Chapter 20

The far field limit of an isolated, stationary object

In this chapter we will derive the metric which describes the gravitational field generated by an isolated, stationary object. Since the source is isolated, in the exterior $T_{\mu\nu} = 0$ and the spacetime is vacuum. Therefore, it is reasonable to assume that far away from the source the metric tends to Minkowski's metric, i.e. the metric must satisfy the *asymptotic flatness* condition.

If the spacetime is asymptotically flat, we can define, in an appropriate coordinate frame, a space coordinate r such that

$$\lim_{r\to\infty} g_{\mu\nu} = \eta_{\mu\nu} \,. \tag{20.1}$$

We call *far field limit* the region of spacetime where $r \gg R$, being R a lengthscale characteristic of the source. In the far field limit,

$$g_{\mu\nu} = \eta_{\mu\nu} + O\left(\frac{1}{r}\right) \,. \tag{20.2}$$

We also assume the metric is stationary, i.e. that it admits a timelike Killing vector so that (see chapter 9), by a suitable choice of coordinates, the metric can be made independent of time; this also implies that the source stress-energy tensor is independent of time.

We shall now show that the metric of a stationary axisymmetric source in the far field limit, up to terms of order $1/r$ in the expansion (20.2), is

$$\begin{aligned} ds^2 \;\; = \;\; & -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 + \frac{2M}{r}\right) dr^2 \\ & + r^2(d\theta^2 + \sin^2\theta d\phi^2) - \frac{4J}{r}\sin^2\theta dt d\phi \\ & + \text{higher order terms in } 1/r \,, \end{aligned} \tag{20.3}$$

where M is the source mass and J its angular momentum.

Let us write the expansion (20.2), which holds at large distance from the source, in a perturbative form

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{20.4}$$

with $|h_{\mu\nu}| \ll 1$. The perturbation $h_{\mu\nu}$ is a solution of the equations of the linearized Einstein equations in vacuum (see Chapter 13, eq. (13.28))

$$\Box_F \bar{h}_{\mu\nu} = 0 \tag{20.5}$$

$$\bar{h}^{\mu}{}_{\nu,\mu} = 0 \tag{20.6}$$

where we remind that $\Box_F$ is the d'Alambertian of the flat spacetime

$$\Box_F = \eta^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\frac{\partial}{\partial x^\beta} = -\frac{\partial^2}{c^2\partial t^2} + \nabla^2,$$

and

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{\alpha}{}_{\alpha}\,. \tag{20.7}$$

Since we are assuming the spacetime to be stationary, therefore (20.5), (20.6) become

$$\nabla^2 \bar{h}_{\mu\nu} = 0, \tag{20.8}$$

$$\bar{h}^{i}{}_{\nu,i} = 0\,. \tag{20.9}$$

We stress that eqs. (20.8) and (20.9) hold only in the far field limit $r \gg R$.

We shall now derive eq. (20.3) in the simple case when the gravitational field generated by the source is weak everywhere, i.e. also inside the source and on its boundary. We shall subsequently show that the metric (20.3) holds in the more general case when the field near the source is strong.

20.1 The weak field case.

If the gravitational field of the source is weak everywhere, as shown in Chapter 12.4.2, Einstein's equations for the metric perturbation $\bar{h}_{\mu\nu}$ inside the source become

$$\begin{aligned} \Box_F \bar{h}_{\mu\nu} &= -16\pi T_{\mu\nu} \\ \bar{h}^{\mu}{}_{\nu,\mu} &= 0\,, \end{aligned} \tag{20.10}$$

and the general solution is (13.29)

$$\bar{h}_{\mu\nu}(t,\mathbf{x}) = 4\int_V \frac{T_{\mu\nu}(t - |\mathbf{x}\text{-}\mathbf{x}'|, \mathbf{x}'\ )}{|\mathbf{x}\text{-}\mathbf{x}'\ |}\ d^3x' \tag{20.11}$$

where V is the source three-volume. On the source boundary, ∂V, by definition the stress-energy tensor vanishes, $T_{\mu\nu} = 0$.

In Chapters 12.4.2 and 14 we were interested in the time-dependent part of the solution (20.11), since we were interested in gravitational waves. Here, instead, we are considering a stationary source, for which $T_{\mu\nu} = T_{\mu\nu}(\mathbf{x}')$, therefore the solution (20.11) becomes

$$\bar{h}_{\mu\nu}(\mathbf{x}) = 4\int_V \frac{T_{\mu\nu}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}d^3x'\,. \tag{20.12}$$

In the following space indices 1, 2, 3 will be denoted by latin letters i, j. As in Chapter 12.4.2 the indices if $h_{\mu\nu}$ will be raised by using Minkowski's metric, thus $h_{i\mu} = h^i{}_\mu$.

Let us consider a reference frame centered on the source center of mass. Be $\mathbf{x}$ a position vector pointing far away from the source, and $\mathbf{x}'$ the position vector of a generic source point; be $|\mathbf{x}| \gg |\mathbf{x}'|$. If we define $r \equiv |\mathbf{x}|$, and Taylor expand the quantity $1/|\mathbf{x}-\mathbf{x}'|$ (this expansion is commonly named *multipolar expansion*) we find

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|} = \frac{1}{r} + \frac{\sum_i x^i x'^i}{r^3} + O\left(\frac{1}{r^3}\right) \qquad i = 1, 3. \tag{20.13}$$

Substituting in (20.12) we find

$$\bar{h}_{\mu\nu}(\mathbf{x}) = \frac{4}{r}\int_V T_{\mu\nu} d^3x' + \frac{4\sum_i x^i}{r^3}\int_V T_{\mu\nu} x'^i d^3x' \,. \tag{20.14}$$

Let us evaluate the 00-component of $\bar{h}_{\mu\nu}$ In the weak field limit $T_{00} \sim \rho c^2$ is the source mass-density, therefore the first integral in eq. (20.14) gives

$$\int_V T_{00} d^3x' = M\,. \tag{20.15}$$

The 00 component of the second integral in eq. (20.14) gives the position of the source center of mass, which coincides with the origin of the coordinates frame; thus

$$\int_V T_{00} x'^i d^3x' = M x'^i_{cdm} = 0\,. \tag{20.16}$$

From eqs. (20.14), (20.15) and (20.16) we find

$$\bar{h}_{00} = \frac{4M}{r} + O\left(\frac{1}{r^3}\right)\,. \tag{20.17}$$

We shall now compute the μi components of $\bar{h}_{\mu\nu}$. To compute the first integral we shall use the divergenceless equation satisfied by $T_{\mu\nu}$ which, for a stationary source, becomes

$$T^{\mu\nu}{}_{,\nu} = T^{\mu 0}{}_{,0} + T^{\mu i}{}_{,i} = T^{\mu i}{}_{,i} = 0 \qquad \mu = 0, 3, i = 1, 3. \tag{20.18}$$

Using (20.18), and the property $\frac{\partial x^i}{\partial x^j} = \delta^i_j$, we find

$$\int_V T^{\mu i} d^3x' = \int_V T^{\mu k}\delta^i_k d^3x' = \int_V T^{\mu k}\frac{\partial x'^i}{\partial x'^k} d^3x' = -\int_V \left(\frac{\partial T^{\mu k}}{\partial x'^k}\right) x'^i d^3x' = 0 \tag{20.19}$$

where we have integrated by parts. Remind: the surface terms do not contribute, because on the boundary of V, $T_{\mu\nu} = 0$. Thus,

$$\int_V T^{\mu i} d^3x' = 0\,. \tag{20.20}$$

We shall compute the second integral using the following property:

$$\begin{aligned}\int_V \left(T^{\mu i}x'^j + T^{\mu j}x'^i\right) d^3x' &= \int_V T^{\mu k}\left(\frac{\partial x'^i}{\partial x'^k}x'^j + \frac{\partial x'^j}{\partial x'^k}x'^i\right) d^3x' \\ &= \int_V T^{\mu k}\frac{\partial}{\partial x'^k}\left(x'^i x'^j\right) d^3x' = -\int_V x'^i x'^j T^{\mu k}{}_{,k} d^3x' = 0;\end{aligned} \tag{20.21}$$

thus

$$\int_V \left(T^{\mu i}x'^j + T^{\mu j}x'^i\right) d^3x' = 0,$$

which implies that the second integral in eq. (20.14) is antisymmetric in the last two indices:

$$\int_V T^{\mu i}x'^j d^3x' = -\int_V T^{\mu j}x'^i d^3x' \,. \tag{20.22}$$

Let us consider now the space components of the second integral in eq. (20.14)

$$\int_V T^{jk}x'^i d^3x' \,. \tag{20.23}$$

this expression is symmetric in the first two indices, and antisymmetric in the last two indices because of eq. (20.22); consequently

$$\begin{aligned} \int_V T^{ki}x'^j d^3x' &= -\int_V T^{kj}x'^i d^3x' = -\int_V T^{jk}x'^i d^3x' \\ &= \int_V T^{ji}x'^k d^3x' = \int_V T^{ij}x'^k d^3x' = -\int_V T^{ik}x'^j d^3x' \,, \end{aligned} \tag{20.24}$$

i.e.

$$\int_V T^{ki}x'^j d^3x' = -\int_V T^{ik}x'^j d^3x'.$$

The only possibility for this equality to be satisfied is that

$$\int_V T^{ki}x'^j d^3x' = 0 \,. \tag{20.25}$$

Consequently, from eqs. (20.14), (20.20) and (20.25) we find

$$\bar{h}_{ik} = O\left(\frac{1}{r^3}\right) \,. \tag{20.26}$$

The last metric components we need to compute are $\bar{h}_{0i}$

$$\bar{h}_{0i}(\mathbf{x}) = \frac{4}{r}\int_V T_{0i} d^3x' + \frac{4\sum_j x^j}{r^3}\int_V T_{0i}x'^j d^3x' \,. \tag{20.27}$$

From eq. (20.20) we know that the first term is zero, whereas eq. (20.22) implies that

$$\int_V T^{0i}x'^j d^3x' = -\int_V T^{0j}x'^i d^3x' \,. \tag{20.28}$$

We shall now show that this integral is related to the source angular momentum. The components T^{0i} are the density of the i-th component of momentum of the source

$$T^{0i} = \mathcal{P}^i; \tag{20.29}$$

a matter element in the volume d^3x', at a distance $\mathbf{x}'$ from the origin, has momentum $\mathcal{P}d^3x'$. According to Newtonian theory, its angular momentum is $\mathbf{dJ} = \mathbf{x}' \times \mathcal{P}d^3x'$, where $\times$ indicates the vector product. Thus the source angular momentum is

$$\mathbf{J} = \int \mathbf{x}' \times \mathcal{P}d^3x' \,. \tag{20.30}$$

The components of $\mathbf{J}$ can be written as follows

$$J^i = -\int_V \epsilon_{ijk} T^{0j} x'^k d^3x', \tag{20.31}$$

where ϵ_{ijk} is the 3-D Levi-Civita tensor density we introduced in section 10.7. We remind that it is completely antisymmetric, since its components change sign under interchange of any pair of indices. Since it is completely antisymmetric, the components with two equal indices are zero, and the only non-vanishing components are those for which the three indices are different. Moreover

$$e_{123} = 1. \tag{20.32}$$

Equation (20.31) can be written as

$$\int_V T^{0i} x'^j d^3x' = -\frac{1}{2}\epsilon_{ijk} J^k \tag{20.33}$$

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PROOF

Be $B^{ij} = -B^{ji}$ an antisymmetric tensor and

$$A^k = \epsilon_{klm} B^{lm}. \tag{20.34}$$

Let us multiply bot members by $\frac{1}{2}\epsilon_{ijk}$

$$\frac{1}{2}\epsilon_{ijk} A^k = \frac{1}{2}\epsilon_{ijk}\epsilon_{klm} B^{lm}; \tag{20.35}$$

the following equality is easy to prove [1]

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{20.36}$$

Consequently,

$$\frac{1}{2}\epsilon_{ijk} A^k = \frac{1}{2}\epsilon_{ijk}\epsilon_{klm} B^{lm} = \frac{1}{2}\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) B^{lm} = B^{ij} \tag{20.37}$$

Using this property, eq. (20.33) follows immediately.

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Thus, using eq. (20.33) the terms in the sum appearing in $\bar{h}_{0i}$ (see eq. (20.27) can be written as

$$\frac{4}{r^3} x^j \int_V T_{0i} x'^j d^3x' = -\frac{4}{r^3} x^j \int_V T^{0i} x'^j d^3x' = \frac{2}{r^3}\epsilon_{ijk} x^j J^k . \tag{20.38}$$

[1] $\epsilon_{ijk} \neq 0$ only if its three indices are all different, thus $\imath \neq k$ and $j \neq k$; similarly for ϵ_{lmk}. Therefore $\epsilon_{ijk}\epsilon_{lmk} \neq 0$ only if the indices ij and lm are the same. If they have the same order, i.e. $ij = lm$, then $\epsilon_{ijk}\epsilon_{lmk} = 1$; if they have the opposite order, i.e. $ij = ml$, then $\epsilon_{ijk}\epsilon_{lmk} = -1$. Consequently, $\epsilon_{ijk}\epsilon_{lmk} = \delta^{il}\delta^{jm} - \delta^{im}\delta^{jl}$.

From eqs. (20.27), (20.20) and (20.38) we find

$$\bar{h}_{0i} = \frac{2}{r^3}\epsilon_{ijk}x^j J^k + O\left(\frac{1}{r^3}\right) \tag{20.39}$$

In summary, the multipolar expansion (20.14) gives

$$\begin{aligned} \bar{h}_{00} &= \frac{4M}{r} + O\left(\frac{1}{r^3}\right) \\ \bar{h}_{0i} &= \frac{2}{r^3}\epsilon_{ijk}x^j J^k + O\left(\frac{1}{r^3}\right) \\ \bar{h}_{ij} &= O\left(\frac{1}{r^3}\right). \end{aligned} \tag{20.40}$$

In terms of $h_{\mu\nu}$ [2]

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\bar{h}^{\alpha}{}_{\alpha}\,, \tag{20.41}$$

we find[3]

$$\begin{aligned} h_{00} &= \frac{2M}{r} + O\left(\frac{1}{r^3}\right) \\ h_{0i} &= \frac{2}{r^3}\epsilon_{ijk}x^j J^k + O\left(\frac{1}{r^3}\right) \\ h_{ij} &= \frac{2M}{r}\delta_{ij} + O\left(\frac{1}{r^3}\right). \end{aligned} \tag{20.42}$$

20.1.1 The far field limit metric in polar coordinates

Let us transform the solution (20.42) in polar coordinates

$$\begin{aligned} x^1 &= r\sin\theta\cos\phi \\ x^2 &= r\sin\theta\sin\phi \\ x^3 &= r\cos\theta\,. \end{aligned} \tag{20.43}$$

Since

$$\sum_i (dx^i)^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\,, \tag{20.44}$$

then

$$h_{ij}dx^i dx^j = \frac{2M}{r}\left(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right). \tag{20.45}$$

The transformation of $h_{0i}dx^0 dx^i$ is less trivial. If we choose the frame orientation such that the angular momentum is directed along the z axis, i.e.

$$\mathbf{J} = (0, 0, J)\,, \tag{20.46}$$

[2]To invert (20.7) we first take the trace of (20.7), finding $\bar{h}^{\lambda}{}_{\lambda} = -h^{\lambda}{}_{\lambda}$, then substitute into (20.7).

[3]Notice that $\frac{x^j}{r^3}$ is an $O\left(\frac{1}{r^2}\right)$ term, because in the far field limit $x^j \sim r$.

then

$$\begin{aligned} h_{0i}dx^0dx^i &= \left(\frac{2}{r^3}dx^0\right)\epsilon_{ijk}x^jJ^kdx^i = -\left(\frac{2}{r^3}dx^0\right)J(x^1dx^2 - x^2dx^1) \\ &= -\left(\frac{2}{r^3}dx^0\right)Jr^2\sin^2\theta d\phi = -\frac{2J}{r}\sin^2\theta dtd\phi, \end{aligned} \tag{20.47}$$

where the equality $x^1dx^2 - x^2dx^1 = r^2\sin^2\theta$ can be found by differentiating eq. (20.43).

In conclusion, the line element is

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{r} + O\left(\frac{1}{r^3}\right)\right)dt^2 \\ &+\left(1 + \frac{2M}{r} + O\left(\frac{1}{r^3}\right)\right)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \\ &+\left(-\frac{4J}{r}\sin^2\theta + O\left(\frac{1}{r^3}\right)\right)dtd\phi\,. \end{aligned} \tag{20.48}$$

This is the solution of the linearized Einstein equations in the weak field limit. If we consider the full, non linear Einstein equations, we have terms of order $O(|h_{\mu\nu}|^2)$, which produce terms of order $\sim M^2/r^2$, $\sim J^2/r^2$ and, due to the non linearity, also to higher order terms. Therefore, with respect to the fully non linear solution, our expansion does not include terms of order $O(1/r^2)$, i.e. it is more correct to write

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{r} + O\left(\frac{1}{r^2}\right)\right)dt^2 \\ &+\left(1 + \frac{2M}{r} + O\left(\frac{1}{r^2}\right)\right)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \\ &+\left(-\frac{4J}{r} + O\left(\frac{1}{r^2}\right)\right)\sin^2\theta dtd\phi\,. \end{aligned} \tag{20.49}$$

Finally, we redefine the radial coordinate as follows:

$$r \to r - M\,. \tag{20.50}$$

Neglecting contributions of order $O(1/r^2)$, the only term which produces a change in the metric is

$$\begin{aligned} &\left(1 + \frac{2M}{r}\right)r^2(d\theta^2 + \sin^2\theta d\phi^2) \to \left(1 + \frac{2M}{r}\right)(r - M)^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &= r^2\left(1 + O\left(\frac{1}{r}\right)\right)(d\theta^2 + \sin^2\theta d\phi^2)\,. \end{aligned} \tag{20.51}$$

With this coordinate definition, we finally reduce the metric (20.49) to the following form:

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 + \frac{2M}{r}\right)dr^2 \\ &+r^2(d\theta^2 + \sin^2\theta d\phi^2) - \frac{4J}{r}\sin^2\theta dtd\phi \\ &+ \text{higher order terms in } 1/r\,. \end{aligned} \tag{20.52}$$

which coincides with eq. (20.3).

20.2 The strong field case

In this section we shall drop the weak field assumption, and we shall assume that near and inside the source the field can be strong. However far away, where we want to find the solution of Einstein's equations, the field is still weak weak, the metric can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{20.53}$$

and we shall neglect terms of order $O(|h_{\mu\nu}|^2)$, and terms that decay with powers larger than of $1/r$, where r is the distance from the source. We seek a solution of the form

$$\bar{h}_{\mu\nu} = \frac{a_{\mu\nu}(\theta,\phi)}{r} + \frac{b_{\mu\nu}(\theta,\phi)}{r^2} + O\left(\frac{1}{r^3}\right) . \tag{20.54}$$

The coefficients $a_{\mu\nu}$, $b_{\mu\nu}$ depend only on the angular variables θ, ϕ, so that they remain finite for $r \to \infty$. The metric perturbation, which we assume to be stationary, satisfies equation (20.8),

$$\nabla^2 \bar{h}_{\mu\nu} = 0 . \tag{20.55}$$

The Laplace operator in spherical coordinates has the form[4]

$$\nabla^2 = \frac{1}{r^2}\partial_r r^2 \partial_r + \frac{\mathbb{L}}{r^2} \tag{20.56}$$

where $\mathbb{L}$ is an operator acting on the angular variables:

$$\mathbb{L} \equiv \partial_\theta^2 + \cot\theta\partial_\theta + \sin^{-2}\theta\partial_\phi^2 . \tag{20.57}$$

By substituting eq. (20.54) in (20.56) we easily find

$$\mathbb{L}a_{\mu\nu}(\theta,\phi) = 0 \tag{20.58}$$

$$\mathbb{L}b_{\mu\nu}(\theta,\phi) = -2b_{\mu\nu}(\theta,\phi) . \tag{20.59}$$

The eigenfunctions of the operator $\mathbb{L}$ are the *spherical harmonics* $Y_{lm}(\theta,\phi)$, with $l = 0, 1, \dots$ and $m = -l, -l+1, \dots, l-1, l$. They are defined by the property

$$\mathbb{L}Y_{lm} = -l(l+1)Y_{lm} . \tag{20.60}$$

Equation (20.59) tells us that $b_{\mu\nu}$ is a linear combination of the spherical harmonics with $l = 1$, which are

$$\begin{aligned} Y_{11} &= -\sqrt{\frac{3}{8\pi}}\sin\theta e^{\mathrm{i}\phi} \\ Y_{11} &= \sqrt{\frac{3}{4\pi}}\cos\theta \\ Y_{1-1} &= \sqrt{\frac{3}{8\pi}}\sin\theta e^{-\mathrm{i}\phi} . \end{aligned} \tag{20.61}$$

[4]The theory of Laplace equation and the properties of spherical harmonics are extensively discussed in the literature. See for instance Jackson's book *Electromagnetism*, Chapter 3.

This is equivalent to say that $b_{\mu\nu}$ is a linear combination of the direction cosines $n^i = x^i/r$, because

$$\begin{aligned} n^1 = \frac{x^1}{r} &= \sin\theta\cos\phi = \sqrt{\frac{8\pi}{3}}\frac{-Y_{11}+Y_{1-1}}{2} \\ n^2 = \frac{x^2}{r} &= \sin\theta\sin\phi = \sqrt{\frac{8\pi}{3}}\frac{-Y_{11}-Y_{1-1}}{2\mathrm{i}} \\ n^3 = \frac{x^3}{r} &= \cos\theta = \sqrt{\frac{4\pi}{3}}Y_{10}\,. \end{aligned} \tag{20.62}$$

Therefore, while $a^{\mu\nu}$ does not depend on the angular variables n^i, $b^{\mu\nu}$ can be written as a linear combination of the n^i's

$$b^{\mu\nu}(n^i) = b^{\mu\nu}_i n^i\,. \tag{20.63}$$

Consequently, the expansion (20.54) can be written as

$$\bar{h}^{\mu\nu} = \frac{a^{\mu\nu}}{r} + \frac{b^{\mu\nu}_i x^i}{r^3} + O\left(\frac{1}{r^3}\right)\,, \tag{20.64}$$

with $a^{\mu\nu}$, $b^{\mu\nu}_i$ constant coefficients.

We now impose on (20.64) the gauge condition (20.6)

$$\bar{h}^{\mu\nu}_{\ \ ,\nu} = 0 \tag{20.65}$$

which, in the case of stationary perturbations, becomes

$$\bar{h}^{\mu i}_{\ \ ,i} = 0\,. \tag{20.66}$$

We get (remember that in linearized gravity it is irrelevant if a space index i is up or down)

$$\bar{h}_{\mu j,j} = -\frac{a_{\mu j}x^j}{r^3} + \frac{b_{\mu ji}\left(\delta^{ij}r^2 - 3x^i x^j\right)}{r^5} = 0 \tag{20.67}$$

which has to be satisfied for all (large) values of r and for all values of $n^i = x^i/r$. Eq. (20.67) is satisfied only if the coefficients of different powers of r vanish, i.e.

$$\begin{aligned} a_{\mu j} &= 0 \\ \left(\delta^{ij} - 3n^i n^j\right) b_{\mu ij} &= 0\,. \end{aligned} \tag{20.68}$$

These equations do not involve a_{00}, b_{00i}, which are in general nonvanishing, free constants; to simplify the notation, we rewrite them as

$$\begin{aligned} a &\equiv a_{00} \\ b_i &\equiv b_{00i}\,. \end{aligned} \tag{20.69}$$

The first of eqs. (20.68) says that all the constant $a_{\mu\nu}$ different from a_{00} vanish. The second equation can be rewritten as

$$H^{ij}b_{0ij} = 0 \tag{20.70}$$

$$H^{ij}b_{kij} = 0 \tag{20.71}$$

where we have defined

$$H^{ij} \equiv \delta^{ij} - 3n^i n^j \,. \qquad (20.72)$$

The general solution of (20.70), (20.71) is

$$\begin{aligned} b_{0ij} &= b\delta_{ij} + c_k \epsilon_{ijk} && (20.73) \\ b_{kij} &= d_k \delta_{ij} + d_i \delta_{kj} - d_j \delta_{ki} && (20.74) \end{aligned}$$

where b, c_k, d_k are constants. A rigorous proof of (20.73), (20.74) would require the use of the structures of Group Theory, which goes beyond the scope of this book. Here we will only give an intuitive, non-rigorous proof of the first solution, (20.73).

Equation (20.70) must be satisfied for any value of n^i, i.e. for any value of the angular variables θ, ϕ. While H^{ij} depends on the angles, $b_{\mu ij}$ *cannot* depend on the angles, therefore equation (20.70) can be satisfied only because of the symmetry properties of H^{ij}, which is symmetric and traceless

$$H^{ij} = H^{ji} \qquad \delta_{ij} H^{ij} = 0 \,. \qquad (20.75)$$

All quantities considered here are tensors in the euclidean three-dimensional space. The only constant tensors which vanish when contracted with H^{ij} are the Kronecker delta, δ_{ij}, and the completely antisymmetric tensor, ϵ_{ijk}: the former vanishes because H^{ij} is traceless, the latter because H^{ij} is symmetric. Thus b_{0ij} must be a combination of these tensors, as shown in eq. (20.73).

Summarizing, by imposing the gauge condition (20.6) on the expansion (20.64) we get

$$\begin{aligned} \bar{h}_{00} &= \frac{a}{r} + \frac{b_i x^i}{r^3} + O\left(\frac{1}{r^3}\right) \\ \bar{h}_{0i} &= \frac{b x^i}{r^3} + \epsilon_{ijk} \frac{x^j c_k}{r^3} + O\left(\frac{1}{r^3}\right) \\ \bar{h}_{ij} &= \frac{1}{r^3}\left(-\delta_{ij} d_k x^k + d_i x^j + d_j x^i\right) + O\left(\frac{1}{r^3}\right); && (20.76) \end{aligned}$$

this solution depends on the constants a, b_i, b, c_k, d_k.

The constants b_i, d_k can be eliminated by a (position dependent) infinitesimal diffeomorphism $x^\mu \to x^\mu + \xi^\mu$ with parameter

$$\xi^\mu = \left(-\frac{b}{r}, -\frac{d^i}{r}\right) \qquad (20.77)$$

(it is infinitesimal in the sense that r is large and $\xi \sim 1/r$).

The change in the metric is (see Chapter **??**)

$$\begin{aligned} g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \;\; &\to \;\; g_{\mu\nu} + g_{\mu\alpha}\xi^\alpha{}_{,\nu} + g_{\nu\alpha}\xi^\alpha{}_{,\mu} + g_{\mu\nu}\xi^\alpha{}_{,\alpha} \\ &= \;\; \eta_{\mu\nu} + h_{\mu\nu} + g_{\mu\alpha}\xi^\alpha{}_{,\nu} + g_{\nu\alpha}\xi^\alpha{}_{,\mu} + g_{\mu\nu}\xi^\alpha{}_{,\alpha}, \end{aligned} \qquad (20.78)$$

therefore, there is a change in the perturbation $h_{\mu\nu}$ given by

$$\delta h_{\mu\nu} = g_{\mu\alpha}\xi^\alpha{}_{,\nu} + g_{\nu\alpha}\xi^\alpha{}_{,\mu} + g_{\mu\nu}\xi^\alpha{}_{,\alpha}. \qquad (20.79)$$

Since $\xi^\mu = O(|h_{\mu\nu}|)$, by neglecting terms quadratic in $h_{\mu\nu}$ we find

$$\delta h_{\mu\nu} = \eta_{\mu\alpha}\xi^\alpha{}_{,\nu} + \eta_{\nu\alpha}\xi^\alpha{}_{,\mu} \,. \tag{20.80}$$

Changing to $\bar{h}_{\mu\nu}$ we find

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}, h_{\alpha\beta} \tag{20.81}$$

thus

$$\begin{aligned} \delta\bar{h}_{\mu\nu} &= \delta h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\delta h_{\alpha\beta} \\ &= \eta_{\mu\alpha}\xi^\alpha{}_{,\nu} + \eta_{\nu\alpha}\xi^\alpha{}_{,\mu} - \eta_{\mu\nu}\xi^\alpha{}_{,\alpha} \,. \end{aligned} \tag{20.82}$$

Since

$$\begin{aligned} \xi^\mu{}_{,0} &= 0 \\ \xi^0{}_{,i} &= \frac{bx^i}{r^3} \\ \xi^k{}_{,i} &= \frac{d^k x^i}{r^3}, \end{aligned} \tag{20.83}$$

then

$$\begin{aligned} \delta\bar{h}_{00} &= -\eta_{00}\xi^k{}_{,k} + O\left(\frac{1}{r^3}\right) = \frac{d^k x^k}{r^3} + O\left(\frac{1}{r^3}\right) \\ \delta\bar{h}_{0i} &= \eta_{00}\xi^0_{,i} + O\left(\frac{1}{r^3}\right) = -\frac{bx^i}{r^3} + O\left(\frac{1}{r^3}\right) \\ \delta\bar{h}_{ij} &= \eta_{ik}\xi^k{}_{,j} + \eta_{jk}\xi^k{}_{,i} - \eta_{ij}\xi^k{}_{,k} = \frac{1}{r^3}\left[d^i x^j + d^j x^j - \eta_{ij} d^k x^k\right]; \end{aligned} \tag{20.84}$$

thus, after the diffeomorphism,

$$\begin{aligned} \bar{h}_{00} &= \frac{a}{r} + \frac{\tilde{b}_i x^i}{r^3} + O\left(\frac{1}{r^3}\right) \\ \bar{h}_{0i} &= \epsilon_{ijk}\frac{x^j c_k}{r^3} + O\left(\frac{1}{r^3}\right) \\ \bar{h}_{ij} &= O\left(\frac{1}{r^3}\right), \end{aligned} \tag{20.85}$$

where we have defined

$$\tilde{b}_i \equiv b_i + d_i \,. \tag{20.86}$$

Furthermore, we can get rid of $\tilde{b}_i$ by performing a (rigid) translation

$$x^i \to x^i + \frac{\tilde{b}_i}{a} \tag{20.87}$$

which produces the following change in the a/r term:

$$\begin{aligned} \frac{a}{r} = a\left((x^i)^2\right)^{-1/2} \quad &\to \quad a\left(\left(x^i + \frac{\tilde{b}_i}{a}\right)^2\right)^{-1/2} \\ &= \quad a\left(r^2\left(1 + 2\frac{\tilde{b}_i x^i}{r^2 a}\right)\right)^{-1/2} + O\left(\frac{1}{r^3}\right) \\ &= \quad \frac{a}{r}\left(1 - \frac{\tilde{b}_i x^i}{r^2 a}\right) + O\left(\frac{1}{r^3}\right) = \frac{a}{r} - \frac{\tilde{b}_i x^i}{r^3} + O\left(\frac{1}{r^3}\right) . \end{aligned} \tag{20.88}$$

Therefore,

$$\begin{aligned} \bar{h}_{00} &= \frac{a}{r} + O\left(\frac{1}{r^3}\right) \\ \bar{h}_{0i} &= \epsilon_{ijk}\frac{x^j c_k}{r^3} + O\left(\frac{1}{r^3}\right) \\ \bar{h}_{ij} &= O\left(\frac{1}{r^3}\right) . \end{aligned} \tag{20.89}$$

Finally, we compute

$$h_{\mu\nu} = \bar{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\bar{h}_{\alpha\beta} \tag{20.90}$$

(which follows from the definition (20.7) because $\eta^{\mu\nu}h_{\mu\nu} = -\eta^{\mu\nu}\bar{h}_{\mu\nu}$, as can easily be seen by taking the trace of (20.7)). We have

$$\frac{1}{2}\eta^{\alpha\beta}\bar{h}_{\alpha\beta} = -\frac{a}{2r} \tag{20.91}$$

therefore

$$\begin{aligned} h_{00} &= \frac{a}{2r} + O\left(\frac{1}{r^3}\right) \\ h_{0i} &= \epsilon_{ijk}\frac{x^j c^k}{r^3} + O\left(\frac{1}{r^3}\right) \\ h_{ij} &= \delta_{ij}\frac{a}{2r} + O\left(\frac{1}{r^3}\right) . \end{aligned} \tag{20.92}$$

With the identifications

$$a = 4M \qquad c^k = 2J^k \tag{20.93}$$

the solution (20.92) coincides with the solution (20.42), which we derived in the case of a weak field source, and that we have already shown to coincide with the solution (20.3).

20.3 Mass and angular momentum of an isolated object

As we have seen, the metric

$$\begin{aligned} ds^2 &= -\left(1-\frac{2M}{r}\right)dt^2 + \left(1+\frac{2M}{r}\right)dr^2 \\ &+r^2(d\theta^2+\sin^2\theta d\phi^2) - \frac{4J}{r}\sin^2\theta dt d\phi \\ &+ \text{higher order terms in } 1/r\,. \end{aligned} \tag{20.94}$$

describes the far field limit of an isolated, stationary source. If the source is weakly gravitating, we have seen that, according to Newtonian physics, M and J have a simple interpretation: they are, respectively, the mass and the angular momentum of the source.

If the source is *not* weakly gravitating M and J arise as integration constants of the general far field solution (20.94), and their physical interpretation needs to be found.

One possibilty is through the study of the motion of test bodies in the metric (20.94). If the test body is far away from a strongly gravitating source characterized by the two constanst, say $\bar{M}$ and $\bar{J}$, its motion cannot be distinguished from that it would have if moving around a *weakly gravitating* source with mass $M = \bar{M}$ and angular momentum $J = \bar{J}$. Thus, an *operational definition* of the mass and angular momentum of the strongly gravitating source can be given by studying geodesic motion far away from the source. The mass will be measured from the orbital frequency of the test mass through Kepler's third law, and the angular momentum by measuring the precession of gyroscopes orbiting around the source.

A different answer is based on the stress-energy pseudotensor, which we have defined in Capter 14. We remind that the stress-energy pseudotensor $t^{\mu\nu}$ describes the energy and momentum carried by the gravitational field, and satisfies, together with the stress-energy tensor $T_{\mu\nu}$, a conservation law:

$$[(-g)(T^{\mu\nu}+t^{\mu\nu})]_{,\nu} = 0\,. \tag{20.95}$$

It can be expressed as a divergence:

$$(-g)(T^{\mu\nu}+t^{\mu\nu}) = \frac{\partial\zeta^{\mu\nu\alpha}}{\partial x^\alpha} \tag{20.96}$$

where

$$\zeta^{\mu\nu\alpha} = \frac{1}{16\pi}\frac{\partial}{\partial x^\beta}\left[(-g)(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta})\right]\,. \tag{20.97}$$

Since we are considering a stationary spacetime, eq. (20.96) becomes

$$(-g)(T^{\mu\nu}+t^{\mu\nu}) = \frac{\partial\zeta^{\mu\nu k}}{\partial x^k}, \qquad k=1,3. \tag{20.98}$$

Let us now consider a spherical three-dimensional volume V centered on the source, with radius r much larger than the source size. WARNING: V is not the source volume, it is much larger!

The total four-momentum P^μ, enclosed in the volume V, is due **both to the source and to the gravitational field**:

$$P^\mu = \int_V d^3x(-g)(T^{0\mu} + t^{0\mu}) \,. \tag{20.99}$$

Substituting (20.98) in (20.99) we find

$$P^\mu = \int_V d^3x \frac{\partial \zeta^{0\mu k}}{\partial x^k} \,. \tag{20.100}$$

By Gauss' theorem, we write this integral as an integral over the spherical surface S surrounding the volume V:

$$P^\mu = \int_S \zeta^{0\mu k} dS_k \,. \tag{20.101}$$

Thus, for instance, the total mass-energy of the system is

$$\boxed{M_{tot} = P^0 = \int_S \zeta^{00k} dS_k} \,. \tag{20.102}$$

As explained in Section 20.1, the three-momentum of the matter element of volume d^3x, located at a point of coordinates x^i, is

$$\mathcal{P}^i d^3x \,, \tag{20.103}$$

and the angular momentum of the matter element is

$$dJ^i = (\mathbf{x} \times \mathcal{P})^i \, d^3x = -\epsilon_{ijk} \mathcal{P}^j x^k d^3x \,. \tag{20.104}$$

Therefore, the total angular momentum which generalizes eq. (20.33) and includes the contribution of the gravitational field, is

$$J^i = -\epsilon_{ijk} \int_V d^3x(-g)(T^{0j} + t^{0j}) x^k \,. \tag{20.105}$$

Using eq. (20.96), eq. (20.105) gives

$$\begin{aligned} J^i &= -\epsilon_{ijk} \int_V d^3x \frac{\partial \zeta^{0jl}}{\partial x^l} x^k = -\epsilon_{ijk} \int_V d^3x \left[\frac{\partial(\zeta^{0jl} x^k)}{\partial x^l} - \zeta^{0jl} \frac{\partial x^k}{\partial x^l} \right] \\ &= -\epsilon_{ijk} \int_V d^3x \left[\frac{\partial(\zeta^{0jl} x^k)}{\partial x^l} - \zeta^{0jk} \right] . \end{aligned} \tag{20.106}$$

We now introduce the quantity $\lambda^{\mu\nu\alpha\beta}$ defined as (see eq. (20.97))

$$\lambda^{\mu\nu\alpha\beta} \equiv \frac{1}{16\pi} \left[(-g)(g^{\mu\nu} g^{\alpha\beta} - g^{\mu\alpha} g^{\nu\beta}) \right] , \tag{20.107}$$

related to $\zeta^{\mu\nu\alpha}$ by the following equation

$$\zeta^{\mu\nu\alpha} = \frac{\partial \lambda^{\mu\nu\alpha\beta}}{\partial x^\beta}, \tag{20.108}$$

which, for a stationary spacetime becomes

$$\zeta^{\mu\nu\alpha} = \frac{\partial \lambda^{\mu\nu\alpha i}}{\partial x^i} \,. \tag{20.109}$$

By replacing the quantity ζ^{0jk} in terms of λ^{0jkl} as given by eq. (20.109), eq. (20.106) becomes

$$\begin{aligned} J^i &= -\epsilon_{ijk} \int_V d^3x \frac{\partial}{\partial x^l} \left(\zeta^{0jl} x^k - \lambda^{0jkl} \right) \\ &= -\epsilon_{ijk} \int_S \left(\zeta^{0jl} x^k - \lambda^{0jlk} \right) dS_l \,. \end{aligned} \tag{20.110}$$

$$\tag{20.111}$$

In conclusion, the total angular momentum of the source is

$$\boxed{J^i = -\epsilon_{ijk} \int_S \left(\zeta^{0jl} x^k - \lambda^{0jlk} \right) dS_l} \,. \tag{20.112}$$

Thus, given the metric, using eqs. (20.102) and (20.112) we can evaluate the total mass-energy and the total angular momentum of the source.

It is possible to show that, using the metric of the far field limit $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is given by eqs. (20.92), i.e.

$$\begin{aligned} h_{00} &= \frac{2M}{r} + O\left(\frac{1}{r^2}\right) \\ h_{0i} &= \frac{2}{r^3} \epsilon_{ijk} x^j J^k + O\left(\frac{1}{r^3}\right) \\ h_{ij} &= \frac{2M}{r} \delta_{ij} + O\left(\frac{1}{r^2}\right) \,, \end{aligned} \tag{20.113}$$

the 4-momentum and the angular momentum of the stationary source are

$$P^\mu = (M, 0, 0, 0) \,, \qquad J^i = (0, 0, J) \,. \tag{20.114}$$

We show explicitly the calculations to find P^0. From eq. (20.102) we have

$$P^0 = \int_S \zeta^{00i} dS_i \equiv \int_S \zeta^{00i} n^i dS, \tag{20.115}$$

where $dS = r^2 d\Omega$ and n^i is the unit vector orthogonal to the surface element dS. Being $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + O(|h_{\mu\nu}|^2)$, the property $g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$ implies

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + O(|h_{\mu\nu}|^2) \tag{20.116}$$

where the indices of $h_{\mu\nu}$ have been raised with Minkowski's metric. Indeed,

$$(\eta^{\mu\nu} - h^{\mu\nu})(\eta_{\nu\rho} + h_{\nu\rho}) = \delta^\mu_\rho + O(|h_{\mu\nu}|^2) \,. \tag{20.117}$$

Therefore,

$$g^{00} = -1 - h^{00} + O(|h_{\mu\nu}|^2) = -1 - \frac{2M}{r} + O(|h_{\mu\nu}|^2) \qquad (20.118)$$

$$g^{ij} = \delta^{ij} - h^{ij} + O(|h_{\mu\nu}|^2) = \left(1 - \frac{2M}{r}\right)\delta^{ij} + O(|h_{\mu\nu}|^2)\,. \qquad (20.119)$$

The determinant of $g_{\mu\nu}$ is

$$g = (-1 + h_{00})(1 + h_{ii}) = -(1 + \frac{4M}{r}) + O(|h_{\mu\nu}|^2)\,. \qquad (20.120)$$

Note that in this expression we neglect the term J/r^3 with respect to M/r, since we are in the far field limit.

From eq. (20.97), neglecting terms $O(|h_{\mu\nu}|^2)$ (like the terms $\sim M^2$, $\sim J^2$), we find

$$\begin{aligned} \zeta^{00i}n^i &= \frac{1}{16\pi}n^i\frac{\partial}{\partial x^j}\left[(-g)\left(g^{00}g^{ij} - g^{0i}g^{0j}\right)\right] \sim \frac{1}{16\pi}n^i\frac{\partial}{\partial x^j}\left[(-g)g^{00}g^{ij}\right] \\ &= \frac{1}{16\pi}n^i\frac{\partial}{\partial x^j}\left[\left(1 + \frac{4M}{r}\right)\left(-1 - \frac{2M}{r}\right)\left(1 - \frac{2M}{r}\right)\delta_{ij}\right] + O(|h_{\mu\nu}|^2) \\ &= -\frac{1}{16\pi}n^i\frac{\partial}{\partial x^j}\frac{4M}{r}\delta_{ij} + O(|h_{\mu\nu}|^2). \end{aligned} \qquad (20.121)$$

Since

$$\frac{\partial}{\partial x^j}\frac{1}{r} = -\frac{n^j}{r^2},$$

then

$$\zeta^{00i}n^i = \frac{1}{4\pi}\frac{M}{r^2}\sum_{i=1,3} n^i n^i = \frac{1}{4\pi}\frac{M}{r^2} \qquad (20.122)$$

and

$$P^0 = \int_S \zeta^{00i}n^i r^2 d\Omega = M\,. \qquad (20.123)$$

The calculation for the angular momentum are similar.

We can conclude that the integration constants M and J appearing in the far field limit metric of an isolated source (20.3) can be correctly interpreted as the mass-energy and the angular momentum of the system. In the case of a weakly gravitating source, the contribution of the gravitational field to the mass and to the angular momentum are negligible; if the source has a strong gravitational field, the field contributes to the total mass and angular momentum, through the stress-energy pseudotensor $t^{\mu\nu}$.

We stress again that, being the source isolated, at large distance the metric tends to Minkowski's metric; this allows us to assume that, for r sufficiently large, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{\mu\nu}$ small. Furthermore, $h_{\mu\nu}$ can be expanded in powers of $1/r$. The dominant contribution in this expansion gives the total mass-energy and angular momentum of the system.