h}_{\mu\nu}$ and $T_{\mu\nu}$

$$\begin{aligned} T_{\mu\nu}(t, x^i) &= \int_{-\infty}^{+\infty} T_{\mu\nu}(\omega, x^i) e^{-i\omega t}\, d\omega, \\ \bar{h}_{\mu\nu}(t, x^i) &= \int_{-\infty}^{+\infty} \bar{h}_{\mu\nu}(\omega, x^i) e^{-i\omega t}\, d\omega, \qquad i = 1, 3 \end{aligned} \tag{14.3}$$

eq. (14.2) becomes

$$\left[\nabla^2 + \frac{\omega^2}{c^2} \right] \bar{h}_{\mu\nu}(\omega, x^i) = -K T_{\mu\nu}(\omega, x^i) \tag{14.4}$$

where

$$K = \frac{16\pi G}{c^4}. \tag{14.5}$$

We shall solve eq. (14.4) outside and inside the source, matching the two solutions across the source boundary.

The exterior solution

Outside the source $T^{\mu\nu} = 0$ and eq. (14.4) becomes

$$\left[\nabla^2 + \frac{\omega^2}{c^2}\right] \bar{h}_{\mu\nu}(\omega, x^i) = 0. \tag{14.6}$$

In polar coordinates, the Laplacian operator ∇^2 is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial}{\partial r}\right] + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta \frac{\partial}{\partial\theta}\right] + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

We shall consider the simplest solution of this equation, i.e. one which does not depend on ϕ and θ:

$$\bar{h}_{\mu\nu}(\omega, r) = \frac{A_{\mu\nu}(\omega)}{r}\ e^{i\frac{\omega}{c}r} + \frac{Z_{\mu\nu}(\omega)}{r}\ e^{-i\frac{\omega}{c}r}.$$

This solution represents a spherical wave, with an *ingoing* part ($\sim e^{-i\frac{\omega}{c}r}$), and an *outgoing* ($\sim e^{i\frac{\omega}{c}r}$) part; indeed, substituting in the second eq. (14.3) $\bar{h}_{\mu\nu}(\omega, x^i)$ by $\sim e^{\pm i\frac{\omega}{c}r}$ the result of the integration over ω gives a function of $(t \mp \frac{r}{c})$ respectively.

Since we are interested only in the wave emitted *from* the source, we shall set $Z_{\mu\nu} = 0$, and consider the solution

$$\bar{h}_{\mu\nu}(\omega, r) = \frac{A_{\mu\nu}(\omega)}{r}\ e^{i\frac{\omega}{c}r}. \tag{14.7}$$

This is the solution outside the source and on its boundary, where $T^{\mu\nu}$ vanishes as well. $A_{\mu\nu}$ is the wave amplitude to be found by solving the equations inside the source.

The interior solution

The wave equation

$$\left[\nabla^2 + \frac{\omega^2}{c^2}\right] \bar{h}_{\mu\nu}(\omega, x^i) = -\mathbf{K} T_{\mu\nu}(\omega, x^i) \tag{14.8}$$

can be solved for each assigned value of the indices μ, ν. To solve eq. (14.8) let us integrate over the source volume

$$\int_V \left[\nabla^2 + \frac{\omega^2}{c^2}\right] \bar{h}_{\mu\nu}(\omega, x^i) d^3x = -\mathbf{K} \int_V T_{\mu\nu}(\omega, x^i) d^3x.$$

The first term can be expanded as follows

$$\int_V \nabla^2\ \bar{h}_{\mu\nu}(\omega, x^i)\ d^3x = \int_V \mathrm{div}[\boldsymbol{\nabla}\ \bar{h}_{\mu\nu}]\ d^3x = \int_S \left(\boldsymbol{\nabla}\ \bar{h}_{\mu\nu}\right)^k\ dS_k \tag{14.9}$$

where $\boldsymbol{\nabla}\ \bar{h}_{\mu\nu}$ is the gradient of $\bar{h}_{\mu\nu}$, S is the surface surrounding the source volume, and we have applied Gauss theorem to $\boldsymbol{\nabla}\ \bar{h}_{\mu\nu}$. Using eq. (14.7) the surface integral can be approximated as follows

$$\int_S \left(\boldsymbol{\nabla}\,\bar{h}_{\mu\nu}\right)^k\, dS_k \;\simeq\; 4\pi\,\epsilon^2\left(\frac{d}{dr}\,\frac{A_{\mu\nu}}{r}\,e^{i\frac{\omega}{c}r}\right)_{r=\epsilon}$$

$$= 4\pi\,\epsilon^2\left[-\,\frac{A_{\mu\nu}}{r^2}\,e^{i\frac{\omega}{c}r}+\frac{A_{\mu\nu}}{r}\,\left(\frac{i\omega}{c}\right)\,e^{i\frac{\omega}{c}r}\right]_{r=\epsilon};$$

if we keep the leading term and discard terms of order ϵ, we find [1]

$$\int_V \nabla^2\,\bar{h}_{\mu\nu}(\omega,x^i)\,d^3x \;\simeq\; -\,4\pi\,A_{\mu\nu}(\omega),$$

and eq. (14.8) becomes

$$-4\pi\,A_{\mu\nu}+\int_V \frac{\omega^2}{c^2}\,\bar{h}_{\mu\nu}(\omega,x^i)\,d^3x = -\mathbf{K}\int_V T_{\mu\nu}(\omega,x^i)\,d^3x. \tag{14.10}$$

The second term

$$\int_V \frac{\omega^2}{c^2}\,\bar{h}_{\mu\nu}(\omega,x^i)\,d^3x$$

satisfies the following inequality

$$\int_V \frac{\omega^2}{c^2}\,\bar{h}_{\mu\nu}(\omega,x^i)\,d^3x \;\lessapprox\; |\bar{h}_{\mu\nu}|_{max}\,\frac{\omega^2}{c^2}\,\frac{4}{3}\pi\epsilon^3, \tag{14.11}$$

where $|\bar{h}_{\mu\nu}|_{max}$ is the maximum reached by $\bar{h}_{\mu\nu}$ in the volume V, and since the right-hand side of eq. (14.11) is of order ϵ^3 it can be neglected. Consequently eq. (14.10) becomes

$$-4\pi A_{\mu\nu}(\omega) = -\mathbf{K}\int_V T_{\mu\nu}(\omega,x^i)\,d^3x \tag{14.12}$$

i.e.

$$A_{\mu\nu}(\omega) = \frac{4G}{c^4}\int_V T_{\mu\nu}(\omega,x^i)\,d^3x.$$

Thus, the solution of the wave equation inside the source gives the wave amplitude $A_{\mu\nu}(\omega)$ as an integral of the stress-energy tensor of the source over the source volume. Knowing $A_{\mu\nu}(\omega)$ we finally find

$$\bar{h}_{\mu\nu}(\omega,r) = \frac{4G}{c^4}\cdot\frac{e^{i\,\omega\frac{r}{c}}}{r}\int_V T_{\mu\nu}(\omega,x^i)\,d^3x, \tag{14.13}$$

or, by the inverse Fourier transform

$$\bar{h}_{\mu\nu}(t,r) = \frac{4G}{c^4}\,\frac{1}{r}\int_V T_{\mu\nu}(t-\frac{r}{c},x^i)\,d^3x. \tag{14.14}$$

This is the gravitational signal emitted by the source.
The integral in (14.14) can be further simplified, but in the meantime note that:

[1] It should be noted that $e^{i\frac{\omega}{c}r}\sim 1$ since we have assumed that $\lambda_{GW} >> \epsilon$.

1) The solution (14.14) for $\bar{h}_{\mu\nu}$ automatically satisfies the second eq. (13.27), i.e. the harmonic gauge condition

$$\frac{\partial}{\partial x^\mu}\bar{h}^\mu{}_\nu = 0.$$

To prove this, we first notice that the solution (14.14) is equivalent to the expression (13.29)

$$\bar{h}_{\mu\nu}(t, \mathbf{x}) = \frac{4G}{c^4}\int_V \frac{T_{\mu\nu}(t - \frac{|\mathbf{x}\text{-}\mathbf{x}'|}{c}, \mathbf{x}')}{|\mathbf{x}\text{-}\mathbf{x}'|}\, d^3x'; \tag{14.15}$$

indeed, since

$$|\mathbf{x}'| < \epsilon\,, \qquad \text{and} \qquad r \gg \epsilon, \tag{14.16}$$

then

$$r \equiv |\mathbf{x}| \simeq |\mathbf{x}\text{-}\mathbf{x}'|. \tag{14.17}$$

By defining the following function

$$g\left(\vec{x} - \vec{x}\,'\right) \equiv \frac{4G}{c^5}\frac{1}{|\mathbf{x}\text{-}\mathbf{x}'|}\delta\left[t' - \left(t - \frac{|\mathbf{x}\text{-}\mathbf{x}'|}{c}\right)\right], \tag{14.18}$$

where $\vec{x} = (ct, \mathbf{x})$ and $\vec{x}\,' = (ct', \mathbf{x}')$, eq. (14.15) can be written as a four-dimensional integral as follows

$$\bar{h}_{\mu\nu}(\vec{x}) = \int_\Omega T_{\mu\nu}(\vec{x}\,')\; g\left(\vec{x} - \vec{x}\,'\right)\; d^4x', \tag{14.19}$$

where $\Omega \equiv V \times I$, and I is the time interval to be taken such that $g(\vec{x} - \vec{x}')$ vanishes at the extrema of I; this happens if I is so large that, for all $\mathbf{x}' \in V$, the expression $t - \frac{|\mathbf{x}\text{-}\mathbf{x}'|}{c}$ is inside I; indeed, from the definition (14.18) g is different from zero only for $t' = t - \frac{|\mathbf{x}\text{-}\mathbf{x}'|}{c}$.

Since g is a function of the *difference* $(\vec{x} - \vec{x}')$, then

$$\frac{\partial}{\partial x^\mu}\left[g\left(\vec{x} - \vec{x}\,'\right)\right] = -\frac{\partial}{\partial x^{\mu\,\prime}}\left[g\left(\vec{x} - \vec{x}\,'\right)\right]. \tag{14.20}$$

Consequently,

$$\frac{\partial}{\partial x^\mu}\bar{h}^{\mu\nu}(\vec{x}) = \int_\Omega T^{\mu\nu}(\vec{x}\,')\frac{\partial}{\partial x^\mu}g\left(\vec{x} - \vec{x}\,'\right)\; d^4x' = -\int_\Omega T^{\mu\nu}(\vec{x}\,')\frac{\partial}{\partial x^{\mu\,\prime}}g\left(\vec{x} - \vec{x}\,'\right)\; d^4x'. \tag{14.21}$$

The last term can be integrated by parts and gives

$$\begin{aligned}\int_\Omega T^{\mu\nu}(\vec{x}\,')\frac{\partial}{\partial x^{\mu\,\prime}}g\left(\vec{x} - \vec{x}\,'\right)\; d^4x' &= \int_\Omega d^4x'\;\frac{\partial}{\partial x^{\mu\,\prime}}\left[T^{\mu\nu}(\vec{x}\,')g\left(\vec{x} - \vec{x}\,'\right)\right]\\ &- \int_\Omega d^4x'\;\left[\frac{\partial}{\partial x^{\mu\,\prime}}T^{\mu\nu}(\vec{x}\,')g\left(\vec{x} - \vec{x}\,'\right)\right]\; d^4x' = 0\,.\end{aligned}$$

The first integral vanishes since $T^{\mu\nu} = 0$ on the boundary of V and $g = 0$ on the boundary of I, the second because the stress-energy tensor satisfies the conservation law $T^{\mu\nu}{}_{,\nu} = 0$. Consequently

$$\frac{\partial}{\partial x^\mu}\bar{h}^{\mu\nu}(\vec{x}) = 0\,.$$

Q.E.D.
2) In order to extract the physical components of the wave we still have to project $\bar{h}_{\mu\nu}$ on the **TT**-gauge.
3) Eq. (14.14) has been derived on two very strong assumptions: weak field ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$) and slow motion ($v_{typical} << c$). For this reason that expression has to be considered as an estimate of the emitted radiation by the system, unless the two conditions are really satisfied.

14.1 The Tensor Virial Theorem

In order to simplify the integral in eq. (14.14) we shall use the conservation law that $T_{\mu\nu}$ satisfies (see chapter 7)

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0, \qquad \rightarrow \qquad \frac{1}{c}\frac{\partial T^{\mu 0}}{\partial t} = -\frac{\partial T^{\mu k}}{\partial x^k}, \qquad \mu = 0..3, \quad k = 1..3. \tag{14.22}$$

Let us integrate this equation over the source volume, assuming the index μ is fixed

$$\frac{1}{c}\frac{\partial}{\partial t}\int_V T^{\mu 0} d^3x = -\int_V \frac{\partial T^{\mu k}}{\partial x^k} d^3x.$$

By Gauss' theorem, the integral over the volume is equal to the flux of $T^{\mu k}$ across the surface S enclosing that volume, thus the right-hand-side becomes

$$\int_V \frac{\partial T^{\mu k}}{\partial x^k} d^3x = \int_S T^{\mu k} dS_k.$$

By definition, on S $T^{\mu\nu} = 0$ and consequently the surface integral vanishes; thus

$$\frac{1}{c}\frac{\partial}{\partial t}\int_V T^{\mu 0} d^3x = 0, \qquad \rightarrow \qquad \int_V T^{\mu 0} d^3x = const. \tag{14.23}$$

From eq. (14.14) it follows that

$$\bar{h}^{\mu 0} = const, \quad \mu = 0..3,$$

and since we are interested in the time-dependent part of the field we shall put

$$\bar{h}^{\mu 0} = 0, \quad \mu = 0..3. \tag{14.24}$$

(Indeed, in the **TT**-gauge $\bar{h}^{\mu 0} = 0$.) We shall now prove the **Tensor-Virial Theorem** which establishes that

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int_V T^{00}\, x^k\, x^n\, d^3x = 2\int_V T^{kn}\, d^3x, \qquad k, n = 1..3. \tag{14.25}$$

Let us consider the space-components of the conservation law (14.22)

$$\frac{\partial T^{n0}}{\partial x^0} = -\frac{\partial T^{ni}}{\partial x^i}, \qquad i, n = 1..3;$$

multiply both members by x^k and integrate over the source volume

$$\begin{aligned}
\frac{1}{c}\frac{\partial}{\partial t}\int_V T^{n0}\, x^k\, d^3x &= -\int_V \frac{\partial T^{ni}}{\partial x^i}\, x^k\, d^3x \\
&= -\left[\int_V \frac{\partial\left(T^{ni}\, x^k\right)}{\partial x^i}\, d^3x - \int_V T^{ni}\, \frac{\partial x^k}{\partial x^i}\, d^3x\right] \\
&= -\int_S \left(T^{ni}\, x^k\right)\, dS_i + \int_V T^{nk}\, d^3x,
\end{aligned}$$

(remember that $\frac{\partial x^k}{\partial x^i} = \delta^k_i$). As before $\int_S \left(T^{ni}\, x^k\right)\, dS_i = 0$, therefore

$$\frac{1}{c}\frac{\partial}{\partial t}\int_V T^{n0}\, x^k\, d^3x = \int_V T^{nk}\, d^3x.$$

Since T^{nk} is symmetric we can rewrite this equation in the following form

$$\frac{1}{2c}\frac{\partial}{\partial t}\int_V \left(T^{n0}\, x^k + T^{k0}\, x^n\right)\, d^3x = \int_V T^{nk}\, d^3x. \tag{14.26}$$

Let us now consider the 0 component of the conservation law

$$\frac{1}{c}\frac{\partial T^{00}}{\partial t} + \frac{\partial T^{0i}}{\partial x^i} = 0, \qquad i = 1..3$$

multiply by $x^k x^n$ and integrate over V

$$\begin{aligned}
\frac{1}{c}\frac{\partial}{\partial t}\int_V T^{00}\, x^k\, x^n\, d^3x &= -\int_V \frac{\partial T^{0i}}{\partial x^i}\, x^k\, x^n\, d^3x \\
&= -\left[\int_V \frac{\partial\left(T^{0i}\, x^k\, x^n\right)}{\partial x^i}\, d^3x - \int_V \left(T^{0i}\, \frac{\partial x^k}{\partial x^i}\, x^n + T^{0i}\, x^k\, \frac{\partial x^n}{\partial x^i}\right)\, d^3x\right] \\
&= -\int_S \left(T^{0i}\, x^k\, x^n\right)\, dS_i + \int_V \left(T^{0k}\, x^n + T^{0n}\, x^k\right)\, d^3x
\end{aligned}$$

the first integral vanishes and this equation becomes

$$\frac{1}{c}\frac{\partial}{\partial t}\int_V T^{00}\, x^k\, x^n\, d^3x = \int_V \left(T^{0k}\, x^n + T^{0n}\, x^k\right)\, d^3x.$$

If we now differentiate with respect to x^0 we find

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int_V T^{00}\, x^k\, x^n\, d^3x = \frac{1}{c}\frac{\partial}{\partial t}\int_V \left(T^{0k}\, x^n + T^{0n}\, x^k\right)\, d^3x,$$

and using eq. (14.26) we finally find

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\int_V T^{00}\, x^k\, x^n\, d^3x = 2\int_V T^{kn}\, d^3x, \qquad k, n = 1, 3. \tag{14.27}$$

The left-hand-side of this equation is the second time derivative of the **quadrupole moment tensor** of the system

$$q^{kn}(t) = \frac{1}{c^2} \int_V T^{00}(t, x^i)\; x^k\; x^n d^3x, \qquad k, n = 1, 3, \tag{14.28}$$

which is a function of time only. Thus, in conclusion

$$\int_V T^{kn}(t, x^i)\; d^3x = \frac{1}{2} \frac{d^2}{dt^2} q^{kn}(t).$$

By using eqs. (14.14) and (14.24) we finally find

$$\begin{cases} \bar{h}^{\mu 0} = 0, \qquad \mu = 0..3 \\ \bar{h}^{ik}(t, r) = \dfrac{2G}{c^4 r} \cdot \left[\dfrac{d^2}{dt^2}\; q^{ik}(t - \dfrac{r}{c}) \right] \end{cases} . \tag{14.29}$$

This is the gravitational wave emitted by a gravitating system evolving in time. It can be composed of masses or of any form of energy, because mass and energy are both sources of the gravitational field.

NOTE THAT

1. $\frac{G}{c^4} \sim 8 \cdot 10^{-50}\; s^2/g\; cm$: this is the reason why gravitational waves are extremely weak!!

2. In order to make the physical degrees of freedom explicitely manifest we still have to transform to the **TT**-gauge.

3. These equations have been derived on very strong assumptions: one is that $T^{\mu\nu}{}_{,\nu} = 0$, i.e. that the motion of the bodies is dominated by non-gravitational forces. However, and remarkably, the result (14.29) depends only on the sources motion and not on the forces acting on them.

4. A system of accelerated charged particles has a time-varying dipole moment

$$\vec{d}_{EM} = \sum_i q_i \vec{r}_i$$

and it will emit dipole radiation, the flux of which depends on the second time derivative of $\vec{d}_{EM}$. For an isolated system of masses we can define a gravitational dipole moment

$$\vec{d}_G = \sum_i m_i \vec{r}_i,$$

which satisfies the conservation law of the total momentum of an isolated system

$$\frac{d}{dt}\vec{d}_G = \vec{0}.$$

For this reason, gravitational waves do not have a dipole contribution. It should be stressed that for a spherical or axisymmetric, stationary distribution of matter (or energy) the quadrupole moment is a constant, even if the body is rotating. Thus, a spherical or axisymmetric star does not emit gravitational waves; similarly a star which collapses in a perfectly spherically symmetric way has a vanishing $\frac{d^2q^{ik}}{dt^2}$ and does not emit gravitational waves. To produce these waves we need a certain degree of asymmetry, as it occurs for instance in the non-radial pulsations of stars, in a non spherical gravitational collapse, in the coalescence of massive bodies etc.

14.2 How to transform to the TT-gauge

The solution (14.29) describes a spherical wave far from the emitting source. Locally, it looks like a plane wave propagating along the direction of the unit vector orthogonal to the wavefront

$$n^\alpha = (0, n^i), \qquad i = 1.., 3 \tag{14.30}$$

where

$$n^i = \frac{x^i}{r}. \tag{14.31}$$

In order to express this waveform in the **TT**-gauge we shall make an infinitesimal coordinate transformation $x^{\mu\,\prime} = x^\mu + \epsilon^\mu$ and choose the vector ϵ^μ which satisfies the wave equation $\Box_F \epsilon^\mu = 0$, so that the harmonic gauge condition is preserved, as explained in chapter 14. The conditions to impose on the perturbed metric are

$$\begin{aligned} &\bar{h}'_{\alpha\beta}\, n^\beta = 0, \qquad \text{trasverse wave condition} \\ &\bar{h}'_{\alpha\beta}\, \eta^{\alpha\beta} = 0, \qquad \text{vanishing trace.} \end{aligned}$$

It should be mentioned that the transverse-wave condition implies that $\bar{h}^{\mu 0} = 0, \quad \mu = 0, 3$ as required in eq. (14.24). Indeed, given the wave-vector $k^\mu = (k^0, k^0 n^i)$ we know by eq. (13.41) that $k^\mu \bar{h}'_{\mu\nu} = 0$, i.e.

$$k^0 \bar{h}'_{0\nu} + k^0 n^i \bar{h}'_{i\nu} = 0.$$

The second term vanishes because of the trasverse wave condition, therefore

$$h'_{0\nu} = 0.$$

We remind here that, as shown in eq. (13.62), in the **TT**-gauge $\bar{h}_{\mu\nu}$ and $h_{\mu\nu}$ coincide.
To hereafter, we shall work in the 3-dimensional euclidean space with metric δ_{ij}. Consequently, there will be no difference between covariant and contravariant indices.

We shall now describe a procedure to project the wave in the TT-gauge, which is equivalent to perform the coordinate transformation mentioned above. As a first step, we define the operator which projects a vector onto the plane orthogonal to the direction of **n**

$$P_{jk} \equiv \delta_{jk} - n_j n_k\,. \tag{14.32}$$

Indeed, it is easy to verify that for any vector V^j, $P_{jk}V^k$ is orthogonal to n^j, i.e. $(P_{jk}V^k)n^j = 0$, and that

$$P^j{}_k P^k{}_l V^l = P^j{}_l V^l \,. \tag{14.33}$$

Note that $P_{jk} = P_{kj}$, i.e. P_{jk} is **symmetric**. The projector is **transverse**, i.e.

$$n^j P_{jk} = 0 \,. \tag{14.34}$$

Then, we define the **transverse–traceless projector**:

$$\mathcal{P}_{jkmn} \equiv P_{jm}P_{kn} - \frac{1}{2}P_{jk}P_{mn} \,. \tag{14.35}$$

which "extracts" the transverse-traceless part of a $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor. In fact, using the definition (14.35), it is easy to see that it satisfies the following properties

- $\mathcal{P}_{jklm} = \mathcal{P}_{lmjk}$
- $\mathcal{P}_{jklm} = \mathcal{P}_{kjml}$
  and
  $$\mathcal{P}_{jkmn}\mathcal{P}_{mnrs} = \mathcal{P}_{jkrs} \,; \tag{14.36}$$
- it is transverse:
  $$n^j\mathcal{P}_{jkmn} = n^k\mathcal{P}_{jkmn} = n^m\mathcal{P}_{jkmn} = n^n\mathcal{P}_{jkmn} = 0 \,; \tag{14.37}$$
- it is traceless:
  $$\delta^{jk}\mathcal{P}_{jkmn} = \delta^{mn}\mathcal{P}_{jkmn} = 0 \,. \tag{14.38}$$

Since h_{jk} and $\bar{h}_{jk}$ differ only by the trace, and since the projector $\mathcal{P}_{jklm}$ extracts the traceless part of a tensor (eq. 14.38), the components of the perturbed metric tensor in the **TT**-gauge can be obtained by applying the projector $\mathcal{P}_{jkmn}$ either to h_{jk} or to $\bar{h}_{jk}$

$$h^{\mathbf{TT}}_{jk} = \mathcal{P}_{jkmn}h_{mn} = \mathcal{P}_{jkmn}\bar{h}_{mn}. \tag{14.39}$$

By applying $\mathcal{P}$ on $\bar{h}_{jk}$ defined in eq. (14.29) we get

$$\begin{cases} h^{\mathbf{TT}}_{\mu 0} = 0, \qquad \mu = 0,3 \\ h^{\mathbf{TT}}_{jk}(t,r) = \dfrac{2G}{c^4 r} \cdot \left[\dfrac{d^2}{dt^2}\, Q^{\mathbf{TT}}_{jk}(t-\dfrac{r}{c})\right] \end{cases} \tag{14.40}$$

where

$$Q^{\mathbf{TT}}_{jk} \equiv \mathcal{P}_{jkmn}q_{mn} \tag{14.41}$$

is the **transverse–traceless part of the quadrupole moment**. Sometimes it is useful to define the **reduced quadrupole moment** Q_{jk}

$$Q_{jk} \equiv q_{jk} - \frac{1}{3}\delta_{jk}q_m^{\ m} \tag{14.42}$$

whose trace is zero by definition, i.e.

$$\delta^{jk}Q_{jk} = 0 \,, \tag{14.43}$$

and from eq. (14.38), it follows that

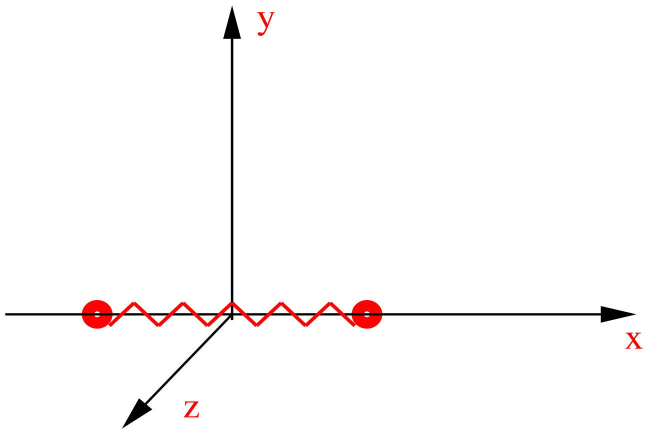
$$Q^{\mathbf{TT}}_{jk} = \mathcal{P}_{jkmn}q_{mn} = \mathcal{P}_{jkmn}Q_{mn} \,. \tag{14.44}$$

14.3 Gravitational wave emitted by a harmonic oscillator

Let us consider a harmonic oscillator composed of two equal masses m oscillating at a frequency $\nu = \frac{\omega}{2\pi}$ with amplitude A. Be l_0 the proper length of the string when the system is at rest. Assuming that the oscillator moves on the x-axis, the position of the two masses will be

$$\begin{cases} x_1 = -\frac{1}{2}l_0 - A\cos\omega t \\ x_2 = +\frac{1}{2}l_0 + A\cos\omega t \end{cases} .$$

The 00-component of the stress-energy tensor of the system is



$$T^{00} = \sum_{n=1}^{2} cp^0 \ \delta(x - x_n) \ \delta(y) \ \delta(z);$$

and since $v << c, \quad \to \quad \gamma \sim 1 \quad \to \quad p^0 = mc$, it reduces to

$$T^{00} = mc^2 \ \sum_{n=1}^{2} \ \delta(x - x_n) \ \delta(y) \ \delta(z);$$

the xx-component of the quadrupole moment $q^{ik}(t) = \frac{1}{c^2} \ \int_V \ T^{00}(t, \mathbf{x}) \ x^i \ x^k dx^3$ is

$$\begin{aligned} q^{xx} = q_{xx} \quad &= \quad m \left[\int_V \ \delta(x - x_1) \ x^2 \ dx \ \delta(y) \ dy \ \delta(z) \ dz \right. \\ &\quad \left. + \int_V \ \delta(x - x_2) \ x^2 \ dx \ \delta(y) \ dy \ \delta(z) \ dz \right] \\ &= \quad m \left[x_1^2 + x_2^2\right] = m \left[\frac{1}{2}l_0^2 + 2A^2 \cos^2 \ \omega t + 2Al_0 \ \cos \ \omega t\right] \\ &= \quad m \left[cost + A^2 \cos \ 2\omega t + 2Al_0 \ \cos \ \omega t\right], \end{aligned} \qquad (14.45)$$

where we have used the trigonometric expression $\cos 2\alpha = 2\cos^2\alpha - 1$.
The zz-component of the quadrupole moment is

$$\begin{aligned} q^{zz} = &= m\left[\int_V \delta(x-x_1)\ dx\ \delta(y)\ dy\ \delta(z)\ z^2\ dz \right.\\ &\left. + \int_V \delta(x-x_2)\ dx\ \delta(y)\ dy\ \delta(z)\ z^2\ dz\right] = 0 \end{aligned}$$

because $\int_V z^2\ \delta(z)\ dz = 0$. Since the motion is confined to the x-axis, all remaining components of q_{ij} vanish.
We shall compute, as an example, the wave emerging in the z-direction; in this case $\mathbf{n} = \frac{\mathbf{x}}{r} \to (0,0,1)$ and

$$P_{jk} = \delta_{jk} - n_j n_k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By applying to Q_{ij} the transverse-traceless projector $\mathcal{P}_{jkmn}$ constructed from P_{jk}, we find

$$\begin{aligned} Q^{\mathbf{TT}}{}_{xx} &= \left(P_{xm}P_{xn} - \frac{1}{2}P_{xx}P_{mn}\right) q_{mn} = \left(P_{xx}P_{xx} - \frac{1}{2}P^2_{xx}\right) q_{xx} = \frac{1}{2}q_{xx}, \quad (14.46)\\ Q^{\mathbf{TT}}{}_{yy} &= \left(P_{ym}P_{yn} - \frac{1}{2}P_{yy}P_{mn}\right) q_{mn} = -\frac{1}{2}P_{yy}P_{xx}q_{xx} = -\frac{1}{2}q_{xx},\\ Q^{\mathbf{TT}}{}_{xy} &= \left(P_{xm}P_{yn} - \frac{1}{2}P_{xy}P_{mn}\right) q_{mn} = 0,\\ Q^{\mathbf{TT}}{}_{zz} &= \left(P_{zm}P_{zn} - \frac{1}{2}P_{zz}P_{mn}\right) q_{mn} = 0. \end{aligned}$$

In addition $Q^{\mathbf{TT}}{}_{zx} = Q^{\mathbf{TT}}{}_{zy} = 0$. Using these expressions eqs. (14.40) become

$$\begin{cases} h^{\mathbf{TT}}{}_{\mu 0} = 0 \\ h^{\mathbf{TT}}{}_{zi} = 0, \qquad h^{\mathbf{TT}}{}_{xy} = 0 \\ h^{\mathbf{TT}}{}_{xx}(t,z) = -h^{\mathbf{TT}}{}_{yy}(t,z) = \dfrac{G}{c^4 z}\ \dfrac{d^2}{dt^2}\ q_{xx}(t-\dfrac{z}{c}), \end{cases} \qquad (14.47)$$

and using eq. (14.45)

$$\begin{aligned} h^{\mathbf{TT}}{}_{xx} &= -h^{\mathbf{TT}}{}_{yy} = \frac{G}{c^4 z}\cdot\left[\frac{d^2}{dt^2}\ q_{xx}(t-\frac{z}{c})\right], \qquad (14.48)\\ &= -\frac{2Gm}{c^4 z}\ \omega^2\ \left[2A^2\cos\ 2\omega(t-\frac{z}{c}) + Al_0\ \cos\ \omega(t-\frac{z}{c})\right]. \end{aligned}$$

Thus, radiation emitted by the harmonic oscillator along the z-axis is *linearly polarized*.
If, for instance, we consider two masses $m = 10^3$ kg, with $l_0 = 1$ m, $A = 10^{-4}$ m, and $\omega = 10^4$ rad/s, the term $[2A^2\cos\ 2\omega t]$ is negligible, and the dominant term is at the same frequency of the oscillations:

$$h^{\mathbf{TT}}{}_{xx} \sim -\frac{2Gm}{c^4 z}\ \omega^2\ Al_0\ \cos\ \omega(t-\frac{z}{c}) \sim \frac{1.6\cdot 10^{-35}}{z},$$

which is, as expected, very very small.
It should be noticed that due to the symmetry of the system, the wave emitted along y will be the same. To find the wave emitted along x, we choose $\mathbf{n} = (1,0,0)$ and use the same procedure: no radiation will be found.

14.4 Gravitational wave emitted by a binary system in circular orbit

We shall now estimate the gravitational signal emitted by a binary system composed of two stars moving on a circular orbit around their common center of mass. For simplicity we shall assume that the two stars of mass m_1 and m_2 are point masses. Be l_0 the orbital separation, M the *total mass*

$$M \equiv m_1 + m_2, \tag{14.49}$$

and μ the *reduced mass*

$$\mu \equiv \frac{m_1 m_2}{M}\,. \tag{14.50}$$

Let us consider a coordinate frame with origin coincident with the center of mass of the system as indicated in figure (14.1) and be

$$l_0 = r_1 + r_2, \qquad r_1 = \frac{m_2 l_0}{M}, \qquad r_2 = \frac{m_1 l_0}{M}. \tag{14.51}$$

The orbital frequency can be found from Kepler's law

$$G\frac{m_1 m_2}{l_0^2} = m_1\, \omega_K^2 \frac{m_2 l_0}{M}, \qquad G\frac{m_1 m_2}{l_0^2} = m_2\, \omega_K^2 \frac{m_1 l_0}{M},$$

from which we find

$$\omega_K = \sqrt{\frac{GM}{l_0^3}} \tag{14.52}$$

is the Keplerian frequency. Be (x_1, x_2) and (y_1, y_2) the coordinates of the masses m_1 and m_2

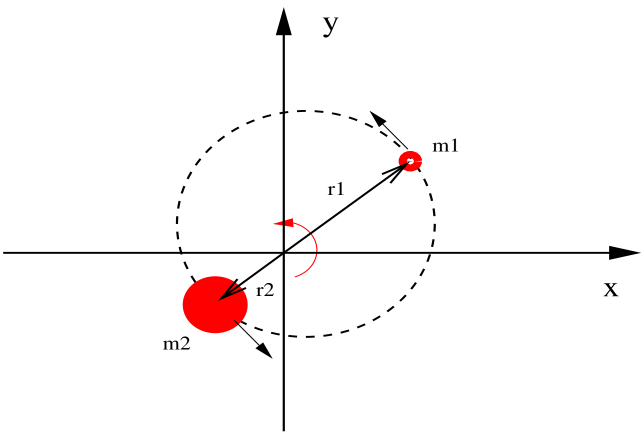


Figure 14.1: Two point masses in circular orbit around the common center of mass

on the orbital plane

$$\begin{aligned} x_1 &= \tfrac{m_2}{M} l_0 \cos\omega_K t & \qquad x_2 &= -\frac{m_1}{M} l_0 \cos\omega_K t \\ y_1 &= \tfrac{m_2}{M} l_0 \sin\omega_K t & \qquad y_2 &= -\frac{m_1}{M} l_0 \sin\omega_K t. \end{aligned} \tag{14.53}$$

The 00-component of the stress-energy tensor of the system is

$$T^{00} = c^2 \sum_{n=1}^{2} m_n \; \delta(x - x_n) \; \delta(y - y_n) \; \delta(z)\,,$$

and the non vanishing components of the quadrupole moment are

$$\begin{aligned} q_{xx} &= m_1 \int_V x^2 \delta(x - x_1) \; dx \; \delta(y - y_1) \; dy \; \delta(z) \; dz \\ &+ m_2 \int_V x^2 \delta(x - x_2) \; dx \; \delta(y - y_2) \; dy \; \delta(z) \; dz = m_1 x_1^2 + m_2 x_2^2 \\ &= \mu \; l_0^2 \; \cos^2 \omega_K t = \frac{\mu}{2} \; l_0^2 \; \cos 2\omega_K t + cost, \end{aligned}$$

$$\begin{aligned} q_{yy} &= m_1 \int_V \delta(x - x_1) \; dx \; y^2 \; \delta(y - y_1) \; dy \; \delta(z) \; dz \\ &+ m_2 \int_V \delta(x - x_2) \; dx \; y^2 \; \delta(y - y_2) \; dy \; \delta(z) \; dz = m_1 y_1^2 + m_2 y_2^2 \\ &= \mu \; l_0^2 \; \sin^2 \omega_K t = -\frac{\mu}{2} \; l_0^2 \; \cos 2\omega_K t + cost1, \end{aligned}$$

and

$$\begin{aligned} q_{xy} &= m_1 \int_V x\delta(x - x_1) \; dx \; y \; \delta(y - y_1) \; dy \; \delta(z) \; dz \\ &+ m_2 \int_V x\delta(x - x_2) \; dx \; y \; \delta(y - y_2) \; dy \; \delta(z) \; dz \\ &= m_1 x_1 y_1 + m_2 x_2 y_2 = \mu \; l_0^2 \; \cos \omega t \; \sin \omega_K t = \frac{\mu}{2} \; l_0^2 \; \sin 2\omega_K t. \end{aligned}$$

(we have used $\cos 2\alpha = 2\cos^2\alpha - 1$, $\sin^2\alpha = \frac{1}{2} - \frac{1}{2}\cos 2\alpha$ and $m_1 m_2 = \mu M$).
In summary

$$\begin{aligned} q_{xx} &= \frac{\mu}{2} \; l_0^2 \; \cos 2\omega_K t + cost \\ q_{yy} &= -\frac{\mu}{2} \; l_0^2 \; \cos 2\omega_K t + cost1 \\ q_{xy} &= \frac{\mu}{2} \; l_0^2 \; \sin 2\omega_K t, \end{aligned}$$

and

$$q^k{}_k = \eta^{kl} \; q_{kl} = q_{xx} + q_{yy} = \text{costant}.$$

Therefore, the time-varying part of q_{ij} and of $Q_{ij} = q_{ij} - \frac{1}{3} \; \delta_{ij} \; q^k{}_k$ are equal:

$$\begin{aligned} q_{xx} &= -q_{yy} = \frac{\mu}{2} \; l_0^2 \quad \cos 2\omega_K t \\ q_{xy} &= \frac{\mu}{2} \; l_0^2 \quad \sin 2\omega_K t, \end{aligned} \tag{14.54}$$

and defining a matrix A_{ij}

$$A_{ij}(t) = \begin{pmatrix} \cos \; 2\omega_K t & \sin \; 2\omega_K t & 0 \\ \sin \; 2\omega_K t & -\cos \; 2\omega_K t & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{14.55}$$

we can write

$$q_{ij} = \frac{\mu}{2}\ l_0^2 \quad A_{ij} + const. \tag{14.56}$$

Since the wave emitted along a generic direction $\mathbf{n}$ in the $\mathbf{TT}$-gauge is

$$h_{ij}^{\mathbf{TT}}(t,r) = \frac{2G}{rc^4}\frac{d^2}{dt^2}\left[Q_{ij}^{\mathbf{TT}}(t-\frac{r}{c})\right] \qquad \text{where} \qquad Q_{ij}^{\mathbf{TT}}(t-\frac{r}{c}) = \mathcal{P}_{ijkl}Q_{kl}(t-\frac{r}{c}) = \mathcal{P}_{ijkl}q_{kl}(t-\frac{r}{c})$$

using eq. (14.52) we find

$$h_{ij}^{\mathbf{TT}} = -\frac{2G}{rc^4}\ \frac{\mu}{2}\ l_0^2\ \ (2\omega_K)^2\ \ [\mathcal{P}_{ijkl}A_{kl}] = -\frac{1}{r}\times\frac{4\ \mu\ M\ G^2}{r\ l_0\ c^4}\ \ [\mathcal{P}_{ijkl}A_{kl}]\,. \tag{14.57}$$

By defining a wave amplitude

$$h_0 = \frac{4\ \mu\ M\ G^2}{l_0\ c^4} \tag{14.58}$$

we can finally write the emitted wave as

$$h_{ij}^{\mathbf{TT}}(t,r) = \ -\ \frac{h_0}{r}\ A_{ij}^{\mathbf{TT}}(t-\frac{r}{c}), \tag{14.59}$$

where

$$A_{ij}^{\mathbf{TT}}(t-\frac{r}{c}) = \left[\mathcal{P}_{ijkl}A_{kl}(t-\frac{r}{c})\right] \tag{14.60}$$

depends on the orientation of the line of sight with respect to the orbital plane.
From these equations we see that the radiation is emitted at **twice the orbital frequency**.

For example, if $\mathbf{n}=\mathbf{z}$, $P_{ij} = diag(1,1,0)$

$$A_{ij}^{\mathbf{TT}}(t) = \begin{pmatrix} \cos\ 2\omega_K t & \sin\ 2\omega_K t & 0 \\ \sin\ 2\omega_K t & -\cos\ 2\omega_K t & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{14.61}$$

and

$$\begin{aligned} h^{\mathbf{TT}}{}_{xx} &= -h^{\mathbf{TT}}{}_{yy} = -\frac{h_0}{z}\ \cos 2\omega_K(t-\frac{z}{c}) \\ h^{\mathbf{TT}}{}_{xy} &= -\frac{h_0}{z}\ \sin 2\omega_K(t-\frac{z}{c}). \end{aligned} \tag{14.62}$$

In this case the wave has both polarizations, and since $h^{\mathbf{TT}}{}_{xx} = h_0/z\ \Re\left\{e^{i\omega(t-\frac{x}{c})}\right\}$ and $h^{\mathbf{TT}}{}_{xy} = h_0/z\ \Im\left\{e^{i\omega(t-\frac{x}{c})}\right\}$, the wave is circularly polarized.
If $\mathbf{n}=\mathbf{x}$, $P_{ij} = diag(0,1,1)$

$$A_{ij}^{\mathbf{TT}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2}\cos\ 2\omega_K t & 0 \\ 0 & 0 & \frac{1}{2}\cos\ 2\omega_K t \end{pmatrix} \tag{14.63}$$

and

$$h^{\mathbf{TT}}{}_{yy} = -h^{\mathbf{TT}}{}_{zz} = +\frac{1}{2}\frac{h_0}{x}\quad \cos 2\omega_K(t-\frac{x}{c}), \tag{14.64}$$

i.e. the wave is a linearly polarized wave.
If $\mathbf{n} = \mathbf{y}$, $P_{ij} = diag(1, 0, 1)$ and

$$A_{ij}^{\mathbf{TT}} = \begin{pmatrix} \frac{1}{2}\cos\ 2\omega_K t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\cos\ 2\omega_K t \end{pmatrix} \tag{14.65}$$

and again the wave is linearly polarized

$$h^{\mathbf{TT}}{}_{xx} = -h^{\mathbf{TT}}{}_{zz} = -\frac{1}{2}\frac{h_0}{y}\ \cos 2\omega_K(t - \frac{y}{c}). \tag{14.66}$$

Eqs. (14.58) can be used to estimate the amplitude of the gravitational signal emitted by the binary system PSR 1913+16 discovered in 1975, (R.A. Hulse and J.H. Taylor, *Discovery Of A Pulsar In A Binary System*, Astrophys. J. **195**, L51, 1975) which consists of two neutron stars orbiting at a very short distance from each other. The data we know from observations are:

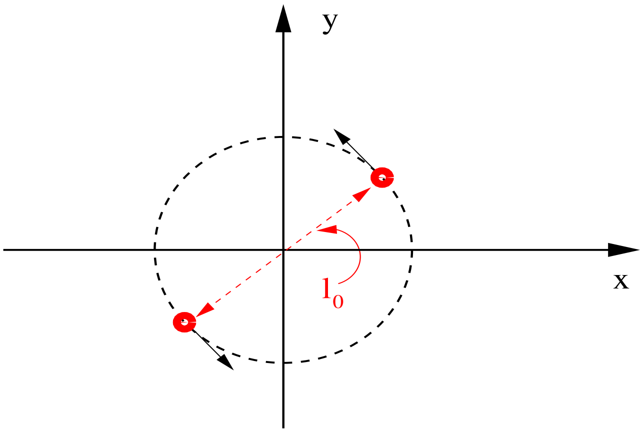


Figure 14.2: Two equal point masses in circular orbit

$$\begin{aligned} m_1 \sim m_2 \sim 1.4 M_\odot, \quad l_0 = 0.19 \cdot 10^{12}\ cm \\ T = 7h\ 45m\ 7s, \qquad \nu_K = \frac{\omega_K}{2\pi} \sim 3.58 \cdot 10^{-5}\ Hz \end{aligned} \tag{14.67}$$

where T is the orbital period. Note that the two stars have nearly equal masses: they are comparable to that of the Sun, and their orbital separation is about twice the radius of the Sun! The orbit is eccentric with eccentricity $\epsilon \simeq 0.617$, however we shall assume it is circular and apply eqs. (14.58). For this system the emission frequency is

$$\nu_{GW} = 2\nu_K \sim 7.16 \cdot 10^{-5}\ Hz, \tag{14.68}$$

therefore the wavelenght of the emitted radiation is

$$\lambda_{GW} = \frac{c}{\nu_{GW}} \sim 10^{14}\ cm \qquad \text{i.e.} \qquad \lambda_{GW} >> l_0. \tag{14.69}$$

Thus, the slow-motion approximation, on which the quadrupole formalism is based, is certainly satisfied in this case even though the two neutron stars are orbiting at such small distance from each other. The distance of the system from Earth is $r = 5\ kpc,$ and since

$$1\ pc = 3.08 \cdot 10^{18}\ cm, \quad \to \quad r = 1.5 \cdot 10^{22}\ cm.$$

The wave amplitude is

$$h_0 = \frac{4\ \mu\ M\ G^2}{r\ l_0\ c^4} \sim 5 \cdot 10^{-23}.$$

A new binary pulsar has more recently been discovered (M. Burgay *et al.*, *An increased estimate of the merger rate of double neutron stars from observations of a highly relativistic system* Nature **426**, 531, 2003) which has an even shorter orbital period and it is closer than PSR 1913+16: it is the double pulsar PSR J0737-3039, whose orbital parameters are

$$\begin{array}{rl} m_1 = 1.337 M_\odot, & m_2 \sim 1.250 M_\odot \\ T = 2.4h, & e = 0.08 \\ r = 500\ pc & l_0 \sim 1.2 R_\odot. \end{array} \tag{14.70}$$

In this case the orbit is nearly circular,

$$\mu = \frac{m_1 m_2}{m_1 + m_2} = 0.646 M_\odot \quad \to \quad h_0 = \frac{4\mu M G^2}{r l_0 c^4} \sim 1.1 \cdot 10^{-21},$$

and waves are emitted at the frequency

$$\nu_{GW} = 2\ \nu_K = 2.3 \cdot 10^{-4}\ Hz.$$

In this section we have considered only circular orbits; the calculations can be generalized to the case of eccentric or open orbits by replacing the equation of motion of the two masses (14.53) by those appropriate to the chosen orbit. By this procedure it is possible to show that when the orbits are ellipses, gravitational waves are emitted at frequencies multiple of the orbital frequency ν_K, and that the number of equally spaced spectral lines increases with the eccentricity.

14.5 Energy carried by a gravitational wave

In order to evaluate how much energy is radiated in gravitational waves by an evolving system, we need to define a tensor that properly describes the energy content of the gravitational field. Our effort will not be completely successful, since we will be able to define a quantity which behaves like a tensor only under linear coordinate transformations. However, this pseudo-tensor will be useful for the purpose we have in mind.

14.5.1 The stress-energy pseudotensor of the gravitational field

In Chapter 7 we have shown that the stress-energy tensor of matter satisfies a divergenceless equation

$$T^{\mu\nu}{}_{;\nu} = 0. \tag{14.71}$$

If we choose a locally inertial frame (LIF), the covariant derivative reduces to the ordinary derivative and eq. (14.71) becomes

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0. \tag{14.72}$$

We shall now try to find a quantity, $\eta^{\mu\nu\gamma}$, such that

$$T^{\mu\nu} = \frac{\partial}{\partial x^\alpha}\eta^{\mu\nu\alpha}; \tag{14.73}$$

In this way, if we impose that $\eta^{\mu\nu\alpha}$ is **antisymmetric** in the indices ν and α, the conservation law (14.72) will automatically be satisfied.

The problem now is: can we find the explicit expression of $\eta^{\mu\nu\gamma}$?

From Einstein's equations we know that

$$T^{\mu\nu} = \frac{c^4}{8\pi G}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right); \tag{14.74}$$

since we are in a locally inertial frame, the Riemann tensor, whose generic expression is

$$\begin{aligned} R_{\gamma\alpha\delta\beta} = \frac{1}{2}\left[\frac{\partial^2 g_{\gamma\beta}}{\partial x^\alpha \partial x^\delta} + \frac{\partial^2 g_{\alpha\delta}}{\partial x^\gamma \partial x^\beta} - \frac{\partial^2 g_{\gamma\delta}}{\partial x^\alpha \partial x^\beta} - \frac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma \partial x^\delta}\right] \\ + g_{\sigma\rho}\left(\Gamma^\sigma_{\alpha\delta}\Gamma^\rho_{\gamma\beta} - \Gamma^\sigma_{\alpha\beta}\Gamma^\rho_{\gamma\delta}\right), \end{aligned} \tag{14.75}$$

reduces to the term in square brackets since all $\Gamma^\sigma_{\alpha\delta}$'s vanish; it follows that in this frame the Ricci tensor becomes

$$\begin{aligned} R^{\mu\nu} &= g^{\mu\alpha}g^{\nu\beta}R_{\alpha\beta} = g^{\mu\alpha}g^{\nu\beta}g^{\gamma\delta}R_{\gamma\alpha\delta\beta} \\ &= \frac{1}{2}g^{\mu\alpha}g^{\nu\beta}g^{\gamma\delta}\left(\frac{\partial^2 g_{\gamma\beta}}{\partial x^\alpha \partial x^\delta} + \frac{\partial^2 g_{\alpha\delta}}{\partial x^\gamma \partial x^\beta} - \frac{\partial^2 g_{\gamma\delta}}{\partial x^\alpha \partial x^\beta} - \frac{\partial^2 g_{\alpha\beta}}{\partial x^\gamma \partial x^\delta}\right). \end{aligned} \tag{14.76}$$

By using this equation, after some cumbersome calculations eq. (14.74) becomes

$$T^{\mu\nu} = \frac{\partial}{\partial x^\alpha}\left\{\frac{c^4}{16\pi G}\frac{1}{(-g)}\frac{\partial}{\partial x^\beta}\left[(-g)\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta}\right)\right]\right\}. \tag{14.77}$$

The term in parentheses is antisymmetric in the indices ν and α and it is the quantity we were looking for:

$$\eta^{\mu\nu\alpha} = \frac{c^4}{16\pi G}\frac{1}{(-g)}\frac{\partial}{\partial x^\beta}\left[(-g)\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta}\right)\right]. \tag{14.78}$$

If we now introduce the quantity

$$\zeta^{\mu\nu\alpha} = (-g)\eta^{\mu\nu\alpha} = \frac{c^4}{16\pi G}\frac{\partial}{\partial x^\beta}\left[(-g)\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta}\right)\right], \tag{14.79}$$

since we are in a locally inertial frame $\frac{\partial}{\partial x^\beta}\frac{1}{(-g)}=0$, therefore we can write eq. (14.77) as

$$\frac{\partial \zeta^{\mu\nu\alpha}}{\partial x^\alpha}=(-g)T^{\mu\nu}. \tag{14.80}$$

This equation has been derived in a LIF, where all first derivatives of the metric tensor vanish, but in any other frame this will not be true and the difference $\frac{\partial \zeta^{\mu\nu\alpha}}{\partial x^\alpha}-(-g)T^{\mu\nu}$ will not be zero, but a quantity which we shall call $(-g)t^{\mu\nu}$ i.e.

$$(-g)t^{\mu\nu}=\frac{\partial \zeta^{\mu\nu\alpha}}{\partial x^\alpha}-(-g)T^{\mu\nu}. \tag{14.81}$$

$t^{\mu\nu}$ is symmetric because both $T^{\mu\nu}$ and $\frac{\partial \zeta^{\mu\nu\alpha}}{\partial x^\alpha}$ are symmetric in μ and ν. The explicit expression of $t^{\mu\nu}$ can be found by substituting in eq. (14.81) the definition of $\zeta^{\mu\nu\alpha}$ given in eq. (14.79), and $T^{\mu\nu}$ computed in terms of the Ricci tensor from eq. (14.74) in an arbitrary frame (i.e. starting from the full expression of the Riemann tensor given in eq. 14.75): after some careful manipulation of the equations it is possible to show that

$$\begin{aligned}
t^{\mu\nu} &= \frac{c^4}{16\pi G}\Big\{\left(2\Gamma^\delta{}_{\alpha\beta}\Gamma^\sigma{}_{\delta\sigma}-\Gamma^\delta{}_{\alpha\sigma}\Gamma^\sigma{}_{\beta\delta}-\Gamma^\delta{}_{\alpha\delta}\Gamma^\sigma{}_{\beta\sigma}\right)\left(g^{\mu\alpha}g^{\nu\beta}-g^{\mu\nu}g^{\alpha\beta}\right)\\
&+ g^{\mu\alpha}g^{\beta\delta}\left(\Gamma^\nu{}_{\alpha\sigma}\Gamma^\sigma{}_{\beta\delta}+\Gamma^\nu{}_{\beta\delta}\Gamma^\sigma{}_{\alpha\sigma}-\Gamma^\nu{}_{\delta\sigma}\Gamma^\sigma{}_{\alpha\beta}-\Gamma^\nu{}_{\alpha\beta}\Gamma^\sigma{}_{\delta\sigma}\right)\\
&+ g^{\nu\alpha}g^{\beta\delta}\left(\Gamma^\mu{}_{\alpha\sigma}\Gamma^\sigma{}_{\beta\delta}+\Gamma^\mu{}_{\beta\delta}\Gamma^\sigma{}_{\alpha\sigma}-\Gamma^\mu{}_{\delta\sigma}\Gamma^\sigma{}_{\alpha\beta}-\Gamma^\mu{}_{\alpha\beta}\Gamma^\sigma{}_{\delta\sigma}\right)\\
&+ g^{\alpha\beta}g^{\delta\sigma}\left(\Gamma^\mu{}_{\alpha\delta}\Gamma^\nu{}_{\beta\sigma}-\Gamma^\mu{}_{\alpha\beta}\Gamma^\nu{}_{\delta\sigma}\right)\Big\}
\end{aligned}$$

This is the stress-energy pseudotensor of the gravitational field we were looking for. Indeed we can rewrite eq. (14.81), valid in any reference frame, in the following form

$$(-g)\left(t^{\mu\nu}+T^{\mu\nu}\right)=\frac{\partial \zeta^{\mu\nu\alpha}}{\partial x^\alpha}, \tag{14.82}$$

and since $\zeta^{\mu\nu\alpha}$ is antisymmetric in μ and α

$$\frac{\partial}{\partial x^\mu}\frac{\partial \zeta^{\mu\nu\alpha}}{\partial x^\alpha}=0,$$

and consequently

$$\frac{\partial}{\partial x^\mu}\left[(-g)\left(t^{\mu\nu}+T^{\mu\nu}\right)\right]=0. \tag{14.83}$$

This equation expresses a conservation law, because, as explained in chapter 7, it has the form of a vanishing ordinary divergence of the quantity $\left[(-g)\left(t^{\mu\nu}+T^{\mu\nu}\right)\right]$. Since $t^{\mu\nu}$ when added to the stress-energy tensor of matter (or fields) satisfies a conservation law, and since it vanishes only in a locally inertial frame where gravity is suppressed, we interpret $t^{\mu\nu}$ as the entity that contains the information on the energy and momentum carried by the gravitational field. Thus eq. (14.83) expresses the conservation law of the total energy and momentum. Unfortunately, $t^{\mu\nu}$ **is not a tensor**; indeed it is a combination of the Γ's that are not tensors.

However, as the Γ's, it behaves as a tensor under linear coordinate transformations.

14.5.2 The energy flux carried by a gravitational wave

Let us consider an emitting source and the associated 3-dimensional coordinate frame O (x, y, z). Be an observer located at $P = (x1, y1, z1)$ as shown in figure 14.3. Be $r = \sqrt{x1^2 + y1^2 + z1^2}$ its distance from the origin. The observer wants to detect the wave coming along the direction identified by the versor $\mathbf{n} = \frac{\mathbf{r}}{|r|}$. As a pedagogical tool, let us consider a second frame O' (x', y', z'), with origin coincident with O, and having the x'-axis aligned with $\mathbf{n}$. Assuming that the wave traveling along x' direction is linearly polarized and has only one polarization, the corresponding metric tensor will be

$$g_{\mu'\nu'} = \begin{array}{cccc} (ct) & (x') & (y') & (z') \end{array}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & [1 + h_+^{\mathbf{TT}}(t, x')] & 0 \\ 0 & 0 & 0 & [1 - h_+^{\mathbf{TT}}(t, x')] \end{pmatrix}.$$

The observer wants to measure the energy which flows per unit time across the unit sur-

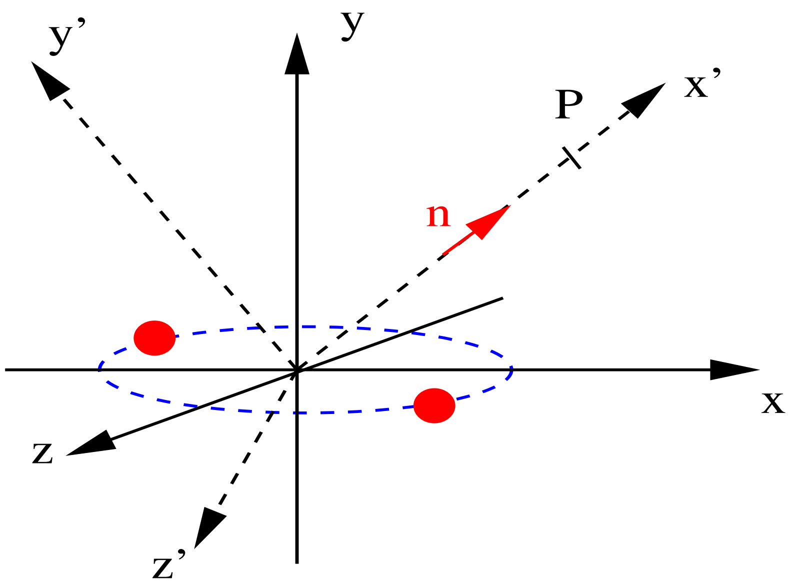


Figure 14.3: A binary system lies in the z-x plane. An observer located at P wants to detect the energy flux of gravitational waves emitted by the system.

face orthogonal to x', i.e. $t^{0x'}$, therefore he needs to compute the Christoffel symbols i.e. the derivatives of $h^{\mathbf{TT}}_{\mu'\nu'}$. According to eq. (14.40) the metric perturbation has the form $h^{\mathbf{TT}}(t, x') = \frac{const}{x'} \cdot \; f(t - \frac{x'}{c})$, and since the only derivatives which matter are those with respect to time and x'

$$\frac{\partial h^{\mathbf{TT}}}{\partial t} \equiv \dot{h}^{\mathbf{TT}} \; = \; \frac{const}{x'} \, \dot{f},$$

$$\frac{\partial h^{\mathbf{TT}}}{\partial x'} \equiv h^{\mathbf{TT}\,\prime} \; = \; -\frac{const}{x'^2} \, f + \frac{const}{x'} \, f' \sim -\frac{1}{c}\frac{const}{x'} \, \dot{f} = -\frac{1}{c}\dot{h}^{\mathbf{TT}},$$

where we have retained only the dominant $1/x'$ term. Thus, the non-vanishing Christoffel symbols are:

$$\Gamma^0{}_{y'y'} = -\Gamma^0{}_{z'z'} = \tfrac{1}{2}\,\dot{h}_+^{\mathbf{TT}} \qquad \Gamma^{y'}{}_{0y'} = -\Gamma^{z'}{}_{0z'} = \frac{1}{2}\,\dot{h}_+^{\mathbf{TT}} \tag{14.84}$$

$$\Gamma^{x'}{}_{y'y'} = -\Gamma^{x'}{}_{z'z'} = \tfrac{1}{2c}\,\dot{h}_+^{\mathbf{TT}} \qquad \Gamma^{y'}{}_{y'x'} = -\Gamma^{z'}{}_{z'x'} = -\frac{1}{2c}\,\dot{h}_+^{\mathbf{TT}}.$$

By substituting the Christoffel symbols in $t^{\mu\nu}$ we find

$$ct^{0x'} = \frac{dE_{GW}}{dtdS} = \frac{c^3}{16\pi G}\left[\left(\frac{dh^{\mathbf{TT}}(t,x')}{dt}\right)^2\right].$$

If both polarizations are present

$$g_{\mu'\nu'} = \begin{pmatrix} (ct) & (x') & (y') & (z') \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & [1+h_+^{\mathbf{TT}}(t,x')] & h_\times^{\mathbf{TT}}(t,x') \\ 0 & 0 & h_\times^{\mathbf{TT}}(t,x') & [1-h_+^{\mathbf{TT}}(t,x')] \end{pmatrix},$$

and

$$\begin{aligned} ct^{0x'} &= \frac{dE_{GW}}{dtdS} = \frac{c^3}{16\pi G}\left[\left(\frac{dh_+^{\mathbf{TT}}(t,x')}{dt}\right)^2 + \left(\frac{dh_\times^{\mathbf{TT}}(t,x')}{dt}\right)^2\right] \\ &= \frac{c^3}{32\pi G}\left[\sum_{jk}\left(\frac{dh_{jk}^{\mathbf{TT}}(t,x')}{dt}\right)^2\right]. \end{aligned} \tag{14.85}$$

This is the energy per unit time which flows across a unit surface orthogonal to the direction x'. However, the direction x' is arbitrary; if the observer is located in a different position and computes the energy flux he receives, he will find formally the same eq. (14.85) but with $h_{jk}^{\mathbf{TT}}$ referred to the **TT**-gauge associated with the new direction. Therefore, if we consider a generic direction $\mathbf{r} = r\mathbf{n}$

$$t^{0r} = \frac{c^2}{32\pi G}\left[\sum_{jk}\left(\frac{dh_{jk}^{\mathbf{TT}}(t,r)}{dt}\right)^2\right]. \tag{14.86}$$

In General Relativity the energy of the gravitational field cannot be defined locally, therefore to find the GW-flux we need to average over several wavelenghts, i.e.

$$\frac{dE_{GW}}{dtdS} = \left\langle ct^{0r}\right\rangle = \frac{c^3}{32\pi G}\left\langle\sum_{jk}\left(\frac{dh_{jk}^{\mathbf{TT}}(t,r)}{dt}\right)^2\right\rangle.$$

Since

$$\begin{cases} h^{\mathbf{TT}}{}_{\mu 0} = 0, & \mu = 0, 3 \\ h^{\mathbf{TT}}_{ik}(t, r) = \dfrac{2G}{c^4 r} \cdot \left[\dfrac{d^2}{dt^2} \, Q^{\mathbf{TT}}_{ik} \left(t - \dfrac{r}{c} \right) \right] \end{cases}$$

by direct substitution we find

$$\frac{dE_{GW}}{dtdS} = \frac{G}{8\pi c^5 \, r^2} \, \left\langle \sum_{jk} \, \left(\dddot{Q}^{\mathbf{TT}}_{jk} \left(t - \frac{r}{c} \right) \right)^2 \right\rangle . \tag{14.87}$$

As explained in section 14.39,

$$Q^{\mathbf{TT}}_{jk} \equiv \mathcal{P}_{jkmn} q_{mn}$$

is the quadrupole tensor projected onto the TT-gauge; moreover, we introduced the reduced quadrupole moment

$$Q_{jk} \equiv q_{jk} - \frac{1}{3} \delta_{jk} q_m{}^m \tag{14.88}$$

which is traceless by definition, and consequently

$$Q^{\mathbf{TT}}_{jk} = \mathcal{P}_{jkmn} q_{mn} = \mathcal{P}_{jkmn} Q_{mn} \, . \tag{14.89}$$

In order to obtain the gravitational luminosity of a source $L_{GW} = \frac{dE_{GW}}{dt}$, i.e. the gravitational energy emitted by the source per unit time, it is more convenient to use the reduced quadrupole moment, therefore we shall write Eq. (14.87) in terms of Q_{jk}, i.e.

$$\frac{dE_{GW}}{dtdS} = \frac{G}{8\pi c^5 \, r^2} \, \left\langle \sum_{jk} \, \left(\mathcal{P}_{jkmn} \dddot{Q}_{mn} \left(t - \frac{r}{c} \right) \right)^2 \right\rangle \, . \tag{14.90}$$

The gravitational luminosity therefore is

$$\begin{aligned} L_{GW} &= \int \frac{dE_{GW}}{dtdS} \, dS = \int \frac{dE_{GW}}{dtdS} \, r^2 d\Omega \qquad (14.91) \\ &= \frac{G}{2c^5} \frac{1}{4\pi} \int d\Omega \left\langle \sum_{jk} \, \left(\mathcal{P}_{jkmn} \dddot{Q}_{mn} \left(t - \frac{r}{c} \right) \right)^2 \right\rangle \, , \end{aligned}$$

where $d\Omega = (d\cos\theta) d\phi$ is the solid angle element. This integral can be computed by using the properties of $\mathcal{P}_{jkmn}$:

$$\begin{aligned} &\sum_{jk} \, \left(\mathcal{P}_{jkmn} \dddot{Q}_{mn} \right)^2 = \sum_{jk} \, \mathcal{P}_{jkmn} \dddot{Q}_{mn} \mathcal{P}_{jkrs} \dddot{Q}_{rs} = \qquad (14.92) \\ &= \left[\sum_{jk} \, \mathcal{P}_{mnjk} \mathcal{P}_{jkrs} \right] \dddot{Q}_{mn} \dddot{Q}_{rs} = \mathcal{P}_{mnrs} \, \dddot{Q}_{mn} \dddot{Q}_{rs} \\ &= \left[(\delta_{mr} - n_m n_r) \, (\delta_{ns} - n_n n_s) - \frac{1}{2} \, (\delta_{mn} - n_m n_n) \, (\delta_{rs} - n_r n_s) \right] \dddot{Q}_{mn} \dddot{Q}_{rs} . \end{aligned}$$

If we expand this expression, and remember that

- $\delta_{mn}\dddot{Q}_{mn} = \delta_{rs}\dddot{Q}_{rs} = 0$
  because the trace of Q_{ij} vanishes by definition, and
- $n_m n_r \delta_{ns}\dddot{Q}_{mn}\dddot{Q}_{rs} = n_n n_s \delta_{mr}\dddot{Q}_{mn}\dddot{Q}_{rs}$
  because Q_{ij} is symmetric,

at the end we find

$$\sum_{jk} \left(\mathcal{P}_{jkmn}\dddot{Q}_{mn}\right)^2 = \dddot{Q}_{rn}\dddot{Q}_{rn} - 2n_m\dddot{Q}_{ms}\dddot{Q}_{sr}n_r + \frac{1}{2}n_m n_n n_r n_s \dddot{Q}_{mn}\dddot{Q}_{rs} \,. \tag{14.93}$$

By substituting this expression in eq. (14.92) we find

$$L_{GW} = \frac{G}{2c^5}\frac{1}{4\pi}\left[\dddot{Q}_{rn}\dddot{Q}_{rn}\int d\Omega - 2\dddot{Q}_{ms}\dddot{Q}_{sr}\int n_m n_r d\Omega + \frac{1}{2}\dddot{Q}_{mn}\dddot{Q}_{rs}\int n_m n_n n_r n_s d\Omega\right] \,. \tag{14.94}$$

Thus, the integrals to be performed over the solid angle are:

$$\frac{1}{4\pi}\int n_m n_r d\Omega, \qquad \text{and} \qquad \frac{1}{4\pi}\int n_m n_n n_r n_s d\Omega. \tag{14.95}$$

Let us compute the first.
In polar coordinates, the versor $\mathbf{n}$ can be written as

$$n^i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \tag{14.96}$$

Thus, for parity reasons

$$\frac{1}{4\pi}\int d\Omega n_m n_r = 0 \quad \text{when } m \neq r. \tag{14.97}$$

Furthermore, since there is no preferred direction in the integration (isotropy), it must be

$$\int d\Omega\ n_1^2 = \int d\Omega\ n_2^2 = \int d\Omega\ n_3^2 \quad \rightarrow \quad \frac{1}{4\pi}\int d\Omega n_m n_r = const \cdot \delta_{mr} \,. \tag{14.98}$$

For instance,

$$\frac{1}{4\pi}\int d\Omega (n_3)^2 = \frac{1}{4\pi}\int d\cos\theta d\phi \cos^2\theta = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta \cos^2\theta = \frac{1}{3} \,, \tag{14.99}$$

and consequently

$$\frac{1}{4\pi}\int d\Omega n_m n_r = \frac{1}{3}\delta_{mr} \,. \tag{14.100}$$

The second integral in (14.95) can be computed in a similar way and gives

$$\frac{1}{4\pi}\int d\Omega n_m n_n n_r n_s = \frac{1}{15}\left(\delta_{mn}\delta_{rs} + \delta_{mr}\delta_{ns} + \delta_{ms}\delta_{nr}\right) . \tag{14.101}$$

By substituting Eqs. (14.100) and (14.101) in Eq. (14.94), we find

$$\begin{aligned} L_{GW} &= \frac{G}{2c^5}\left[\dddot{Q}_{rn}\dddot{Q}_{rn} - \frac{2}{3}\dddot{Q}_{ms}\dddot{Q}_{sr}\delta_{mr} + \frac{1}{30}\dddot{Q}_{mn}\dddot{Q}_{rs}\left(\delta_{mn}\delta_{rs} + \delta_{mr}\delta_{ns} + \delta_{ms}\delta_{nr}\right)\right] \\ &= \frac{G}{2c^5}\left[\dddot{Q}_{rn}\dddot{Q}_{rn} - \frac{2}{3}\dddot{Q}_{rs}\dddot{Q}_{sr} + \frac{1}{30}\left(\dddot{Q}_{mn}\delta_{mn}\dddot{Q}_{rs}\delta_{rs} + \dddot{Q}_{rn}\dddot{Q}_{rn} + \dddot{Q}_{sn}\dddot{Q}_{ns}\right)\right] \\ &= \frac{G}{2c^5}\dddot{Q}_{rn}\dddot{Q}_{rn}\left[1 - \frac{2}{3} + \frac{2}{30}\right] = \frac{G}{2c^5}\dddot{Q}_{rn}\dddot{Q}_{rn} \times \frac{2}{5} = \frac{G}{5c^5}\dddot{Q}_{rn}\dddot{Q}_{rn} \,, \end{aligned}$$

where we have used the property $Q_{mn}\delta_{mn} = Q_{rs}\delta_{rs} = 0$ due to the fact that the reduced quadrupole tensor is traceless. Finally, the gravitational wave luminosity is

$$L_{GW} = \frac{G}{5c^5} \left\langle \sum_{k,n=1}^{3} \dddot{Q}_{kn}\left(t - \frac{r}{c}\right) \dddot{Q}_{kn}\left(t - \frac{r}{c}\right) \right\rangle . \tag{14.102}$$

This formula was derived by A. Einstein in the paper *Über Gravitationswellen* published in 1918[2].

[2]The original article can be found on the website http://adsabs.harvard.edu/abs/1918SPAW.......154E