Chapter 21

The Kerr solution

As shown in Chapter 10, the solution of Einstein's equations describing the exterior of an isolated, spherically symmetric, static object is quite simple. Indeed, the Schwarzschild solution was found in 1916, immediatly after the derivation of Einstein's equation. Finding the solution describing a rotating body (all astrophysical objects do rotate!) is a much more difficult problem; indeed we do not know any analytic, exact solution describing the exterior of a rotating star, even though approximate solutions are known.

However, there exists an exact solution of Einstein's equations in vacuum ($T_{\mu\nu} = 0$), which describes a rotating, stationary, axially symmetric *black hole*. It was derived 1963 by R. Kerr, and it is known as the **Kerr solution**. This solution describes a black hole, because, as for a Scharzschild black hole, it describes the spacetime generated by a curvature singularity concealed by a *horizon*.

We stress that while, thanks to Birkoff's theorem, the Schwarzschild metric for $r > 2M$ describes the exterior of any spherically symmetric, static, isolated object (a star, a planet, a stone, etc.), the Kerr metric outside the horizon can only describe the exterior of a black hole.[1]

21.1 The Kerr metric in Boyer-Lindquist coordinates

The explicit form of the Kerr metric is the following:

$$\begin{aligned} ds^2 &= -dt^2 + \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) + (r^2+a^2)\sin^2\theta d\phi^2 \\ &\quad + \frac{2Mr}{\Sigma}(a\sin^2\theta d\phi - dt)^2 \end{aligned} \tag{21.1}$$

where

$$\begin{aligned} \Delta(r) &\equiv r^2 - 2Mr + a^2 \\ \Sigma(r,\theta) &\equiv r^2 + a^2\cos^2\theta\,. \end{aligned} \tag{21.2}$$

[1]Actually, there is no proof that it cannot exist a stellar model matching with Kerr metric at the surface of the star, but such a model has never been found.

The coordinates (t, r, θ, ϕ), in terms of which the metric has the form (21.1), are the *Boyer-Lindquist coordinates*.

The Kerr metric depends on two parameters, M and a; comparing (21.1) with the far field limit metric of an isolated object (20.3), we see that M is the black hole mass, and Ma its angular momentum.

Some properties of the Kerr metric can be deduced from the line element (21.1):

- It is not static: it is not invariant for time reversal $t \to -t$.
- It is *stationary*: it does not depend explicitly on time.
- It is *axisymmetric*: it does not depend explicitly on ϕ.
- It is invariant for simultaneous inversion of t and ϕ,

$$\begin{aligned} t &\to -t \\ \phi &\to -\phi\,; \end{aligned} \tag{21.3}$$

  this property follows from the fact that the time reversal of a rotating object implies an object which rotates in the opposite direction.

- In the limit $r \to \infty$, the Kerr metric (21.1) reduces to Minkowski's metric in polar coordinates; then, the Kerr spacetime is *asymptotically flat*.
- In the limit $a \to 0$ (with $M \neq 0$), $\Delta \to r^2 - 2Mr$ and $\Sigma \to r^2$, then (21.1) reduces to the Schwarzschild metric

$$ds^2 \to -(1 - 2M/r)dt^2 + (1 - 2M/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{21.4}$$

- In the limit $M \to 0$ (with $a \neq 0$), (21.1) reduces to

$$ds^2 = -dt^2 + \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2}dr^2 + (r^2 + a^2\cos^2\theta)d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2 \tag{21.5}$$

  which is the metric of flat spacetime in spheroidal coordinates:

$$\begin{aligned} x &= \sqrt{r^2 + a^2}\sin\theta\cos\phi \\ y &= \sqrt{r^2 + a^2}\sin\theta\sin\phi \\ z &= r\cos\theta\,. \end{aligned} \tag{21.6}$$

  Indeed,

$$\begin{aligned} dx &= \frac{r}{\sqrt{r^2 + a^2}}\sin\theta\cos\phi dr + \sqrt{r^2 + a^2}\cos\theta\cos\phi d\theta - \sqrt{r^2 + a^2}\sin\theta\sin\phi d\phi \\ dy &= \frac{r}{\sqrt{r^2 + a^2}}\sin\theta\sin\phi dr + \sqrt{r^2 + a^2}\cos\theta\sin\phi d\theta + \sqrt{r^2 + a^2}\sin\theta\cos\phi d\phi \\ dz &= \cos\theta dr - r\sin\theta d\theta \end{aligned} \tag{21.7}$$

thus

$$
\begin{aligned}
ds^2 = \quad & dx^2 + dy^2 + dz^2 = \left(\frac{r^2}{r^2+a^2}\sin^2\theta + \cos^2\theta\right) dr^2 \\
& + \left((r^2+a^2)\cos^2\theta + r^2\sin^2\theta\right) d\theta^2 + (r^2+a^2)\sin^2\theta d\phi^2 \\
& = \frac{r^2+a^2\cos^2\theta}{r^2+a^2} dr^2 + (r^2+a^2\cos^2\theta)d\theta^2 + (r^2+a^2)\sin^2\theta d\phi^2 \,.
\end{aligned}
\tag{21.8}
$$

- The metric (21.1) is singular for $\Delta = 0$ and for $\Sigma = 0$. The curvature invariants ($R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$, $R_{\mu\nu}R^{\mu\nu}$, etc.) are regular on $\Delta = 0$, and singular on $\Sigma = 0$. Thus, $\Delta = 0$ is a coordinate singularity, while $\Sigma = 0$ is a true singularity of the manifold.

  Note that in the Schwarzschild limit ($a = 0$), $\Sigma = r^2 = 0$ is the curvature singularity, while (for $r \neq 0$) $\Delta = r(r - 2M) = 0$ is the coordinate singolarity corresponding to the black hole horizon.

The metric has the form

$$
g_{\mu\nu} = \begin{pmatrix} g_{tt} & 0 & 0 & g_{t\phi} \\ 0 & \frac{\Sigma}{\Delta} & 0 & 0 \\ 0 & 0 & \Sigma & 0 \\ g_{t\phi} & 0 & 0 & g_{\phi\phi} \end{pmatrix}
\tag{21.9}
$$

with

$$
\begin{aligned}
g_{tt} &= -\left(1 - \frac{2Mr}{\Sigma}\right), \qquad g_{rr} = \frac{\Sigma}{\Delta} \\
g_{t\phi} &= -\frac{2Mr}{\Sigma} a \sin^2\theta, \qquad g_{\theta\theta} = \Sigma \\
g_{\phi\phi} &= \left[r^2 + a^2 + \frac{2Mra^2}{\Sigma}\sin^2\theta\right]\sin^2\theta \,.
\end{aligned}
\tag{21.10}
$$

To compute the inverse metric $g^{\mu\nu}$, we only need to invert the $t\phi$ block, while the inversion of the $r\theta$ part is trivial. The $t\phi$ block is

$$
\tilde{g}_{ab} = \begin{pmatrix} g_{tt} & g_{t\phi} \\ g_{t\phi} & g_{\phi\phi} \end{pmatrix}
\tag{21.11}
$$

and its determinant is

$$
\begin{aligned}
\tilde{g} &= g_{tt}g_{\phi\phi} - g_{t\phi}^2 \\
&= -\left(1 - \frac{2Mr}{\Sigma}\right)\left[r^2 + a^2 + \frac{2Mra^2}{\Sigma}\sin^2\theta\right]\sin^2\theta - \frac{4M^2r^2a^2}{\Sigma^2}\sin^4\theta \\
&= -\left[r^2 + a^2 + \frac{2Mra^2}{\Sigma}\sin^2\theta\right]\sin^2\theta + (r^2+a^2)\frac{2Mr}{\Sigma}\sin^2\theta \\
&= -(r^2+a^2)\sin^2\theta + \frac{2Mr}{\Sigma}\sin^2\theta\left[-a^2\sin^2\theta + r^2 + a^2\right] \\
&= -(r^2+a^2)\sin^2\theta + 2Mr\sin^2\theta = -\Delta\sin^2\theta
\end{aligned}
\tag{21.12}
$$

therefore

$$\tilde{g}^{ab} = -\frac{1}{\Delta \sin^2\theta} \begin{pmatrix} g_{\phi\phi} & -g_{t\phi} \\ -g_{t\phi} & g_{tt} \end{pmatrix} \tag{21.13}$$

and

$$g^{\mu\nu} = \begin{pmatrix} g^{tt} & 0 & 0 & g^{t\phi} \\ 0 & \frac{\Delta}{\Sigma} & 0 & 0 \\ 0 & 0 & \frac{1}{\Sigma} & 0 \\ g^{t\phi} & 0 & 0 & g^{\phi\phi} \end{pmatrix} \tag{21.14}$$

with

$$\begin{aligned} g^{tt} &= -\frac{1}{\Delta}\left[r^2 + a^2 + \frac{2Mra^2}{\Sigma}\sin^2\theta\right] \\ g^{t\phi} &= -\frac{2Mr}{\Sigma\Delta}a \\ g^{\phi\phi} &= \frac{\Delta - a^2\sin^2\theta}{\Sigma\Delta\sin^2\theta} \end{aligned} \tag{21.15}$$

where we have used the following equality

$$\frac{\Sigma - 2Mr}{\Sigma\Delta\sin^2\theta} = \frac{r^2 + a^2\cos^2\theta - 2Mr}{\Sigma\Delta\sin^2\theta} = \frac{\Delta - a^2\sin^2\theta}{\Sigma\Delta\sin^2\theta}\,. \tag{21.16}$$

21.2 Symmetries of the metric

Being stationary and axisymmetric, the Kerr metric admits two Killing vector fields:

$$\vec{k} \equiv \frac{\partial}{\partial t} \qquad \vec{m} \equiv \frac{\partial}{\partial \phi} \tag{21.17}$$

or equivalently, in coordinates (t, r, θ, ϕ),

$$k^\mu \equiv (1, 0, 0, 0) \qquad m^\mu \equiv (0, 0, 0, 1)\,. \tag{21.18}$$

As a consequence, there are two conserved quantities associated to test particles motion:

$$E \equiv -u^\mu k_\mu = -u_t \qquad L = u^\mu m_\mu = u_\phi\,, \tag{21.19}$$

where u^μ is the particle four-velocity. For massive particles, the four-momentum is $P^\mu = mu^\mu$, and the conserved quantities are the energy and the angular momentum per unit mass, measured at radial infinity. For massless particle, we can choose the affine parameter (as we will always do in the following) such that the four-momentum coincides with the four-velocity, i.e. $P^\mu = u^\mu$; thus, for massless particles E is the energy and L the angular momentum at infinity.

It can be shown that $\vec{k}$, $\vec{m}$ are the only Killing vector fields admitted by the Kerr metric; thus, any Killing vector field is a linear combination of them.

21.3 Frame dragging and ZAMO

Let us consider an observer, with timelike four-velocity u^μ, which falls toward the black hole from infinity, with zero angular momentum

$$L = u_\phi = 0\,. \tag{21.20}$$

Such observer is conventionally named ZAMO, which stands for "**zero angular momentum observer**". Eq. (21.20) implies that, since when $r \to \infty$ the metric becomes flat, also $u^\phi = \eta^{\phi\mu} u_\mu = 0$. Consequently the ZAMO angular velocity Ω, defined as

$$\Omega \equiv \frac{d\phi}{dt} = \frac{\frac{d\phi}{d\tau}}{\frac{dt}{d\tau}} = \frac{u^\phi}{u^t}, \tag{21.21}$$

in the limit $r \to \infty$ also vanishes. However, it vanishes only at infinity, since along the ZAMO's trajectory

$$u^\phi = g^{\phi t} u_t \neq 0 \qquad \longrightarrow \qquad \Omega \neq 0\,. \tag{21.22}$$

To compute Ω in terms of the metric (21.1), we use the condition

$$u_\phi = 0 = g_{\phi\phi} u^\phi + g_{\phi t} u^t\,, \tag{21.23}$$

thus, from the definition (21.21)

$$\Omega = \frac{u^\phi}{u^t} = -\frac{g_{\phi t}}{g_{\phi\phi}}\,. \tag{21.24}$$

We have

$$g_{\phi t} = -\frac{2Mra}{\Sigma}\sin^2\theta \tag{21.25}$$

and

$$\begin{aligned} g_{\phi\phi} &= (r^2+a^2)\sin^2\theta + \frac{2Mra^2\sin^4\theta}{\Sigma} \\ &= \frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2\cos^2\theta)(r^2+a^2) + 2Mra^2\sin^2\theta\right] \\ &= \frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)^2 - (r^2+a^2)a^2\sin^2\theta + 2Mra^2\sin^2\theta\right] \\ &= \frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)^2 - a^2\sin^2\theta\Delta\right] \end{aligned} \tag{21.26}$$

therefore the ZAMO angular velocity is

$$\boxed{\Omega = \frac{2Mar}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}\,.} \tag{21.27}$$

Note that, since

$$(r^2+a^2)^2 > a^2\sin^2\theta(r^2+a^2-2Mr)\,, \tag{21.28}$$

$\Omega/(Ma) > 0$ always; this means that the angular velocity has the same sign of the black hole angular momentum Ma, i.e., the ZAMO always **corotates** with the black hole.

Therefore, an observer which moves toward a Kerr black hole starting at radial infinity with zero angular momentum (which implies zero angular velocity) is *dragged* by the black hole gravitational, and acquires an angular velocity which forces the ZAMO to corotate with the black hole.

21.4 Black hole horizons

In this section we will show that the singularity on the surface $\Delta = 0$ exhibited by the Kerr metric in the Boyer-Lindquist coordinates is a coordinate singularity, which can be removed by an appropriate coordinate transformation. Furthermore, we shall show that $\Delta = 0$ identify the black hole horizons, and we shall discuss their structure.

21.4.1 How to remove the singularity at $\Delta = 0$

To show that $\Delta = 0$ is a coordinate singularity, we make a coordinate transformation that brings the metric into a form which is not singular at $\Delta = 0$; the new coordinates are called *Kerr coordinates*. Following the same procedure as in Chapter 10 for the Schwarzschild spacetime, we look for a family of null geodesics, and choose a coordinate system such that the null geodesics are coordinate lines in the new system. In the case of Kerr geometry, the spacetime cannot be decomposed in a product of two-dimensional manifolds, thus the study of null geodesics is more complex than in the Schwarzschild case. The Kerr metric admits two special families of null geodesics, named *principal null geodesics*, given by

$$u^\mu = \frac{dx^\mu}{d\lambda} = \left(\frac{dt}{d\lambda}, \frac{dr}{d\lambda}, \frac{d\theta}{d\lambda}, \frac{d\phi}{d\lambda}\right) = \left(\frac{r^2+a^2}{\Delta}, \pm 1, 0, \frac{a}{\Delta}\right) , \tag{21.29}$$

where the sign plus (minus) corresponds to outgoing (ingoing) geodesics. In the Schwarzschild limit these are the usual outgoing and ingoing geodesics (with $\frac{dr}{dt} = \pm\frac{r^2}{\Delta} = \pm\frac{r}{r-2M}$, see Section 11.7), but in the Kerr case they acquire an angular velocity $d\phi/d\lambda$ proportional to a and diverging when $\Delta = 0$.

We will not prove explicitly that (21.29) are geodesics; we only show that they are null, i.e. that:

$$g_{\mu\nu}u^\mu u^\nu = 0 . \tag{21.30}$$

We have

$$\begin{aligned} g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} &= -\left(\frac{dt}{d\lambda}\right)^2 + \Sigma\left(\frac{1}{\Delta}\left(\frac{dr}{d\lambda}\right)^2 + \left(\frac{d\theta}{d\lambda}\right)^2\right) \\ &\quad +(r^2+a^2)\sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2 + \frac{2Mr}{\Sigma}\left(a\sin^2\theta\frac{d\phi}{d\lambda} - \frac{dt}{d\lambda}\right)^2 . \end{aligned} \tag{21.31}$$

First, we notice that

$$\frac{dt}{d\lambda} - a\sin^2\theta\frac{d\phi}{d\lambda} = \frac{r^2+a^2-a^2\sin^2\theta}{\Delta} = \frac{\Sigma}{\Delta} . \tag{21.32}$$

Then,

$$\begin{aligned} g_{\mu\nu}u^\mu u^\nu &= -\frac{(r^2+a^2)^2}{\Delta^2} + \frac{\Sigma}{\Delta} + (r^2+a^2)\sin^2\theta\frac{a^2}{\Delta^2} + \frac{2Mr\Sigma}{\Delta^2} \\ &= \frac{1}{\Delta^2}\Big[-(r^2+a^2)(r^2+a^2) + (r^2+a^2\cos^2\theta)(r^2+a^2-2Mr) \\ &\quad +\sin^2\theta a^2(r^2+a^2) + (r^2+a^2\cos^2\theta)2Mr\Big] = 0 . \end{aligned} \tag{21.33}$$

Consequently, the tangent vector (21.29) is null.

Let us consider the ingoing geodesics, and indicate the tangent vector as l^μ

$$l^\mu = \left(\frac{r^2+a^2}{\Delta}, -1, 0, \frac{a}{\Delta}\right) ; \tag{21.34}$$

let us parametrize the geodesics in terms of r:

$$\frac{dt}{dr} = -\frac{r^2+a^2}{\Delta} \qquad \frac{d\phi}{dr} = -\frac{a}{\Delta} . \tag{21.35}$$

We want these geodesics to be coordinate lines of our new system; thus, one of our coordinates is r, while the others are quantities which are constant along each geodesic belonging to the family. One of these is θ; the remaining two coordinates are given by

$$\begin{aligned} v &\equiv t + T(r) \\ \bar{\phi} &\equiv \phi + \Phi(r) \end{aligned} \tag{21.36}$$

where $T(r)$ and $\Phi(r)$ are solutions of[2]

$$\begin{aligned} \frac{dT}{dr} &= \frac{r^2+a^2}{\Delta} \\ \frac{d\Phi}{dr} &= \frac{a}{\Delta} \end{aligned} \tag{21.37}$$

so that, along a geodesic of the family,

$$\frac{dv}{dr} = \frac{d\bar{\phi}}{dr} \equiv 0 \tag{21.38}$$

and the tangent vector of the ingoing principal null geodesics (21.34) is, in the new coordinates, simply

$$l^\mu = (0, +1, 0, 0) . \tag{21.39}$$

We can now compute the metric tensor in the coordinate system $(v, r, \theta, \bar{\phi})$. We recall that, in Boyer-Lindquist coordinates,

$$ds^2 = -dt^2 + \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) + (r^2+a^2)\sin^2\theta d\phi^2 + \frac{2Mr}{\Sigma}(a\sin^2\theta d\phi - dt)^2 . \tag{21.40}$$

We have

$$\begin{aligned} dv &= dt + \frac{r^2+a^2}{\Delta}dr \quad ; \quad dt = dv - \frac{r^2+a^2}{\Delta}dr \\ d\bar{\phi} &= d\phi + \frac{a}{\Delta}dr \quad ; \quad d\phi = d\bar{\phi} - \frac{a}{\Delta}dr , \end{aligned} \tag{21.41}$$

[2]Note that eqs. (21.37) have a unique solution; since the right-hand sides of (21.37) depend on r only, the only freedom consists in the choice of the origins of v and $\bar{\phi}$.

then

$$\begin{aligned} -dt^2 &= -dv^2 - \frac{(r^2+a^2)^2}{\Delta^2}dr^2 + 2\frac{r^2+a^2}{\Delta}dvdr \\ (r^2+a^2)\sin^2\theta d\phi^2 &= (r^2+a^2)\sin^2\theta d\bar{\phi}^2 + (r^2+a^2)\frac{a^2}{\Delta^2}\sin^2\theta dr^2 \\ &\quad -2(r^2+a^2)\frac{a}{\Delta}\sin^2\theta dr d\bar{\phi}\,, \end{aligned} \tag{21.42}$$

$\frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2$ does not change (r, θ are also coordinates in the new frame), the parenthesis in the last term of (21.40) reduces to

$$\begin{aligned} dt - a\sin^2\theta d\phi &= dv - a\sin^2\theta d\bar{\phi} - \frac{r^2+a^2-a^2\sin^2\theta}{\Delta}dr \\ &= dv - a\sin^2\theta d\bar{\phi} - \frac{\Sigma}{\Delta}dr\,, \end{aligned} \tag{21.43}$$

thus

$$\begin{aligned} \frac{2Mr}{\Sigma}(dt - a\sin^2\theta d\phi)^2 &= \frac{2Mr}{\Sigma}dv^2 + \frac{2Mr}{\Sigma}a^2\sin^4\theta d\bar{\phi}^2 \\ &\quad + \frac{2Mr\Sigma}{\Delta^2}dr^2 - \frac{4Mr}{\Sigma}a\sin^2\theta dv d\bar{\phi} \\ &\quad - \frac{4Mr}{\Delta}dvdr + \frac{4Mra\sin^2\theta}{\Delta}d\bar{\phi}dr \end{aligned} \tag{21.44}$$

and, collecting all terms, we find

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2Mr}{\Sigma}\right)dv^2 + 2dvdr + \Sigma d\theta^2 \\ &\quad + \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\Sigma}\sin^2\theta d\bar{\phi}^2 \\ &\quad -2a\sin^2\theta dr d\bar{\phi} - \frac{4Mra}{\Sigma}\sin^2\theta dv d\bar{\phi}\,. \end{aligned} \tag{21.45}$$

The coordinates $(v, r, \theta, \bar{\phi})$ are the **Kerr coordinates**. In this frame, the metric is not singular on $\Delta = 0$. This means that, while the Boyer-Lindquist coordinates are defined in all spacetime except the submanifolds $\Delta = 0$ and $\Sigma = 0$ [3], the Kerr coordinates can also be defined in such submanifold. Then, after changing coordinates to the Kerr frame, we *extend* the manifold, to include the submanifold $\Delta = 0$.

We note, for later use, that being

$$g_{vr} = 1 \qquad g_{rr} = g_{\theta r} = 0 \qquad g_{\bar{\phi}r} = -a\sin^2\theta\,, \tag{21.46}$$

we find

$$l_\mu = (1, 0, 0, -a\sin^2\theta)\,. \tag{21.47}$$

[3]To be precise, one should subtract also the extrema of the domain of angular coordinates, $\theta = 0, \pi$, $\phi = 0, 2\pi$, as usual when polar-like coordinates are considered. Anyway, the related "pathologies" are much easier to cure, by simple coordinate redefinitions, so we will limit our analysis to the "pathologies" at $\Delta = 0$ and $\Sigma = 0$.

Notice also that

$$
\begin{aligned}
& \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \sin^2\theta \\
& = \frac{1}{\Sigma}\left[(r^2+a^2)^2 - (r^2+a^2-2Mr)a^2\sin^2\theta\right]\sin^2\theta \\
= \; & \frac{r^2+a^2}{\Sigma}(r^2+a^2-a^2\sin^2\theta)\sin^2\theta + \frac{2Mr}{\Sigma}a^2\sin^4\theta \\
& = (r^2+a^2)\sin^2\theta + \frac{2Mr}{\Sigma}a^2\sin^4\theta
\end{aligned} \tag{21.48}
$$

and

$$
\frac{2Mr}{\Sigma}(dv - a\sin^2\theta d\bar{\phi})^2 = \frac{2Mr}{\Sigma}\left[dv^2 + a^2\sin^4\theta d\bar{\phi}^2 - 2a\sin^2\theta dv d\bar{\phi}\right] \tag{21.49}
$$

therefore the metric in Kerr coordinates can also be written in the simpler form

$$
\begin{aligned}
ds^2 &= -dv^2 + 2dvdr + \Sigma d\theta^2 + (r^2+a^2)\sin^2\theta d\bar{\phi}^2 - 2a\sin^2\theta dr d\bar{\phi} \\
&\quad + \frac{2Mr}{\Sigma}(dv - a\sin^2\theta d\bar{\phi})^2 .
\end{aligned} \tag{21.50}
$$

If we want an explicit time coordinate, we can define

$$
\bar{t} \equiv v - r \tag{21.51}
$$

so that the metric (21.50) becomes

$$
\begin{aligned}
ds^2 &= -d\bar{t}^2 + dr^2 + \Sigma d\theta^2 + (r^2+a^2)\sin^2\theta d\bar{\phi}^2 - 2a\sin^2\theta dr d\bar{\phi} \\
&\quad + \frac{2Mr}{\Sigma}(d\bar{t} + dr - a\sin^2\theta d\bar{\phi})^2 .
\end{aligned} \tag{21.52}
$$

21.4.2 Horizon structure

We shall now study the submanifold

$$
\Delta = r^2 + a^2 - 2Mr = 0 \,, \tag{21.53}
$$

where the Kerr metric, written in the Boyer-Lindquist coordinates, has a coordinate singularity

$$
ds^2 = -dt^2 + \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right) + (r^2+a^2)\sin^2\theta d\phi^2 + \frac{2Mr}{\Sigma}(a\sin^2\theta d\phi - dt)^2 . \tag{21.54}
$$

Writing Δ in the form

$$
\Delta(r) = (r-r_+)(r-r_-), \tag{21.55}
$$

with

$$
\begin{aligned}
r_+ &\equiv M + \sqrt{M^2-a^2} \\
r_- &\equiv M - \sqrt{M^2-a^2}
\end{aligned} \tag{21.56}
$$

solutions of Eq. (21.53), the surfaces where there is a coordinate singularity are then $r = r_+$ and $r = r_-$.

When $a^2 > M^2$, eq. (21.53) has no real solution, and the Kerr metric does not describe a black hole. Indeed, in this case there is no horizon concealing the singularity at $\Sigma = 0$ and the singularity is said "naked". [4] It should be mentioned that numerical simulation of astrophysical processes leading to black hole formation show that the final object cannot have $a > M$. In addition, theoretical studies on the mathematical structure of spacetime indicate that when $a^2 > M^2$ there are problems which would be too difficult to explain here. Thus, in general, the solution with $a > M$ is considered unphysical. However, we remark that this is still an open issue. To hereafter, we will restrict our analysis to the case

$$a^2 \leq M^2. \tag{21.57}$$

The limiting case $a^2 = M^2$ is called *extremal black hole.*

In section 11.5 we showed how to establishing whether a hypersurface is spacelike, timelike or null by studying the norm of it normal unit vector. We briefly recall the results. If for instance we consider a family of hypersurfaces $\Theta \equiv r - constant = 0$, being

$$n_\mu = \Theta_{,\mu} = (0, 1, 0, 0)\,, \tag{21.58}$$

we have that

- if $n_\mu n^\mu < 0$, the hypersurface is spacelike, and can be crossed by physical objects only in one direction.
- If $n_\mu n^\mu > 0$, the hypersurface is timelike, and can be crossed in both direction.
- If $n_\mu n^\mu = 0$, the hypersurface is null, and can be crossed only in one direction.

Null hypersurfaces separate regions of spacetime where $r = const$ are timelike hypersurfaces, from regions where $r = const$ are spacelike hypersurfaces; therefore, an object crossing a null hypersurface $r = const$ can never go back, and for this reason null hypersurfaces are called **horizons**.

From eqs. (21.58) and (21.14) we find

$$n_\mu n^\mu = n_\mu n_\nu g^{\mu\nu} = g^{rr} = \frac{\Delta}{\Sigma}\,. \tag{21.59}$$

Thus, the vector normal to the surfaces $r = r_+$ and $r = r_-$, where $\Delta = 0$, is a null vector, $n_\mu n^\mu = 0$. Consequently these surfaces are *null hypersurfaces*, and a Kerr black hole admits two horizons. Since $r_+ > r_-$, we call $r = r_+$ the *outer horizon*, and $r = r_-$ the *inner horizon.*

The two horizons separate the spacetime in three regions:

I. $r > r_+$. Here the $r = const.$ hypersurfaces are timelike. The asymptotic region $r \to \infty$, where the metric becomes flat, is in this region, which can be considered as the black hole exterior.

[4] According to Roger Penrose's cosmic censorship conjecture, naked singularity cannot exist.

II. $r_- < r < r_+$. Here the $r = const.$ hypersurfaces are spacelike. An object which falls inside the outer horizon, can only continue falling to decreasing values of r, until it reaches the inner horizon and pass to region III.

III. $r < r_-$. Here the $r = const.$ hypersurfaces are timelike. This region contains the singularity, which will be studied in section 21.6.

In the case of extremal black holes, when $a^2 = M^2$, the two horizons coincide, and region II disappears.

If we consider the outer horizon r_+ as a sort of "black hole surface" then we could conventionally consider the angular velocity of an observer which falls radially from infinity - i.e., an observer with zero angular momentum, or ZAMO - as a sort of "black hole angular velocity". The ZAMO's angular velocity is given by (21.27):

$$\Omega = \frac{d\phi}{dt} = \frac{2Mar}{(r^2+a^2)^2 - a^2\Delta\sin^2\theta} . \tag{21.60}$$

At $r = r_+$, $\Delta = 0$ thus

$$\Omega = \frac{2Mar_+}{(r_+^2+a^2)^2} \equiv \Omega_H \tag{21.61}$$

which is a constant, i.e. it does not depend on θ and ϕ. In this sense, we can say that a black hole *rotates rigidly*.

The quantity $\Omega_H = \Omega(r_+)$ can be expressed in a simpler form. Since $\Delta = 0$ on r_+

$$r_+^2 + a^2 = 2Mr_+ \tag{21.62}$$

and

$$\boxed{\Omega_H = \frac{a}{2Mr_+} = \frac{a}{r_+^2+a^2}} . \tag{21.63}$$

21.5 The infinite redshift surface and the ergosphere

While in Schwarzschild's spacetime the horizon is also the surface where g_{tt} changes sign, in Kerr spacetime these surfaces do not coincide. Indeed

$$\begin{aligned} g_{tt} &= -1 + \frac{2Mr}{\Sigma} = -\frac{1}{\Sigma}\left(r^2 - 2Mr + a^2\cos^2\theta\right) \\ &= -\frac{1}{\Sigma}(r - r_{S+})(r - r_{S-}) = 0 \ , \end{aligned} \tag{21.64}$$

where

$$r_{S\pm} \equiv M \pm \sqrt{M^2 - a^2\cos^2\theta} . \tag{21.65}$$

These surfaces are called *infinite redshift surfaces*, because, as discussed in section 12.5, if a source located at a point P_{em} near the black hole emits a light signal with frequency ν_{em}, it will be observed at infinity with frequency

$$\nu_{obs} = \sqrt{\frac{g_{tt}(P_{em})}{g_{tt}(P_{obs})}}\nu_{em} \ ; \tag{21.66}$$

thus, if at P_{em} $g_{tt} = 0$, $\nu_{obs} = 0$ and the redshift is infinite.

Since the coefficient of r^2 in eq. (21.64) is negative, $g_{tt} < 0$ outside $[r_{S_-}, r_{S_+}]$, and $g_{tt} > 0$ inside that interval. In addition, being $\sqrt{M^2 - a^2\cos^2\theta} > \sqrt{M^2 - a^2}$, the horizons, which is located at

$$r_\pm = M \pm \sqrt{M^2 - a^2}\,, \tag{21.67}$$

are inside the interval $[r_{S_-}, r_{S_+}]$:

$$r_{S-} \leq r_- < r_+ \leq r_{S+}\,. \tag{21.68}$$

They coincide at $\theta = 0, \pi$, i.e. on the symmetry axis, while at the equatorial plan $r_{S+} = 2M$

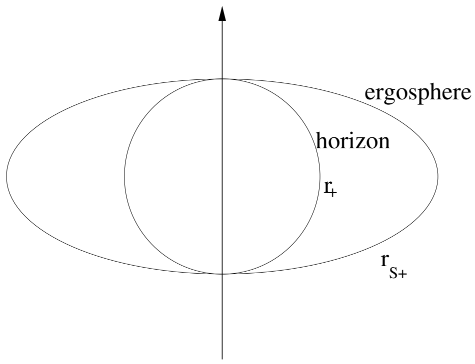

Figure 21.1: The ergosphere and the outer horizon

and $r_{S-} = 0$.

Therefore, there is a region *outside the outer horizon* where $g_{tt} > 0$ [5]. This region, i.e.

$$r_+ < r < r_{S+} \tag{21.69}$$

is said *ergoregion*, and its outer boundary, the surface $r = r_{S+}$, is said *ergosphere*. Note that, being the ergoregion outside the outer horizon, an object arriving from infinity may cross the ergosphere, enter in the ergoregion, and then cross the ergosphere in the opposite direction to return at infinity.

In the ergoregion the killing vector $k^\mu = (1, 0, 0, 0)$ becomes spacelike:

$$k^\mu k^\nu g_{\mu\nu} = g_{tt} > 0\,. \tag{21.70}$$

21.5.1 Static and stationary observers

We define *static observer*, an observer moving on a timelike worldline, with tangent vector (i.e. with four-velocity) proportional to k^μ. Remember that on the worldlines of k^μ the coordinates r, θ, ϕ are constant. Since inside the ergosphere k^μ becomes spacelike, in that region a static observer *cannot exist*. In other words, an observer inside the ergosphere cannot stay still, but is forced to move.

[5] This does not happen in Schwarzschild spacetime, where $g_{tt} > 0$ only inside the horizon

A *stationary observer* is one who does not see the metric changing while he is moving. Then, its tangent vector must be a Killing vector, i.e. it must be a combination of the two Killing vectors of the Kerr metric, $k^\mu = \partial/\partial t$ and $m^\mu = \partial/\partial\phi$:

$$u^\mu = \frac{k^\mu + \omega m^\mu}{|\vec{k} + \omega\vec{m}|} = (u^t, 0, 0, u^\phi) = u^t(1, 0, 0, \omega) \tag{21.71}$$

where ω is the observer angular velocity

$$\omega \equiv \frac{d\phi}{dt} = \frac{u^\phi}{u^t} \,. \tag{21.72}$$

Said differently, the worldline of a stationary observer has constant r and θ. He can only move along circles, with angular velocity ω, since on such orbits it does not see the metric changing, being the spacetime axially symmetric.

A stationary observer can exist provided

$$u^\mu u^\nu g_{\mu\nu} = (u^t)^2 \left[g_{tt} + 2\omega g_{t\phi} + \omega^2 g_{\phi\phi}\right] = -1 \ , \tag{21.73}$$

i.e.

$$\omega^2 g_{\phi\phi} + 2\omega g_{t\phi} + g_{tt} < 0 \,. \tag{21.74}$$

To solve (21.74), let us consider the equation

$$\omega^2 g_{\phi\phi} + 2\omega g_{t\phi} + g_{tt} = 0 \ , \tag{21.75}$$

with solutions

$$\omega_\pm = \frac{-g_{t\phi} \pm \sqrt{g_{t\phi}^2 - g_{tt}g_{\phi\phi}}}{g_{\phi\phi}} \,. \tag{21.76}$$

The discriminant is $g_{t\phi}^2 - g_{tt}g_{\phi\phi}$, which is the opposite of the determinant $\tilde{g}$ we computed in eq. (21.12). Thus, using that result we find

$$g_{t\phi}^2 - g_{tt}g_{\phi\phi} = \sin^2\theta[r^2 + a^2 - 2Mr] = \Delta\sin^2\theta \,. \tag{21.77}$$

From this equation we see that a stationary observer cannot exist when $\Delta < 0$

$$r_- < r < r_+ \ ,$$

i.e., no stationary observer can exist in the region between the two horizons.

Being (see eq. (21.28))

$$g_{\phi\phi} = \frac{\sin^2\theta}{\Sigma}[(r^2 + a^2)^2 - a^2\sin^2\theta\Delta] > 0 \,, \tag{21.78}$$

the coefficient of ω^2 in eq. (21.74) is positive, and the inequality (21.74) is satisfied, outside the outer horizon (where $r > r_+$, so that $\Delta > 0$ and then $\omega_- < \omega_+$), for

$$\omega_- \leq \omega \leq \omega_+ \,. \tag{21.79}$$

Thus a stationary observer must have an angular velocity in this range.

Note that, on the outer horizon $r = r_+$, $\Delta = 0$ and $\omega_- = \omega_+$; therefore eq. (21.74) has coincident solutions

$$\omega = -\frac{g_{t\phi}}{g_{\phi\phi}} = \Omega_H \,. \tag{21.80}$$

Since Ω_H is the angular velocity of a ZAMO observer moving on the outer horizon, eq. (21.80) shows that the only stationary observer who can move on the outer horizon is the ZAMO. This is another reason why the ZAMO's angular velocity on the outer horizon is considered the black hole angular velocity.

On the infinite redshift surface, $g_{tt} = 0$ so (being $g_{t\phi} < 0$)

$$\omega_- = \frac{-g_{t\phi} - \sqrt{g_{t\phi}^2}}{g_{\phi\phi}} = 0\,. \tag{21.81}$$

Outside the ergosphere, i.e. for $r \geq r_{S_+}$,

$$g_{tt} < 0, \qquad g_{\phi\phi} > 0, \qquad \text{therefore} \qquad \omega_- < 0 \quad \text{and} \quad \omega_+ > 0\,. \tag{21.82}$$

Thus, otside the ergosphere a stationary observer can be both co-rotating and counter-rotating with the black hole. Conversely, in the ergoregion, where $r_+ < r < r_{S_+}$,

$$g_{tt} > 0, \qquad g_{\phi\phi} > 0, \qquad \text{therefore} \qquad \omega_- > 0 \quad \text{and} \quad \omega_+ > 0\,. \tag{21.83}$$

Thus, inside the ergoregion a stationary observer can exist only if he corotates with the black hole.

21.6 The singularity of the Kerr metric

The Kerr metric is singular on the surface

$$\Sigma = r^2 + a^2\cos^2\theta = 0\,, \tag{21.84}$$

i.e. for $r = 0$ and $\theta = \pi/2$. For the Schwarzschild spacetime, where the coordinates t, r, θ, ϕ were interpreted as spherical polar coordinates, the curvature singularity was at $r = 0$ for any value of the angular coordinates. If we interpret the Boyer-Lindquist coordinates t, r, θ, ϕ as spherical polar coordinates, the singularity at $r = 0, \theta = \pi/2$ is in a quite strange location!

21.6.1 The Kerr-Schild coordinates

In order to understand the singularity structure, we now change coordinate frame, to the so-called Kerr-Schild coordinates, which are well defined in $r = 0$. Let us start with the metric in Kerr coordinates $(\bar{t}, r, \theta, \bar{\phi})$, given in eq. (21.52):

$$\begin{aligned} ds^2 &= -d\bar{t}^2 + dr^2 + \Sigma d\theta^2 + (r^2 + a^2)\sin^2\theta d\bar{\phi}^2 - 2a\sin^2\theta dr d\bar{\phi} \\ &\quad + \frac{2Mr}{\Sigma}(d\bar{t} + dr - a\sin^2\theta d\bar{\phi})^2\,. \end{aligned} \tag{21.85}$$

The Kerr-Schild coordinates $(\bar{t}, x, y, z)$ are a cartesian frame defined by

$$\begin{aligned} x &= \sqrt{r^2+a^2}\sin\theta\cos\left(\bar{\phi}+\arctan\frac{a}{r}\right) \\ y &= \sqrt{r^2+a^2}\sin\theta\sin\left(\bar{\phi}+\arctan\frac{a}{r}\right) \\ z &= r\cos\theta\,. \end{aligned} \tag{21.86}$$

The transformation of the metric in Kerr-Schild coordinates will be derived in the next section; here we use this coordinate frame to give a picture of the structure of Kerr spacetime near the singularity.

We have

$$\begin{aligned} x^2+y^2 &= (r^2+a^2)\sin^2\theta \\ z^2 &= r^2\cos^2\theta \end{aligned} \tag{21.87}$$

thus

$$\frac{x^2+y^2}{r^2+a^2}+\frac{z^2}{r^2}=1\,, \tag{21.88}$$

then the surfaces with constant r are ellipsoids (Figure 21.2), and

$$\frac{x^2+y^2}{a^2\sin^2\theta}-\frac{z^2}{a^2\cos^2\theta}=1\,, \tag{21.89}$$

then the surfaces with constant θ are half-hyperboloids (Figure 21.3). In Figures 21.2, 21.3

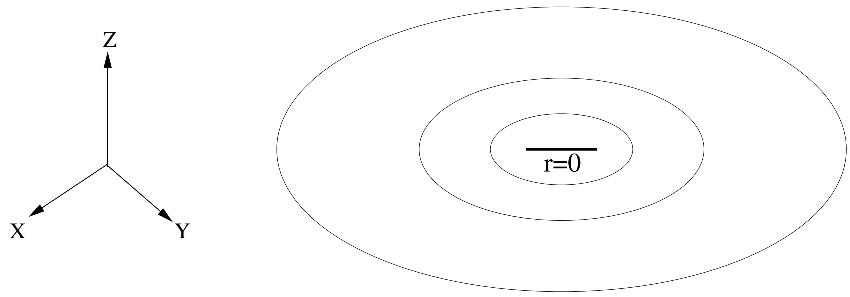

Figure 21.2: $r = const$ ellipsoidal surfaces in the Kerr-Schild frame; the thick line represents the $r = 0$ disk.

we show the $r = const$, $\theta = const$ surfaces in the Kerr-Schild $(\bar{t}, x, y, z)$ frame. This means that x, y, z are represented as Euclidean coordinates, and r, θ are considered as functions of x, y, z.

If we consider the Kerr spacetime for r sufficiently large, the r, θ coordinates behave like ordinary polar coordinates. But near the black hole their nature changes: $r = 0$ is not a single point but a disk,

$$x^2+y^2\le a^2\,, \qquad z=0 \tag{21.90}$$

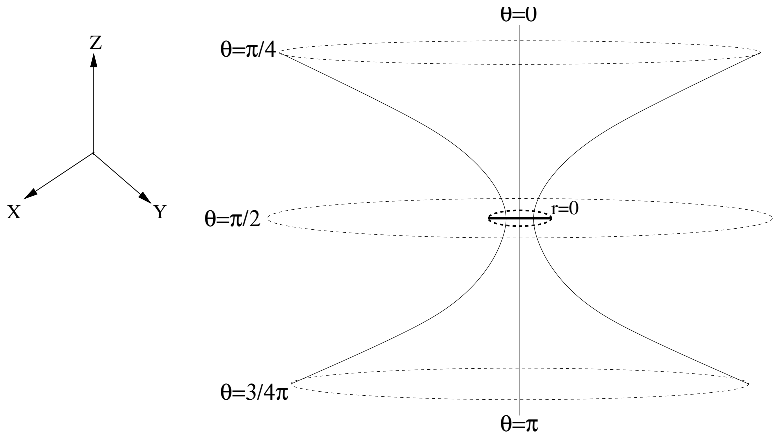

Figure 21.3: $\theta = const$ half-hyperboloidal surfaces in the Kerr-Schild frame; the thick ring represents the $r = 0, \theta = \pi/2$ singularity.

and this disk is parametrized by the coordinate θ. In particular,

$$r = 0 \qquad \theta = \frac{\pi}{2} \tag{21.91}$$

corresponds to the *ring*

$$x^2 + y^2 = a^2\,, \qquad z = 0\,. \tag{21.92}$$

This is the strucure of the singularity of the Kerr metric: it is a ring singularity. Inside the ring, the metric is perfectly regular.

21.6.2 The metric in Kerr-Schild coordinates

By introducing $\alpha = \arctan a/r$, we have

$$r^2 \sin^2 \alpha = a^2 \cos^2 \alpha \tag{21.93}$$

thus

$$\begin{aligned} r^2 &= (r^2 + a^2) \cos^2 \alpha \\ a^2 &= (r^2 + a^2) \sin^2 \alpha \end{aligned} \tag{21.94}$$

and, rewriting (21.86) as

$$\begin{aligned} x &= \sin\theta\sqrt{r^2 + a^2}(\cos\bar{\phi}\cos\alpha - \sin\bar{\phi}\sin\alpha) \\ y &= \sin\theta\sqrt{r^2 + a^2}(\sin\bar{\phi}\cos\alpha + \cos\bar{\phi}\sin\alpha) \\ z &= r\cos\theta \end{aligned} \tag{21.95}$$

and substituting (21.94) we have

$$\begin{aligned} x &= \sin\theta(r\cos\bar{\phi} - a\sin\bar{\phi}) \\ y &= \sin\theta(r\sin\bar{\phi} + a\cos\bar{\phi}) \\ z &= r\cos\theta\,. \end{aligned} \tag{21.96}$$

Differentiating,

$$\begin{aligned} dx &= \cos\theta(r\cos\bar\phi - a\sin\bar\phi)d\theta + \sin\theta\cos\bar\phi dr - \sin\theta(r\sin\bar\phi + a\cos\bar\phi)d\bar\phi \\ dy &= \cos\theta(r\sin\bar\phi + a\cos\bar\phi)d\theta + \sin\theta\sin\bar\phi dr + \sin\theta(r\cos\bar\phi - a\sin\bar\phi)d\bar\phi \\ dz &= -r\sin\theta d\theta + \cos\theta dr \end{aligned} \tag{21.97}$$

thus

$$\begin{aligned} dx^2 + dy^2 + dz^2 &= dr^2 + \left(r^2\sin^2\theta + (r^2+a^2)\cos^2\theta\right)d\theta^2 \\ &\quad + (r^2+a^2)\sin^2\theta d\bar\phi^2 - 2\sin^2\theta a dr d\bar\phi \\ &= dr^2 + \Sigma d\theta^2 + (r^2+a^2)\sin^2\theta d\bar\phi^2 - 2a\sin^2\theta dr d\bar\phi\,. \end{aligned} \tag{21.98}$$

Then, the metric (21.85) is the Minkowski metric plus the term

$$\frac{2Mr}{\Sigma}(d\bar t + dr - a\sin^2\theta d\bar\phi)^2\,. \tag{21.99}$$

Being

$$\Sigma = r^2 + a^2\cos^2\theta = r^2 + \frac{a^2z^2}{r^2}\,, \tag{21.100}$$

the factor $2Mr/\Sigma$ is easily expressed in Kerr-Schild coordinates:

$$\frac{2Mr}{\Sigma} = \frac{2Mr^3}{r^4 + a^2z^2}\,. \tag{21.101}$$

The one-form $d\bar t + dr - a\sin^2\theta d\bar\phi$ is more complicate to transform. We will prove that

$$d\bar t + dr - a\sin^2\theta d\bar\phi = d\bar t + \frac{r(xdx + ydy) - a(xdy - ydx)}{r^2 + a^2} + \frac{zdz}{r}\,. \tag{21.102}$$

First of all, let us express the differentials (21.97) as

$$\begin{aligned} dx &= \frac{\cos\theta}{\sin\theta}xd\theta + \sin\theta\cos\bar\phi dr - yd\bar\phi \\ dy &= \frac{\cos\theta}{\sin\theta}yd\theta + \sin\theta\sin\bar\phi dr + xd\bar\phi \\ dz &= -r\sin\theta d\theta + \cos\theta dr\,. \end{aligned} \tag{21.103}$$

We have

$$\begin{aligned} xdx + ydy &= \frac{\cos\theta}{\sin\theta}(x^2+y^2)d\theta + \sin\theta(x\cos\bar\phi + y\sin\bar\phi)dr \\ &= \sin\theta\cos\theta(r^2+a^2)d\theta + \sin^2\theta r dr \end{aligned} \tag{21.104}$$

$$\begin{aligned} ydx - xdy &= -(x^2+y^2)d\bar\phi + \sin\theta(y\cos\bar\phi - x\sin\bar\phi)dr \\ &= -(r^2+a^2)\sin^2\theta d\bar\phi + \sin^2\theta a dr \end{aligned} \tag{21.105}$$

$$zdz = -r^2\sin\theta\cos\theta d\theta + r\cos^2\theta dr \tag{21.106}$$

then

$$
\begin{aligned}
& (xdx+ydy)\frac{r}{r^2+a^2}+(ydx-xdy)\frac{a}{r^2+a^2}+\frac{zdz}{r} \\
= & \left(r\sin\theta\cos\theta d\theta+\frac{r^2}{r^2+a^2}\sin^2\theta dr\right) \\
& +\left(-a\sin^2\theta d\bar{\phi}+\frac{a^2}{r^2+a^2}\sin^2\theta dr\right) \\
& +\left(-r\sin\theta\cos\theta d\theta+\cos^2\theta dr\right) \\
= & \; dr-a\sin^2\theta d\bar{\phi}
\end{aligned}
\tag{21.107}
$$

which proves (21.102). The metric in Kerr-Schild coordinates is then

$$
\begin{aligned}
ds^2 = & -d\bar{t}^2+dx^2+dy^2+dz^2 \\
& +\frac{2Mr^3}{r^4+a^2z^2}\left[d\bar{t}+\frac{r(xdx+ydy)-a(xdy-ydx)}{r^2+a^2}+\frac{zdz}{r}\right]^2 .
\end{aligned}
\tag{21.108}
$$

Note that the metric has the form

$$g_{\mu\nu}=\eta_{\mu\nu}+Hl_\mu l_\nu \tag{21.109}$$

with

$$H\equiv\frac{2Mr^3}{r^4+a^2z^2} \tag{21.110}$$

and, in Kerr-Schild coordinates,

$$l_\mu dx^\mu=d\bar{t}+\frac{r(xdx+ydy)-a(xdy-ydx)}{r^2+a^2}+\frac{zdz}{r} \tag{21.111}$$

while in Kerr coordinates

$$l_\alpha dx^\alpha=d\bar{t}+dr-a\sin^2\theta d\bar{\phi}=dv-a\sin^2\theta d\bar{\phi} \tag{21.112}$$

thus l_μ is exactly the null vector (21.47), i.e. the generator of the principal null geodesics which have been used to define the Kerr coordinates.

21.7 General black hole solutions

In general, we can define a black hole as an asymptotically flat solution of Einstein's equations in vacuum, curvature singularity concealed by the horizon. Black holes form in the gravitational collapse of stars, if they are sufficiently massive.

When a black hole forms in a gravitational collapse, since gravitational waves emission and other dissipative processes damp its violent oscillations, we can expect that, after some time, it settles down to a stationary state. Thus, **stationary** black holes are considered the final outcome of gravitational collapse.

There are some remarkable theorems on stationary black holes, derived by S. Hawking, W. Israel, B. Carter, which prove the following:

- A stationary black hole is axially symmetric (while, as we know from Birkoff's theorem, a static black hole is spherically symmetric).
- Any stationary, axially symmetric black hole, with no electric charge, is described by the Kerr solution.
- Any stationary, axially symmetric black hole described by the so-called **Kerr-Newman solution**, which is the generalization of the Kerr solution with nonvanishing electric charge, is characterized by only three parameters: the mass M, the angular momentum aM, and the charge Q.

  All other features the star possessed before collapsing, such as a particular structure of the magnetic field, mountains, matter current, differential rotations etc, disappear in the final black hole which forms. This result has been summarized with the sentence: "**A black hole has no hair**", and for this reason the unicity theorems are also called *no hair theorems*.