Chapter 22

Geodesic motion in Kerr spacetime

Let us consider a geodesic with affine parameter λ and tangent vector

$$u^\mu = \frac{dx^\mu}{d\lambda} \equiv \dot{x}^\mu . \tag{22.1}$$

In this section we shall use Boyer-Lindquist's coordinates, and the dot will indicate differentiation with respect to λ. The tangent vector u^μ is solution of the geodesic equations

$$u^\mu u^\nu{}_{;\mu} = 0 \,, \tag{22.2}$$

which, as shown in Chapter 11, is equivalent to the Euler-Lagrange equations

$$\frac{d}{d\lambda}\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} = \frac{\partial \mathcal{L}}{\partial x^\alpha} \tag{22.3}$$

associated to the Lagrangian

$$\mathcal{L}\left(x^\mu, \dot{x}^\mu\right) = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \,. \tag{22.4}$$

By defining the *conjugate momentum* p_μ as

$$p_\mu \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = g_{\mu\nu} \dot{x}^\nu \,, \tag{22.5}$$

the Euler-Lagrange equations become

$$\frac{d}{d\lambda} p_\mu = \frac{\partial \mathcal{L}}{\partial x^\mu} \,. \tag{22.6}$$

Note that, if the metric does not depend on a given coordinate x^μ, the conjugate momentum p_μ is a constant of motion and coincides with the constant of motion associated to the Killing vector tangent to the corresponding coordinate lines. The Kerr metric in Boyer-Lindquist coordinates is indepentent of t and ϕ, therefore

$$p_t = \dot{x}_t \equiv u_t = const \qquad \text{and} \qquad p_\phi = \dot{x}_\phi \equiv u_\phi = const; \tag{22.7}$$

these quantities coincide with the constant of motion associated to the Killing vectors $k^\mu = (1, 0, 0, 0)$ and $m^\mu = (0, 0, 0, 1)$, i.e. $k^\mu u_\mu = u_t$ and $m^\mu u_\mu = u_\phi$.

Therefore, geodesic motion in Kerr geometry is characterized by two constants of motion, which we indicate as:

$$E \equiv -k^\mu u_\mu = -u_t = -p_t \quad \text{constant along geodesics} \tag{22.8}$$

$$L \equiv m^\mu u_\mu = u_\phi = p_\phi \quad \text{constant along geodesics}\,. \tag{22.9}$$

As explained in Section 11.2, for massive particles E and L are, respectively, the energy and the angular momentum per unit mass, as measured at infinity with respect to the black hole. For massless particles, E and L are the energy and the angular momentum at infinity.

Equations (22.2) (or, equivalently, (22.3)) in Kerr spacetime are very complicate to solve directly. To simplify the problem we hall use the conserved quantities, as we did in Chapter 11 in when we studied geodesic motion in Schwarzschild's spacetime. For this, we need four algebraic relations involving u^μ.

Furthermore

$$g_{\mu\nu}u^\mu u^\nu = \kappa \tag{22.10}$$

where

$$\begin{aligned} \kappa = -1 &\qquad \text{for timelike geodesics} \\ \kappa = 1 &\qquad \text{for spacelike geodesics} \\ \kappa = 0 &\qquad \text{for null geodesics}\,. \end{aligned} \tag{22.11}$$

Eqs. (22.8), (22.9), (22.10) give three algebraic relations involving u^μ, but they are not sufficient to to determine the *four* unknowns u^μ. In Schwarzschild spacetime a fourth equation is provided by the planarity of the orbit ($u^\theta = 0$ if $\theta(\lambda = 0) = \pi/2$); in Kerr spacetime orbits are planar only in the equatorial plane therefore, in general, geodesic motion cannot be studied in a simple way, using eqs. (22.8), (22.9), (22.10) only, as we did for Scharzschild. However, as we shall briefly explain in the last section of this chapter, there exists a further conserved quantity, the *Carter constant*, which allows to find the tangent vector u^μ using algebraic relations.

22.1 Equatorial geodesics

In this section we study geodesic motion in the equatorial plane, i.e. geodesics with

$$\theta \equiv \frac{\pi}{2}\,. \tag{22.12}$$

First of all, let us prove that such geodesics exist, i.e. that eq. (22.12) is solution of the Euler-Lagrange equations. The Lagrangian is

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \frac{1}{2}\left\{-\left(1-\frac{2Mr}{\Sigma}\right)\dot{t}^2 - \frac{4Mr}{\Sigma}a\sin^2\theta\;\dot{t}\dot{\phi} + \frac{\Sigma}{\Delta}\,\dot{r}^2\right. \\ &+ \left.\Sigma\,\dot{\theta}^2 + \left[r^2+a^2+\frac{2Mra^2}{\Sigma}\sin^2\theta\right]\sin^2\theta\;\dot{\phi}^2\right\} \end{aligned} \tag{22.13}$$

and the θ component of Euler-Lagrange's equations is

$$\frac{d}{d\lambda}(g_{\theta\mu}\dot{x}^\mu) = \frac{d}{d\lambda}(\Sigma\dot{\theta}) = \Sigma\ddot{\theta} + \Sigma_{,\mu}\dot{x}^\mu\dot{\theta} = \frac{1}{2}g_{\mu\nu,\theta}\dot{x}^\mu\dot{x}^\nu \,. \tag{22.14}$$

The right-hand side is

$$\begin{aligned}\frac{1}{2}g_{\mu\nu,\theta}\dot{x}^\mu\dot{x}^\nu &= \frac{1}{2}\left\{\Sigma_{,\theta}\left(\frac{(\dot{r})^2}{\Delta} + (\dot{\theta})^2\right) + 2\sin\theta\cos\theta(r^2+a^2)(\dot{\phi})^2\right. \\ &- \left.\frac{2Mr}{\Sigma^2}\Sigma_{,\theta}\left(a\sin^2\theta\dot{\phi} - \dot{t}\right)^2 + \frac{4Mr}{\Sigma}\left(a\sin^2\theta\dot{\phi} - \dot{t}\right)2a\sin\theta\cos\theta\dot{\phi}\right\}\end{aligned} \tag{22.15}$$

where $\Sigma_{,\theta} = -2a^2\sin\theta\cos\theta$ and $\Sigma_{,r} = 2r$. It is easy to check that, when $\theta = \pi/2$, equation (22.14) reduces to

$$\ddot{\theta} = -\frac{2}{r}\dot{r}\dot{\theta} \,. \tag{22.16}$$

Therefore, if $\dot{\theta} = 0$ and $\theta = \pi/2$ at $\lambda = 0$, then for $\lambda > 0$ $\dot{\theta} \equiv 0$ and $\theta \equiv \pi/2$. Thus, a geodesic which starts in the equatorial plane, remains in the equatorial plane at later times.

This also occurs in Schwarzschild spacetime, and in that case, due to the spherical symmetry, it is possible to generalize the result to any orbit, and prove that all geodesics are planar. This generalization is not possible for the Kerr metric which is axially symmetric. In this case only equatorial geodesics are planar.

On the equatorial plane, $\Sigma = r^2$, therefore

$$\begin{aligned} g_{tt} &= -\left(1 - \frac{2M}{r}\right) \\ g_{t\phi} &= -\frac{2Ma}{r} \\ g_{rr} &= \frac{r^2}{\Delta} \\ g_{\phi\phi} &= r^2 + a^2 + \frac{2Ma^2}{r} \end{aligned} \tag{22.17}$$

and

$$E = -g_{t\mu}u^\mu = \left(1 - \frac{2M}{r}\right)\dot{t} + \frac{2Ma}{r}\dot{\phi} \tag{22.18}$$

$$L = g_{\phi\mu}u^\mu = -\frac{2Ma}{r}\dot{t} + \left(r^2 + a^2 + \frac{2Ma^2}{r}\right)\dot{\phi} \,. \tag{22.19}$$

To solve eqs. (22.18), (22.19) for $\dot{t}$, $\dot{\phi}$ we define

$$\begin{aligned} A &\equiv 1 - \frac{2M}{r} \\ B &\equiv \frac{2Ma}{r} \\ C &\equiv r^2 + a^2 + \frac{2Ma^2}{r} \end{aligned} \tag{22.20}$$

and write eqs. (22.18), (22.19) as

$$\begin{aligned} E &= A\dot{t} + B\dot{\phi} && (22.21) \\ L &= -B\dot{t} + C\dot{\phi}\,. && (22.22) \end{aligned}$$

Furthermore, the following relation can be used

$$AC + B^2 = \left(1 - \frac{2M}{r}\right)\left(r^2 + a^2 + \frac{2Ma^2}{r}\right) + \frac{4M^2a^2}{r^2} = r^2 - 2Mr + a^2 = \Delta\,. \quad (22.23)$$

Therefore,

$$\begin{aligned} CE - BL &= [AC + B^2]\dot{t} = \Delta\dot{t} \\ AL + BE &= [AC + B^2]\dot{\phi} = \Delta\dot{\phi} && (22.24) \end{aligned}$$

i.e.

$$\boxed{\begin{aligned} \dot{t} &= \frac{1}{\Delta}\left[\left(r^2 + a^2 + \frac{2Ma^2}{r}\right)E - \frac{2Ma}{r}L\right] \\ \dot{\phi} &= \frac{1}{\Delta}\left[\left(1 - \frac{2M}{r}\right)L + \frac{2Ma}{r}E\right] \end{aligned}}\,. \quad (22.25)$$

The quantity C defined in eq. (22.20) can be written in a different form, which will be useful in the following:

$$\begin{aligned} \frac{(r^2 + a^2)^2 - a^2\Delta}{r^2} &= \frac{1}{r^2}[(r^2 + a^2)(r^2 + a^2) - a^2(r^2 + a^2 - 2Mr)] \\ &= \frac{1}{r^2}[(r^2 + a^2)r^2 + 2Mra^2] = r^2 + a^2 + \frac{2Ma^2}{r} \\ &\equiv C\,. && (22.26) \end{aligned}$$

Note that C is always positive.

Let us now derive the equation for the radial component of the four-velocity. Equation (22.10) can be written in terms of A, B, C:

$$\begin{aligned} g_{\mu\nu}u^\mu u^\nu &= \kappa \\ &= -A\dot{t}^2 - 2B\dot{t}\dot{\phi} + C\dot{\phi}^2 + \frac{r^2}{\Delta}\dot{r}^2 \\ &= -[A\dot{t} + B\dot{\phi}]\dot{t} + [-B\dot{t} + C\dot{\phi}]\dot{\phi} + \frac{r^2}{\Delta}\dot{r}^2 \\ &= -E\dot{t} + L\dot{\phi} + \frac{r^2}{\Delta}\dot{r}^2 && (22.27) \end{aligned}$$

where we have used eqs. (22.21), (22.22). Therefore,

$$\dot{r}^2 = \frac{\Delta}{r^2}(E\dot{t} - L\dot{\phi} + \kappa) = \frac{1}{r^2}\left[CE^2 - 2BLE - AL^2\right] + \frac{\kappa\Delta}{r^2}\,. \quad (22.28)$$

The polynomial $[CE^2 - 2BLE - AL^2]$ has zeros

$$V_\pm = \frac{BL \pm \sqrt{B^2L^2 + ACL^2}}{C} = \frac{L}{C}[B \pm \sqrt{\Delta}]\,. \tag{22.29}$$

Consequently, eq. (22.28) can be written as

$$\dot{r}^2 = \frac{C}{r^2}(E - V_+)(E - V_-) + \frac{\kappa\Delta}{r^2}\,. \tag{22.30}$$

Using eq. (22.26), eqs. (22.30) and (22.29) finally become

$$\boxed{\dot{r}^2 = \frac{(r^2+a^2)^2 - a^2\Delta}{r^4}(E - V_+)(E - V_-) + \frac{\kappa\Delta}{r^2}}\,, \tag{22.31}$$

and

$$\boxed{V_\pm = \frac{2Mar \pm r^2\sqrt{\Delta}}{(r^2+a^2)^2 - a^2\Delta}L}\,. \tag{22.32}$$

In the Schwarzschild limit $a \to 0$ and

$$V_+ + V_- \propto a \to 0\,, \qquad V_+V_- \to \frac{L^2\Delta}{r^4} \tag{22.33}$$

therefore, if we define $V \equiv -V_+V_-$, eqs. (22.31), (22.32) reduce to the well known form

$$\dot{r}^2 = E^2 - V(r), \qquad \text{where} \qquad V(r) = -\frac{\kappa\Delta}{r^2} + \frac{L^2\Delta}{r^4} = \left(1 - \frac{2M}{r}\right)\left(-\kappa + \frac{L^2}{r^2}\right) \tag{22.34}$$

where we recall that $\kappa = -1$ for timelike geodesics, $\kappa = 0$ for null geodesics, $\kappa = 1$ for spacelike geodesics.

22.1.1 Kerr's potentials for equatorial geodesics

$$V_\pm = \frac{2MLar \pm Lr^2\sqrt{\Delta}}{(r^2+a^2)^2 - a^2\Delta}\,. \tag{22.35}$$

In principle we would have four possibilities, corresponding to L positive and negative and a positive and negative. In practice, there are only two interesting cases: $La > 0$ and $La < 0$, i.e. the test particle is either corotating or counterrotating with the black hole. If the signs of L and a change simultaneously, the potentials $V_\pm$ interchange: V_+ becomes V_- and viceversa. To avoid this, it is better to redefine the names of the potentials as follows

$$V_\pm = \frac{2MLar \pm |L|r^2\sqrt{\Delta}}{(r^2+a^2)^2 - a^2\Delta}\,, \tag{22.36}$$

so that the following inequality is always true

$$V_+ \geq V_-\,. \tag{22.37}$$

In general, we find that:

- V_+ and V_- coincide for $\Delta = 0$, i.e. for

$$r = r_+ = M + \sqrt{M^2 - a^2} \tag{22.38}$$

while for $r > r_+$, $\Delta > 0$ and then $V_+ > V_-$. Furthermore,

$$V_+(r_+) = V_-(r_+) = \frac{2Mr_+La}{(r_+^2 + a^2)^2}\,, \tag{22.39}$$

which is positive if $La > 0$, negative if $La < 0$.

- In the limit $r \to \infty$, $V_\pm \to 0$.

- If $La > 0$ (corotating orbits), the potential V_+ is definite positive; V_- (which is positive at r_+) vanishes when

$$r\sqrt{\Delta} = 2Ma \quad \Rightarrow \quad r^2(r^2 - 2Mr + a^2) = 4M^2a^2 \tag{22.40}$$

which gives

$$r^4 - 2Mr^3 + a^2r^2 - 4M^2a^2 = (r - 2M)(r^3 + a^2r + 2Ma^2) = 0; \tag{22.41}$$

thus V_- vanishes at $r = 2M$, which is the location of the ergosphere in the equatorial plane.

- If $La < 0$ (counterrotating orbits), the potential V_- is definite negative and V_+ (which is negative at r_+) vanishes at $r = 2M$.

- The study of the derivatives of $V_\pm$, which is too long to be reported here, shows that both potentials, V_+ and V_-, have only one stationary point.

In summary, $V_+(r)$ and $V_-(r)$ have the shapes shown in Figure 22.1 where the upper and lower panels refer, respectively, to the case $La > 0$ and $La < 0$ cases.

22.1.2 Null geodesics

In the case of null geodesics the radial equation (22.31) becomes

$$\dot{r}^2 = \frac{C}{r^2}(E - V_+)(E - V_-) = \frac{(r^2 + a^2)^2 - a^2\Delta}{r^4}(E - V_+)(E - V_-) \tag{22.42}$$

Since $\dot{r}^2$ must be positive, from eq. (22.42) we see that, and being $(r^2 + a^2)^2 - a^2\Delta > 0$, null geodesics are possible for massless particle whose constant of motion E satisfies the following inequalities

$$\boxed{E < V_- \qquad \text{or} \qquad E > V_+}\,. \tag{22.43}$$

Thus, the region $V_- < E < V_+$, corresponding to the dashed regions in Figure 22.1, is forbidden.

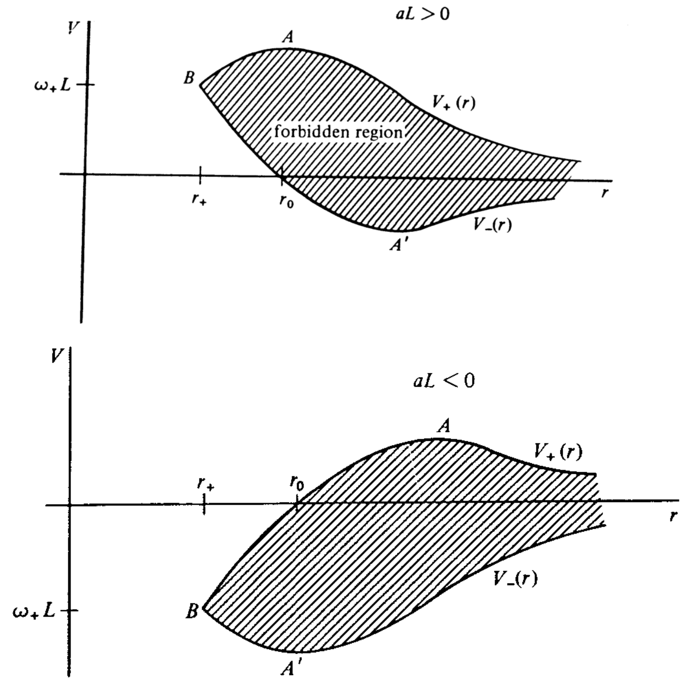

Figure 22.1: The potentials $V_+(r)$ and $V_-(r)$, for corotating ($aL > 0$) and counterrotating ($aL < 0$) orbits. The shadowed region is not accessible to the motion of photons or other massless particles.

In order to study the orbits, it is useful to compute the radial acceleration. By differentiating eq. (22.42) with respect to the affine parameter λ, we find

$$2\dot{r}\ddot{r} = \left[\left(\frac{C}{r^2}\right)'(E - V_+)(E - V_-) - \frac{C}{r^2}V_+'(E - V_-) - \frac{C}{r^2}V_-'(E - V_+)\right]\dot{r} \tag{22.44}$$

i.e.

$$\ddot{r} = \frac{1}{2}\left(\frac{C}{r^2}\right)'(E - V_+)(E - V_-) - \frac{C}{2r^2}\left[V_+'(E - V_-) + V_-'(E - V_+)\right], \tag{22.45}$$

where the prime indicates differentiation with respect to r. Let us evaluate the radial acceleration in a point where the radial velocity $\dot{r}$ is zero, i.e. when $E = V_+$ or $E = V_-$:

$$\begin{aligned} \ddot{r} &= -\frac{C}{2r^2}V_+'(V_+ - V_-) \qquad \text{if } E = V_+ \\ \ddot{r} &= -\frac{C}{2r^2}V_-'(V_- - V_+) \qquad \text{if } E = V_- \, . \end{aligned} \tag{22.46}$$

Since

$$V_+ - V_- = \frac{2|L|r^2\sqrt{\Delta}}{(r^2 + a^2)^2 - a^2\Delta} = \frac{2|L|\sqrt{\Delta}}{C}, \tag{22.47}$$

we find

$$\ddot{r} = \mp\frac{|L|\sqrt{\Delta}}{r^2}V'_{\pm} \qquad \text{if} \;\; E = V_{\pm}\,. \tag{22.48}$$

- **Unstable circular orbits**
  If $E = V_+(r_{max})$, where r_{max} is the stationary point of V_+ (i.e. $V'_+(r_{max}) = 0$), the radial acceleration vanishes; since when $E = V_+(r_{max})$ the radial velocity also vanishes, a massless particle with that value of E can be captured on a circular orbit, but the orbit is unstable, as it is the orbit at $r = 3M$ for the Schwarzschild metric.
  It is possible to show that r_{max} is solution of the equation

  $$r(r-3M)^2 - 4Ma^2 = 0\,. \tag{22.49}$$

  Note that the value of r_{max} is independent of L. The solution of (22.49) is a decreasing function of a, and, in particular,

  $$\begin{aligned} r_{max} &= 3M \qquad \text{for } a = 0 \\ r_{max} &= M \qquad \text{for } a = M \\ r_{max} &= 4M \qquad \text{for } a = -M\,. \end{aligned} \tag{22.50}$$

  Therefore, while for a Schwarzschild black hole the unstable circular orbit of a photon is at $r = 3M$, for a Kerr black hole it can be much closer; in particular, in the extremal case $a = M$, for corotating orbits $r_{max} = M$ coincides with the outer horizon.

- **Photon capture**
  A photon falling from infinity with constant of motion $E > V_+(r_{max})$, crosses the horizon and falls toward the singularity.

- **Deflection**
  If $0 < E < V_+(r_{max})$, the particle reaches the turning point where $E = V_+(r)$ and $\dot{r} = 0$; eq. (22.48) shows that at the turning point $\ddot{r} > 0$, therefore the particle reverts its motion and escapes free at infinity. In this case the particle is deflected.

In the above cases the constant of motion E associated to the timelike Killing vector is assumed to be positive.

It remains to consider the case $E < V_-$, and in particular to see whether negative values of E, admitted in principle admitted by eq. (22.43), have a physical meaning.

22.1.3 How do we measure the energy of a particle

The energy of a particle is an observer-dependent quantity. In special relativity, the energy of a particle with four-momentum P^μ, measured by an observer with four-velocity u^μ, is defined as

$$\mathcal{E}^{(u)} = -\eta_{\mu\nu}u^\mu P^\nu = -u^\mu P_\nu\,. \tag{22.51}$$

For instance, the energy measured by a static observer $u^\mu_{st} = (1,0,0,0)$ is

$$\mathcal{E}^{(u_{st})} = -P_0\;. \tag{22.52}$$

A negative energy would correspond to a particle moving backwards in time, and causality would be violated. Thus, energy is always positive; if measured by a different observer it will be different, but still positive. Eq. (22.51) is a tensor equation; it holds in a locally inertial frame, where $g_{\mu\nu} \equiv \eta_{\mu\nu}$, therefore it can be written as

$$\boxed{\mathcal{E}^{(u)} = -g_{\mu\nu} u^\mu P^\nu = -u^\mu P_\mu} \,. \tag{22.53}$$

Thus, by the principle of general covariance, eq. (22.53), is the definition of energy valid in any frame, and consequently $\mathcal{E}$ must be positive in any frame.

Let us now consider a static observer with $u^\mu_{st} = (1, 0, 0, 0)$, in Kerr spacetime, located at radial infinity, where such observer can exist. According to the definition (22.53), the energy measured by the static observer is $\mathcal{E}^{(u_{st})} = -P_0$. Let us now compare this quantity with the constant of motion $E = -u_0$ given in eq. (22.8). If the particle is massless we can always parametrize the geodesic in such a way that $P_0 \equiv u_0$. Thus:

$$\mathcal{E}^{(u_{st})} = -P_0 = E \tag{22.54}$$

We conclude that *for a particle starting (or ending) its motion at radial infinity with respect to the black hole, the constant of motion E is the particle energy, as measured by a static observer located at infinity* [1]. For such particles orbits with negative values of E are not allowed. Thus, referring to Figure 22.1, orbits with $E < V_-$ and E negative impinging from radial infinity are forbidden, even though for such values $\dot{r}^2 > 0$ (see eq. (22.42)).

Let us now consider a massless particle which starts its motion in the egoregion, i.e. between r_+ and r_0 (see Figure 22.1). In this region static observers cannot exist, therefore we need to consider a different observer, for instance a stationary ZAMO (i.e. an observer for which $\dot{r} = 0$ and $L = u_\phi = 0$), whose four-velocity can be written as

$$u^\mu_{ZAMO} = const(1, 0, 0, \Omega) \tag{22.55}$$

where the ZAMO angular velocity Ω on the equatorial plane is (see eq. (21.27))

$$\Omega = \frac{2Mar}{(r^2 + a^2)^2 - a^2\Delta} \tag{22.56}$$

and the constant is found by imposing $g_{\mu\nu} u^\mu u^\nu = -1$. The constant must be positive, otherwise the ZAMO would move backwards in time.

The particle energy measured by the ZAMO is

$$\mathcal{E}^{ZAMO} = -P_\mu u^\mu_{ZAMO} = const(E - \Omega L) \,, \tag{22.57}$$

where we have used eqs. (22.8) and (22.9). Thus, the requirement $\mathcal{E}^{ZAMO} > 0$ is equivalent to

$$E > \Omega L \,. \tag{22.58}$$

[1] similarly, for massive particles E is the energy per unit mass as measured by a static observer at infinity.

By comparing (22.56) with the expression of the potentials $V_\pm$ given by eq. (22.36) we find that

$$V_- < \Omega L < V_+ . \tag{22.59}$$

Therefore, geodesics with

$$E > V_+ \tag{22.60}$$

satisfy the positive energy condition (22.58), and are allowed, whereas those with $E < V_-$ are forbidden, since do not satisfy eq. (22.58).

Thus, referring to Figure 22.36:

- a corotating particle ($La > 0$) can move within the ergoregion only if the costant of motion E is positive and is in the range

  $$V_+(r_+) < E < V_+(r_{max}) . \tag{22.61}$$

  If $E > V_+(r_{max})$ the particle can cross the ergosphere and escape at infinity.

- For counterrotating particles ($La < 0$), since in the ergoregion V_+ is negative the requirement $E > V_+$ (necessary and sufficient to ensure that $\mathcal{E} > 0$) allows negative values of the constant of motion E. Thus, counterrotating particles moving in the ergoregion can have negative E, provided

  $$V_+(r_+)(= V_-(r_+)) < E < 0 . \tag{22.62}$$

  As we shall show in the next section, this possibility has an interesting consequence.

  It should be stressed that this is not a contradiction, because it is only at infinity that E represents the particle energy; the geodesics we are considering never reach infinity.

22.1.4 Penrose's process

In this section we will use a slightly different notation for the constants of motion E, L, which have been shown to be the energy and angular momentum *per unit mass*, for massive particles, and the energy and angular momentum for massless particles, as measured by a static observer at infinity. Here we define E and L to be the energy and angular momentum at infinity, both for massive and massless particles, so that eqs. (22.8) and (22.9) become

$$E = -k^\mu P_\mu \,, \qquad L = m^\mu P_\mu \,. \tag{22.63}$$

This simply means that, for massive particles, E and L have been multiplied by the particle mass m.

We shall now show that since particles with negative E can exist in the ergoregion, we can imagine a process through which it may be possible to extract rotational energy from a Kerr black hole; this is named *Penrose's process*.

In what follows we shall set $a > 0$. Assuming $a < 0$ would lead to the same conclusions. Suppose that we shoot a massive particle with energy E and angular momentum L from infinity, so that it falls towards the black hole in the equatorial plane. Its four-momentum covariant components are

$$P_\mu = (-E, P_r, 0, L)\,. \tag{22.64}$$

Along the geodesic the particle four-momentum changes, but the covariant components $P_t = k^\mu P_\mu = -E$, and $P_\phi = m^\mu P_\mu = L$ remain constant, i.e.,

$$P_\mu = (-E, P_r, 0, L)\,. \tag{22.65}$$

When the particle enters the ergoregion, it decays in two photons, with momenta

$$P_{1\,\mu} = (-E_1, P_{1\,r}, 0, L_1) \qquad P_{2\,\mu} = (-E_2, P_{2\,r}, 0, L_2)\,. \tag{22.66}$$

Since the four-momentum is conserved in this decay, we have

$$P^\mu = P^{1\,\mu} + P^{2\,\mu} \quad \text{or equivalently} \quad P_\mu = P_{1\,\mu} + P_{2\,\mu}\ ,$$

from which it follows that

$$E = E_1 + E_2\ , \qquad\qquad L = L_1 + L_2\,. \tag{22.67}$$

Let us assume that $\dot r_1 < 0$, so that the photon 1 falls into the black hole, and that it has negative constants of motion, i.e. $E_1 < 0$ and $L_1 < 0$, with (see eq. (22.62))

$$V_+(r_+)(= V_-(r_+)) < E_1 < 0\ .$$

We further assume that $\dot r_2 > 0$, i.e. the photon 2 comes back to infinity. Note that, as explained in section 22.1.3, this is possible only if

$$E_2 > V_+(r_{max})\ .$$

Its energy and angular momentum are

$$\begin{aligned} E_2 &= E - E_1 > E \\ L_2 &= L - L_1 > L\,, \end{aligned} \tag{22.68}$$

thus, at the end of the process the particle we find at infinity is more energetic than the one we sent in. It is possible to show that, since $E_1 < 0, L_1 < 0$, the capture of photon 1 by the black hole reduces its mass-energy M and its angular momentum $J = Ma$; indeed their values M_{fin}, J_{fin} are respectively:

$$M_{fin} = M + E_1 < M \tag{22.69}$$
$$J_{fin} = J + L_1 < J\,. \tag{22.70}$$

To prove the inequality (22.69), we note that, as shown in Chapter 17, the total mass-energy of the system is

$$P^0_{tot} = \int_V d^3x(-g)(T^{00} + t^{00})\ , \tag{22.71}$$

where V is the volume of a $t = const.$ three-surface. If we neglect the gravitational field generated by the particle, t^{00} is due to the black hole only, thus

$$P^0_{tot} = \int_V d^3x(-g)T^{00}_{particle} + M\,. \tag{22.72}$$

Let us compute this integral when the process starts, i.e. at a time when the massive particle is shoot into the black hole; the spacetime is flat, the particle energy is E, and the 00-component of the stress-energy tensor of a point particle with energy E, in Minkowskian coordinates is

$$T^{00}_{particle} = E\delta^3(\mathbf{x} - \mathbf{x}(t))\,. \tag{22.73}$$

Thus eq. (22.72) gives

$$P^0_{tot\ in} = E + M_{in}\ . \tag{22.74}$$

Repeating the computation at the end of the process, namely when the photon 2 reaches infinity, we find

$$P^0_{tot\ fin} = E_2 + M_{fin}\,. \tag{22.75}$$

Due to the stationarity of the Kerr metric, if we neglect the outgoing gravitational flux generated by the particle, P^0_{tot} is a conserved quantity; therefore by equating the initial and final momentum we find

$$P^0_{tot\ in} = P^0_{tot\ fin} \qquad \to \qquad M_{fin} = M_{in} + (E - E_2) \qquad \to \qquad M_{fin} = M_{in} + E_1 < M_{in}\ . \tag{22.76}$$

This proves the relation (22.69), and eq. (22.70) can be proved accordingly.

In conclusion, by this process we have *extracted* rotational energy from the black hole.

22.1.5 Innermost stable circular orbit for timelike geodesics

The study of timelike geodesics is much more complicate, because equation (22.30) which, when $\kappa = -1$, becomes

$$\dot{r}^2 = \frac{C}{r^2}(E - V_+)(E - V_-) - \frac{\Delta}{r^2}\,, \tag{22.77}$$

does not allow a simple qualitative study as in the case of null geodesics. Therefore, here we only report some results of a detailed study of geodesics equation in this general case.

A very relevant quantity (of astrophysical interest) is the location of the innermost stable circular orbit (ISCO), which, in the Schwarzschild case, is at $r = 6M$. In Kerr spacetime, the expression for r_{ISCO} is quite complicate, but its qualitative behaviour is simple: there are two solutions

$$r^{\pm}_{ISCO}(a)\,, \tag{22.78}$$

one corresponding to corotating and counterrotating orbits. For $a = 0$, the two solutions coincide to $6M$, as expected; by increasing $|a|$, the ISCO moves closer to the black hole for corotating orbits, and farther for counterrotating orbits. When $a = \pm M$, the corotating ISCO coincides with the outer horizon, at $r = r_+ = M$. This behaviour is very similar to that we have already seen in the case of unstable circular orbits for null geodesics.

In Figure 22.1.5 we show (for $a \geq 0$) the locations of the last stable and unstable circular orbits for timelike geodesics, and of the unstable circular orbit for null geodesics. This figure is taken from the article where these orbits have been studied (*J. Bardeen, W. H. Press, S. A. Teukolsky, Astrophys. J. 178, 347, 1972*).

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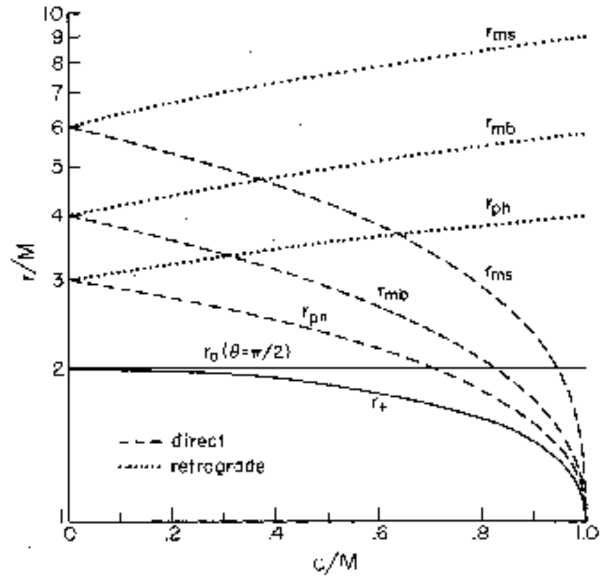

FIG. 1.—Radii of circular, equatorial orbits around a rotating black hole of mass M, as functions of the hole's specific angular momentum a. Dashed and dotted curves (for direct and retrograde orbits) plot the Boyer-Lindquist coordinate radius of the innermost stable (ms), innermost bound (mb), and photon (ph) orbits. Solid curves indicate the event horizon (r_+) and the equatorial boundary of the ergosphere (r_0).

22.1.6 3^{rd} Kepler's law

Let us consider a circular timelike geodesic in the equatorial plane. We remind that the Lagrangian (22.4) is

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \tag{22.79}$$

and the r-component of the Euler-Lagrange equation is

$$\frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{\partial \mathcal{L}}{\partial r} \, . \tag{22.80}$$

Being $g_{r\mu} = 0$ if $\mu \neq r$, we have

$$\frac{d}{d\lambda}(g_{rr}\dot{r}) = \frac{1}{2} g_{\mu\nu,r}\dot{x}^\mu \dot{x}^\nu \,. \tag{22.81}$$

For circular geodesic, $\dot{r} = \ddot{r} = 0$, and this equation reduces to

$$g_{tt,r}\dot{t}^2 + 2g_{t\phi,r}\dot{t}\dot{\phi} + g_{\phi\phi,r}\dot{\phi}^2 = 0\,. \tag{22.82}$$

The angular velocity is $\omega = \dot{\phi}/\dot{t}$, thus

$$g_{\phi\phi,r}\omega^2 + 2g_{t\phi,r}\omega + g_{tt,r} = 0\,. \tag{22.83}$$

We remind that on the equatorial plane

$$\begin{aligned} g_{tt} &= -\left(1 - \frac{2M}{r}\right) \\ g_{t\phi} &= -\frac{2Ma}{r} \\ g_{\phi\phi} &= r^2 + a^2 + \frac{2Ma^2}{r}\,, \end{aligned} \tag{22.84}$$

then

$$2\left(r - \frac{Ma^2}{r^2}\right)\omega^2 + \frac{4Ma}{r^2}\omega - \frac{2M}{r^2} = 0\,. \tag{22.85}$$

The equation

$$(r^3 - Ma^2)\omega^2 + 2Ma\omega - M = 0 \tag{22.86}$$

has discriminant

$$M^2a^2 + M(r^3 - Ma^2) = Mr^3 \tag{22.87}$$

and solutions

$$\begin{aligned} \omega_\pm &= \frac{-Ma \pm \sqrt{Mr^3}}{r^3 - Ma^2} = \pm\sqrt{M}\,\frac{r^{3/2} \mp a\sqrt{M}}{r^3 - Ma^2} \\ &= \pm\sqrt{M}\frac{r^{3/2} \mp a\sqrt{M}}{(r^{3/2} + a\sqrt{M})(r^{3/2} - a\sqrt{M})} \\ &= \pm\frac{\sqrt{M}}{r^{3/2} \pm a\sqrt{M}}\,. \end{aligned} \tag{22.88}$$

This is the relation between angular velocity and radius of circular orbits, and reduces, in Schwarzschild limit $a = 0$, to

$$\omega_\pm = \pm\sqrt{\frac{M}{r^3}}\,, \tag{22.89}$$

which is Kepler's 3^{rd} law.

22.2 General geodesic motion: the Carter constant

To study geodesics in Kerr spacetime, it is convenient to use the *Hamilton-Jacobi approach*, which allows to indentify a further constant of motion.

It should be stressed that this constant is not associated to a spacetime symmetry.

Given the Lagrangian of the system

$$\mathcal{L}(x^\mu, \dot{x}^\mu) = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \tag{22.90}$$

and given the conjugate momenta[2]

$$p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = g_{\mu\nu} \dot{x}^\nu \,, \tag{22.91}$$

by inverting eq. (22.91), we can express $\dot{x}^\mu$ in terms of the conjugate momenta:

$$\dot{x}^\mu = g^{\mu\nu} p_\nu \,. \tag{22.92}$$

The *Hamiltonian* is a functional of the coordinate functions $x^\mu(\lambda)$ and of their conjugate momenta $p_\mu(\lambda)$, defined as

$$H(x^\mu, p_\nu) = p_\mu \, \dot{x}^\mu(p_\nu) - \mathcal{L}\left(x^\mu, \dot{x}^\mu(p_\nu)\right) \,. \tag{22.93}$$

Thus, in our case

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu \,. \tag{22.94}$$

Geodesic equations are equivalent to the Euler-Lagrange equations for the Lagrangian functional (22.90), which are equivalent to the Hamilton equations for the Hamiltonian functional:

$$\begin{aligned} \dot{x}^\mu &= \frac{\partial H}{\partial p_\mu} \\ \dot{p}_\mu &= -\frac{\partial H}{\partial x^\mu} \,. \end{aligned} \tag{22.95}$$

Solving eqs. (22.95) presents the same difficulties as solving Euler-Lagrange's equations. However, in the Hamilton-Jacobi approach, which we briefly recall, the further constant of motion emerges quite naturally.

In the Hamilton-Jacobi approach, we look for a function of the coordinates and of the curve parameter λ,

$$S = S(x^\mu, \lambda) \tag{22.96}$$

which is solution of the *Hamilton-Jacobi equation*

$$H\left(x^\mu, \frac{\partial S}{\partial x^\mu}\right) + \frac{\partial S}{\partial \lambda} = 0 \,. \tag{22.97}$$

In general such solution depends on four integration constants.

[2]Not to be confused with the four-momentum of the particle, which we denote with P^μ.

It can be shown that, if S is a solution of the Hamilton-Jacobi equation, then

$$\frac{\partial S}{\partial x^\mu} = p_\mu \,. \tag{22.98}$$

Therefore, once eq. (22.97) is solved, the expressions of the conjugate momenta (and of $\dot{x}^\mu$) follows in terms of the four constants, and allows to write the solutions of geodesic equations in a closed form, through integrals.

First of all, we can use what we already know, i.e.

$$\begin{aligned} H &= \frac{1}{2}g^{\mu\nu}p_\mu p_\nu = \frac{1}{2}\kappa \\ p_t &= -E \quad \text{constant} \\ p_\phi &= L \qquad \text{constant}\,. \end{aligned} \tag{22.99}$$

These conditions require that

$$S = -\frac{1}{2}\kappa\lambda - Et + L\phi + S^{(r\theta)}(r,\theta) \tag{22.100}$$

where $S^{(r\theta)}$ is a function of r and θ to be determined.

Furthermore, we look for a separable solution, by making the ansatz

$$S = -\frac{1}{2}\kappa\lambda - Et + L\phi + S^{(r)}(r) + S^{(\theta)}(\theta)\,. \tag{22.101}$$

Substituting (22.101) into the Hamilton-Jacobi equation (22.97), and using the expression (21.14) for the inverse metric, we find

$$\begin{aligned} &-\kappa + \frac{\Delta}{\Sigma}\left(\frac{dS^{(r)}}{dr}\right)^2 + \frac{1}{\Sigma}\left(\frac{dS^{(\theta)}}{d\theta}\right)^2 \\ &-\frac{1}{\Delta}\left[r^2 + a^2 + \frac{2Mra^2}{\Sigma}\sin^2\theta\right]E^2 + \frac{4Mra}{\Sigma\Delta}EL + \frac{\Delta - a^2\sin^2\theta}{\Sigma\Delta\sin^2\theta}L^2 = 0\,. \end{aligned} \tag{22.102}$$

Using the relation (21.26)

$$(r^2 + a^2) + \frac{2Mra^2}{\Sigma}\sin^2\theta = \frac{1}{\Sigma}\left[(r^2+a^2)^2 - a^2\sin^2\theta\Delta\right] \tag{22.103}$$

and multiplying by $\Sigma = r^2 + a^2\cos^2\theta$, we get

$$\begin{aligned} &-\kappa(r^2 + a^2\cos^2\theta) + \Delta\left(\frac{dS^{(r)}}{dr}\right)^2 + \left(\frac{dS^{(\theta)}}{d\theta}\right)^2 \\ &-\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]E^2 + \frac{4Mra}{\Delta}EL + \left(\frac{1}{\sin^2\theta} - \frac{a^2}{\Delta}\right)L^2 = 0 \end{aligned} \tag{22.104}$$

i.e.

$$\begin{aligned}
&\Delta\left(\frac{dS^{(r)}}{dr}\right)^2 - \kappa r^2 - \frac{(r^2+a^2)^2}{\Delta}E^2 + \frac{4Mra}{\Delta}EL - \frac{a^2}{\Delta}L^2 \\
= \;& -\left(\frac{dS^{(\theta)}}{d\theta}\right)^2 + \kappa a^2\cos^2\theta - a^2\sin^2\theta E^2 - \frac{1}{\sin^2\theta}L^2 \,.
\end{aligned} \tag{22.105}$$

We rearrange equation (22.105) by adding to both sides the constant quantity $a^2E^2 + L^2$:

$$\begin{aligned}
&\Delta\left(\frac{dS^{(r)}}{dr}\right)^2 - \kappa r^2 - \frac{(r^2+a^2)^2}{\Delta}E^2 + \frac{4Mra}{\Delta}EL - \frac{a^2}{\Delta}L^2 + a^2E^2 + L^2 \\
= \;& -\left(\frac{dS^{(\theta)}}{d\theta}\right)^2 + \kappa a^2\cos^2\theta + a^2\cos^2\theta E^2 - \frac{\cos^2\theta}{\sin^2\theta}L^2 \,.
\end{aligned} \tag{22.106}$$

In equation (22.106), the left-hand side does not depend on θ, and is equal to the right-hand side which does not depend on r; therefore, this quantity must be a constant $\mathcal{C}$:

$$\begin{aligned}
&\left(\frac{dS^{(\theta)}}{d\theta}\right)^2 - \cos^2\theta\left[(\kappa+E^2)a^2 - \frac{1}{\sin^2\theta}L^2\right] = \mathcal{C} \\
&\Delta\left(\frac{dS^{(r)}}{dr}\right)^2 - \kappa r^2 - \frac{(r^2+a^2)^2}{\Delta}E^2 + \frac{4Mra}{\Delta}EL - \frac{a^2}{\Delta}L^2 + E^2a^2 + L^2 \\
= \;& \Delta\left(\frac{dS^{(r)}}{dr}\right)^2 - \kappa r^2 + (L-aE)^2 - \frac{1}{\Delta}\left[E(r^2+a^2) - La\right]^2 = -\mathcal{C} \,.
\end{aligned} \tag{22.107}$$

Note that in rearranging the terms in the last two lines, we have used the relation

$$-2aLE + 2aLE\frac{r^2+a^2}{\Delta} = -\frac{4aMr}{\Delta}LE \,. \tag{22.108}$$

If we define the functions $R(r)$ and $\Theta(\theta)$ as

$$\begin{aligned}
\Theta(\theta) &\equiv \mathcal{C} + \cos^2\theta\left[(\kappa+E^2)a^2 - \frac{1}{\sin^2\theta}L^2\right] \\
R(r) &\equiv \Delta\left[-\mathcal{C} + \kappa r^2 - (L-aE)^2\right] + \left[E(r^2+a^2) - La\right]^2 \,,
\end{aligned} \tag{22.109}$$

then

$$\begin{aligned}
\left(\frac{dS^{(\theta)}}{d\theta}\right)^2 &= \Theta \\
\left(\frac{dS^{(r)}}{dr}\right)^2 &= \frac{R}{\Delta^2}
\end{aligned} \tag{22.110}$$

and the solution of the Hamilton-Jacobi equation has the form

$$S = -\frac{1}{2}\kappa\lambda - Et + L\phi + \int \frac{\sqrt{R}}{\Delta} dr + \int \sqrt{\Theta} d\theta \,. \tag{22.111}$$

Thus, the constant $\mathcal{C}$, which is called Carter's constant, from its discoverer B. Carter, emerges as a separation constant and characterize, together with E and L, geodetic motion in Kerr spacetime. We stress again that, unlike E and L, it is not associated to a spacetime symmetry.

Once we have the solution of the Hamilton-Jacobi equations, depending on four constants $(\kappa, E, L, \mathcal{C})$, it is possible to find the particle trajectory. Indeed, from (22.98) we know the expressions of the conjugate momenta

$$\begin{aligned} p_\theta^2 &= (\Sigma\dot{\theta})^2 = \Theta(\theta) \\ p_r^2 &= \left(\frac{\Sigma}{\Delta}\dot{r}\right)^2 = \frac{R(r)}{\Delta^2} \end{aligned} \tag{22.112}$$

therefore

$$\begin{aligned} \dot{\theta} &= \pm\frac{1}{\Sigma}\sqrt{\Theta} \\ \dot{r} &= \pm\frac{1}{\Sigma}\sqrt{R} \end{aligned} \tag{22.113}$$

which can be solved by numerical integration.