Chapter 17

Neutron Stars

When a star reaches the end of its thermonuclear evolution, which terminates with the formation of the heavier element that its progenitor mass allows to form, the internal pressure can no longer sustain the gravitational attraction, and the star collapses. Current theories of stellar evolution show that if the progenitor mass is $\lessapprox 8\ M_{\odot}$ the collapse proceeds until it is halted by the pressure arising from electron degeneracy, as explained in Chapter 16; then, if the progenitor mass is sufficiently high to ignite at least hydrogen burning, a white dwarf can form. They are usually composed of carbon and oxygen. If the progenitor has a mass in the range $\sim 8 - 10\ M_{\odot}$ oxygen-neon-magnesium stars can form, but they are rare.

White dwarfs are necessarily less massive than the Chandrasekhar limit, $M_{CH} \sim 1.4\ M_{\odot}$: the outer layers of matter are expelled by the progenitor star near the end of the nuclear burning process, and create a planetary nebula surrounding the hot core which collapses to form the white dwarf. The number of white dwarfs in the Galaxy is expected to be of the order of 10^{10}.

If the mass of the progenitor is in the range $\sim 8M_{\odot} < M < 20 - 30M_{\odot}$ the evolutionary path is different. Nuclear processes are able to burn elements heavier than carbon and oxygen, and exothermic nuclear reactions can proceed all the way to ^{56}Fe, which is the most stable element in nature; indeed, no element heavier than ^{56}Fe can be generated by fusion of lighter elements through exothermic reactions.

The process which produces ^{56}Fe starts with silicon burning, and goes this way:

$$^{28}Si +^{28} Si \to^{56} Ni + \gamma, \tag{17.1}$$
$$^{56}Ni \to^{56} Co + e^{+} + \nu_e, \tag{17.2}$$
$$^{56}Co \to^{56} Fe + e^{+} + \nu_e. \tag{17.3}$$

It should be reminded that the atomic number Z (number of protons), and the mass number A (number of protons + neutrons) of the elements involved in the process are[1]:

- Si, $Z = 14$, $A = 28,0855 \sim 28$
- Ni, $Z = 28$, $A = 58,6934 \sim 59$

[1]The digits appearing in the mass numbers are due to the fact that in the stable state a gas of a given element is a mixture of isotopes of that element.

- Co, $Z = 27$, $A = 59,9332 \sim 60$
- Fe, $Z = 26$, $A = 55,845 \sim 56$

element. Note that the ^{56}Ni which forms in the reaction (17.1) has tree neutrons less than required for stability, therefore, it decays as indicated in the reaction (17.2), forming ^{56}Co; this is again unstable, being in defect of 4 neutrons, and decays forming ^{56}Fe, which is stable.

In addition, as the core density increases, the inverse β-decay process, through which electrons are captured by protons forming neutrons and neutrinos

$$e^- + p \rightarrow n + \nu_e, \tag{17.4}$$

becomes more and more efficient and nuclei richer of neutrons than ^{56}Fe can form, like ^{118}Kr. However, they are formed through endothermic reactions and subtract energy to the star. Elements heavier than kripton can form during the subsequent phase of collapse as explained below.

As the iron core forms the pressure provided by nuclear burning is able to maintain the star in equilibrium. However, there are several processes which tend to destabilize the star:

- a large number of neutrinos are produced both in the silicon burning reactions (17.1)-(17.3) and by the inverse β-decay process. Since neutrinos interact with matter very weakly, they diffuse from the core to the surface and leave the star, subtracting energy from the core.
- Heavier elements production beyond Fe and Ni, which mainly occurs through neutron capture, subtract energy to the star.
- In the reaction reactions (17.1) γ-photons are produced. At temperatures of the order of 10^{10} k, the number of high energy photons (> 8 MeV) is sufficient to ignite the iron photodisintegration process

$$\gamma +^{56} Fe \rightarrow 13\ ^4He + 4n.$$

  This is an endothermic process which subtracts further energy to the core and, in addition, produces a large number of neutrons.

All these processes tend to destabilize the star; when the core mass becomes larger than the Chandrasekhar limit the internal pressure gradient can no more balance the gravitational attraction and the core collapses, reaching densities typical of atomic nuclei, $\sim 10^{14}\ g/cm^3$, in a fraction of a second. The core is now composed mainly of neutrons, an so rigid that infalling matter bounces back producing a violent shock wave that ejects, in a spectacular explosion, most of the material external to the core in the outer space. This phenomenon is called *supernova explosion*: the luminosity of the star suddenly increases to values of the order of $\sim 10^9\ L_\odot$, where $L_\odot$ is the Sun luminosity, and it is in this phase that elements heavier than ^{56}Fe are created. The remnant of this explosion is a nebula, in the middle of which sits what remains of the core, i.e. a neutron star.

Neutron stars, whose structure and composition will be described shortly, are often observed as pulsars, i.e. radio sources whose emission exhibits a very sharp periodicity; pulsars are rapidly rotating neutron stars with strong magnetic fields ($B \sim 10^{11} - 10^{13}$ Gauss), which emit beams of radio waves from the magnetic poles; the observed periodicity is due to the fact that since the star rotates and the magnetic field is in general not aligned with the rotation axis, the beam is visible only when it points in the direction of the detector. However, not all neutron stars are detectable as pulsars, since their beams may not point toward the Earth, or their magnetic field may not be sufficiently strong. Moreover, pulsars slow down during their life because electromagnetic and gravitational emission processes subtract a substantial fraction of its rotational energy. The estimated number of neutron stars in our Galaxy is $\sim 10^9$.

That neutron star could exist was first suggested by Landau (*L. Landau, Zeits. Sowjetunion* **1**, *285, 1932*) and by Baade and Zwicky (*W. Baade, F. Zwicky, Proc. Nat. Acad. Sci.* **20**, *255, 1934, Phys. Rev.* **46**, *76, 1934*), who introduced the idea that neutron stars should be the leftover of a supernova explosion. It is interesting to follow the hystorical path that lead to the discovery of pulsars and that allowed to establish the connection between supernova explosions and neutron stars. In 1942 Baade identified the Crab Nebula and the star in its center as the remnant of the supernova explosion occurred in 1054 and observed by the chinese astronomer Yang Wei-te (*W. Baade, Astrophysical Journal,* **96**, *188, 1942*). Twenty years later Hewish discovered a radio source in the Crab Nebula, whose position corresponded to that of the star observed by Baade (*A. Hewish, S. E. Okoye, Nature* **207**, *59, 1965*) and three years later Bell and Hewish discovered four pulsars; however, none of them was surrounded by a nebula left by a supernova explosion (*A. Hewish, S. J. Bell, J. D. H. Pilkington, P.F. Scott, R. A. Collins, Nature* **217**, *709, 1968*). The correlation between pulsars and supernovae was firmly established only when a pulsar was discovered in the remnant of the Vela supernova (*M. I. Large, A. F. Vaughan, B. Y. Mills, Nature* **220**, *340, 1968*) and a very fast pulsar (period=0.033 seconds) was discovered in the Crab Nebula (*D. H. Staelin, E. C. Reifenstein, Science* **162**, *1481, 1968*).

17.1 The internal structure of a neutron star

The interior of a neutron star can be modeled, as shown in figure 17.1, as a sequence of layers of different composition and thickness surrounding an innermost, secret core. Proceeding from the exterior, we first encounter an outer crust, $\sim 0.3km$ thick, an inner crust, $\sim 0.5km$ thick, and a core extending over about 10 km. We shall assume that the temperature of matter in the neutron star interior is $T = 0$ and that matter is transparent to neutrinos. The first assumption is justified because at the densities typical of neutron stars, neutrons have a Fermi energy $E_F = kT_F$ which corresponds to temperatures $T_F \sim 3 \cdot 10^{11} - 10^{12}\ K$, whereas shortly after their birth ($\sim$ a year after) neutron stars reach temperatures $T \leq 10^9\ K << T_F$. The second assumption is based on the fact that the mean free path of neutrinos in nuclear matter at $T \lesssim 10^9\ K$ is much larger than the typical radius of a neutron star.

- *The outer crust.*
  The matter density ranges within $\sim 10^7$ g/cm^3 to the neutron drip density, $\rho_d = 4 \cdot 10^{11}$ g/cm^3. It is composed of a heavy nuclei lattice immersed in an electron gas. Proceeding from the external boundary to the internal one, as the density increases the inverse

β-decay process becomes more and more efficient and neutrons are produced in large number according to eq. 17.4. The produced neutrinos, as usual, leave the star. In this region pressure is mainly due to the degenerate electron gas. At $\rho = 4 \cdot 10^{11} g/cm^3$ all bound states available in the nuclei for neutrons are filled, neutrons can no longer live bound to nuclei and start leaking out (neutron drip).

- *The inner crust.*
  Density ranges between ρ_d and the nuclear density $\rho_0 = 2.67 \cdot 10^{14} g/cm^3$ and the dominant contribution to pressure is due to the neutron gas. Matter is composed of a mixture of two phases: one, with density comparable to ρ_0, is rich of protons and is indicated as PRM (Proton Rich Matter); the second phase is a neutron gas (NG). In addition, the electron gas is present to ensure charge neutrality. In order to determine the fundamental state of matter in this region one has to specify the density of the two phases, ρ_{PRM} and ρ_{NG}, which determines the fraction of volume each phase occupies, the proton fraction in the PRM and the geometrical properties of the structures that are formed by the two phases and which strongly depend on surface effects at the interface bewteen different phases. For $\rho_d < \rho < 0.35\ \rho_0$ the minimum energy configuration is formed by spherical drops of nuclei, surrounded by a gas of electrons and neutrons. For higher densities the separation between spheres decrease up to the touching limit. Thus, for $0.35\ \rho_0 < \rho < 0.5\ \rho_0$ the spheres merge, forming bar-type structures, called "spaghetti" and for $0.5\ \rho_0 < \rho < 0.56\ \rho_0$ bars merge to form slab-type structures, called "lasagne". When the density reaches the nuclear density ρ_0 the two phases are no longer separated and form a homogenoeus fluid of protons, neutrons and electrons.

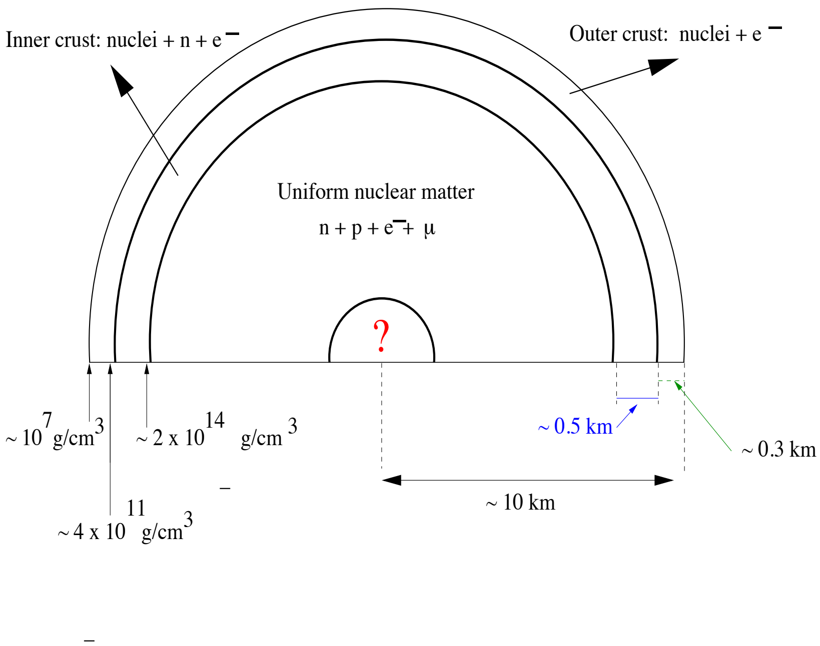

Figure 17.1: Neutron stars internal structure (not in scale!).

There is a quite general consensus on the equation of state (EOS) of matter in the outer and inner crust, because at these densities the properties of matter can be obtained from experimental data on neutron rich nuclei. Conversely, densities as those prevailing in the core are presently unreachable in a laboratory, and consequently the available models of EOS at supranuclear densities are based on theoretical models only partially constrained by empirical data. To describe in detail the equations of state proposed to describe matter in the neutron star core is beyond the scopes of this lectures. In the following we shall just mention a number of phenomena that may occur in the core.

- *The core*
  For $\rho > 2.67 \cdot 10^{14} g/cm^3$ matter is composed of a homogenoeus fluid of $p,\ n,\ e^-$, in β-equilibrium. By minimizing the free energy one finds that matter in the core is stable only if protons are about 10% of the total.

  Several processes may develop at higher density. For instance, electrons become more energetic and their kinetic energy increases. So does the chemical potential; indeed the chemical potential is the energy needed to insert in a gas in equilibrium a new particle in the same state, and this energy increases if the state is more energetic. Thus, at some density the electrons chemical potential becomes larger than the rest mass of the muon $m_{\mu^-} = 105 MeV$. At this point the decay of a neutron through the reaction

  $$n- > p + \mu^- + \bar{\nu}_\mu.$$

  which creates the muon μ^-, becomes energetically more convenient than the β-decay

  $$n- > p + e^- + \bar{\nu}_e.$$

  As usual neutrinos escape, and $n,\ p,\ e^-, \mu^-$ remain trapped in the core. It should be stressed that the main contribution to pressure in the core comes from neutrons; since they are more massive than electrons, the total energy also will be now provided by the neutrons themselves. Moreover, since the density extremely high (no stable neutron star can form until the central density exceeds $\rho \sim 10^{13}\ \ g/cm^3$), $p_F >> m_N c^2$, and neutrons must be treated as ultrarelativistic particles.

  Neutrons, as electrons, are fermions. However, the pressure they generate cannot be associated only to Pauli exclusion principle, because at the core densities we can no longer treat neutrons as non interacting particles, as we did for the degenerate electron gas in white dwarfs. Indeed, if we would assume that the neutron star is composed of non interacting neutrons, we would find a maximum mass of 0.7 $M_\odot$, which is exceedingly too low: lower than the Chandrasekhar limit, and lower than the mass of any observed neutron star.

  Depending on the particular way we choose to model neutrons interaction, we shall have a different composition. For instance in some models heavy barions may form through the transition

  $$n + e^- \rightarrow \Sigma^- + \nu_e.$$

  Or, when $\rho \sim 2 - 3\ \rho_0$, π or K mesons may form, which are bosons and therefore not subjected to Pauli's exclusion principle; in this case a Bose-Einstein condensate may form in the innermost regions. Or further, since nucleons are known to be composite objects of size $\sim 0.5 - 1.0$ fm, corresponding to a density $\sim 10^{15}$ g/cm^3, if the density in the core reaches this value matter undergoes a transition to a new phase, in which quarks are no longer confined into nucleons or hadrons.

Thus, at the densities that are expected to occur in the inner core of a neutron star the EOS of matter at supranuclear densities depends on the modeling of neutron interactions, and the typical mass and radius these model predict range within $M \sim [1-3]\ M_\odot$ and $R \sim [9-15]\ km$; thus, the surface gravity is of the order of $\frac{GM}{c^2R} \sim 10^{-1}$, (we remind that $\frac{GM_\odot}{c^2} \sim 1.47\ km$) and therefore general relativity must be used to determine the structure of such stars. In what follows we shall first introduce the tools that are needed to describe pefect fluids in general relativity, then we shall derive the equilibrium equations of a compact star, on the assumption that the fluid in the interior behaves as a perfect fluid.

17.2 Thermodynamics of perfect fluids in General Relativity

Let us consider a perfect fluid with fixed chemical composition and in thermodynamical equilibrium. The motion of the fluid is described by a vector field, the four-velocity u^α.

Let us consider a small fluid volume and be P_0 a point in the volume, for instance coincident with its center of mass. Since the volume moves in spacetime it will describe a worldtube, defined as the congruence of geodesics described by the particles in the volume. The worldtube is plotted in figure 17.2 in 2+1 dimensions (time+2 spacelike dimensions).

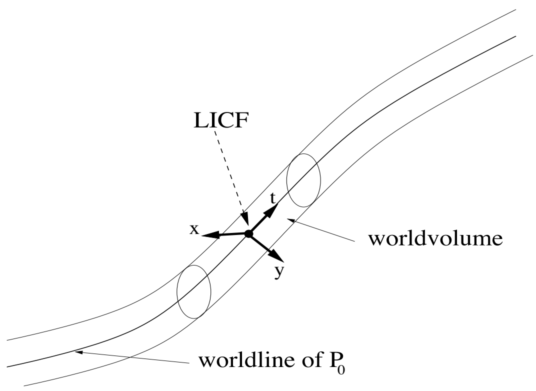

Figure 17.2: A worldtube and a worldvolume associated to the fluid element with center of mass in P_0 is plotted in 2+1 dimensions (2 for space, 1 for time)

At some time $t = t_0$, let us consider a frame with origin in P_0 which is locally inertial and such that P_0 is at rest, i.e. the inertial frame is also *comoving* with P_0. In the following we shall indicate this frame as LICF. Since P_0 is at rest, its four velocity is $\vec{u} = (1, 0, 0, 0)$, and consequently the coordinate time t of the LICF coincides with the proper time of P_0. Note that:

- It is always possible to define such a locally inertial, comoving frame; indeed, given a locally inertial frame, we can make a boost and transform to a new locally inertial frame with respect to which P_0 is at rest at a given time.
- Since this frame is locally inertial, the fluid in the neighborhood of P_0 is freely falling and it does not feel gravity, provided we restrict to a sufficiently short time interval.

We shall define a *worldvolume*, as a portion of the 4-dimensional worldtube (see figure 17.2) which is small enough to be covered by the LICF, but is large with respect to the typical scales of the dynamics of microphysical interactions. This requires that gravity does not affect the dynamics of microscopical interactions, an hypothesis which is generally accepted, because the typical length scales of microscopical interactions (for instance nuclear interactions) are much smaller than typical gravitational lengthscales (for instance curvature radius).

Under these hypotheses a *fluid element*, i.e. the portion of fluid enclosed in a worldvolume is described by the following thermodynamical variables [2]:

- the *particle number density* n
- the *energy density* ϵ
- the *pressure* p
- the *temperature* T
- the *entropy per particle* s,

which are scalar fields.

The *equation of state* (EOS) of the fluid is a relation which gives one of the thermodynamical variables in terms of two of the others, for instance

$$\epsilon = \epsilon(p, s)\,; \tag{17.5}$$

it encodes the information on the microphysics of the system. Given the values of two variables and the EOS, the values of all other thermodynamical variables can be determined, as we will later show with some example; thus, in our approach a thermodynamical state depends on *two* variables. Of course, this is not true for non-perfect fluids or for fluids whose chemical composition is allowed to change.

[2] we also assume, as usual in fluid mechanics, that the fluid element is large enough to contain a sufficiently large number of particles so that thermodynamical variables can properly be defined

17.2.1 Baryon number conservation law

A fundamental equation in the study of stellar structure is the conservation of particles number. Let us consider a fluid element of volume V and let us choose a LICF as described before. The volume V contains nV particles, therefore, if τ is the proper time associated to the LICF origin, P_0, the conservation of particles number can be written as

$$\frac{d}{d\tau}(nV) = 0 \ . \tag{17.6}$$

Note that this equation is not covariant, because V is not a scalar quantity. We shall now show that the generalization of this law, valid in any reference frame, is

$$(nu^\alpha)_{;\alpha} = 0 \, . \tag{17.7}$$

In the case of a star, n is the *baryon number density*; indeed, the baryon number is conserved by all interactions. If we assume that the star does not contain antimatter and that the mesons content is negligible, the baryon number [3] coincides with the number of baryons. Since baryons are much heavier than electrons and neutrinos, the star "rest mass" is considered as due to baryons only. Therefore, in the following we will refer to n as the baryon number density.

Proof of equation (17.7)

To hereafter we shall use geometric units $G = c = 1$.
Let us assume for simplicity that V is a cube of edges $\Delta x = \Delta y = \Delta z = L$ and that the LICF origin, P_0, is chosen as in figure 17.3. In P_0 the fluid is at rest, but within the volume it has a small velocity

$$v^i = \frac{dx^i}{dt} \, , \tag{17.8}$$

where t is the coordinate time of the chosen LICF. In order to show that the covariant form of the barion number conservation law is (17.7), we firstly expand eq. (17.6) valid in a LICF

$$\begin{aligned} \frac{d}{d\tau}(nV) = 0 \quad \rightarrow \quad u^\alpha \left(nV\right)_{,\alpha} \quad &= \quad 0 \quad \rightarrow \quad n_{,\alpha}Vu^\alpha + nV_{,\alpha}u^\alpha = 0 \\ &\Downarrow \\ n_{,\alpha}Vu^\alpha \quad &+ \quad n[V_{,0}u^0 + V_{,i}u^i] \, , \qquad i = 1,3 \, . \end{aligned} \tag{17.9}$$

Let us first evaluate the order of the various terms. In a LICF the metric reduces to $\eta_{\mu\nu}$ and its first derivatives vanish, therefore in the small volume V $g_{\mu\nu} = \eta_{\mu\nu} + O(|x^\alpha|^2)$. Furthemore,

$$d\tau = \sqrt{-g_{\alpha\beta}dx^\alpha dx^\beta} = dt\sqrt{1 + O(|x^\alpha|^2)} = dt + O(|x^\alpha|^2) \, . \tag{17.10}$$

The components of u^α consequently are

$$\begin{aligned} u^0 \quad &= \quad \frac{dt}{d\tau} = 1 + O(|x^\alpha|^2) \\ u^i \quad &= \quad \frac{dx^i}{d\tau} = v^i + O(|x^\alpha|^2) \ . \end{aligned} \tag{17.11}$$

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[3]The baryon number is $B = \dfrac{n_q - n_{\bar{q}}}{3}$ where where n_q is the number of quarks, and $n_{\bar{q}}$ is the number of antiquarks, and is a conserved quantum number.

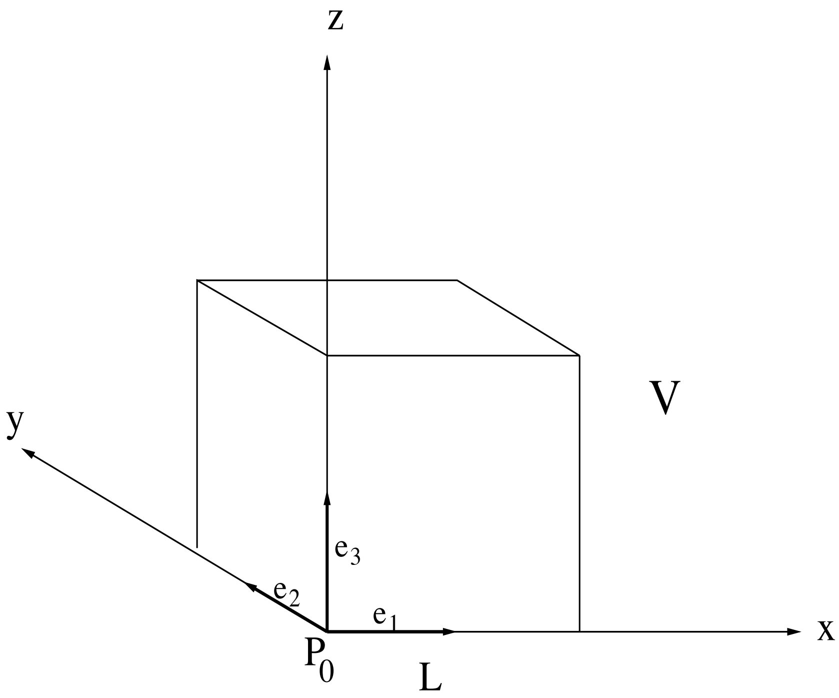

Figure 17.3: The fluid volume element V in a LICF at some fixed instant of time.

Since the LICF is a comoving frame, $v^i(P_0) = 0$ and if we assume that $\partial v^i/\partial x^j$ is finite, we find

$$\lim \frac{v^i}{x^j} = \lim \frac{v^i - v^i(P_0)}{x^j - x^j(P_0)} = \frac{\partial v^i}{\partial x^j} \quad \to \quad v^i = \frac{\partial v^i}{\partial x^j} x^j \ , \tag{17.12}$$

i.e.

$$v^i = O(|x^\alpha|) \qquad \text{and} \qquad u^i = O(|x^\alpha|)\,. \tag{17.13}$$

Using eqs. (17.11) and (17.13), the term $[V_{,0}u^0 + V_{,i}u^i]$ in eq. (17.9) becomes

$$V_{,0}u^0 + V_{,i}u^i = V_{,0}[1 + O(|x^\alpha|^2)] + O(|x^\alpha|) = V_{,0} + O(|x^\alpha|) \ . \tag{17.14}$$

Let us evaluate how the volume V changes in a time interval δt. The edges of each face of the cube change as follows

$$\begin{aligned} \delta(\Delta x) &= \left[\frac{dx}{dt}\delta t\right]_{\text{front face}} - \left[\frac{dx}{dt}\delta t\right]_{\text{back face}} = \frac{\partial v^x}{\partial x} L\delta t \\ \delta(\Delta y) &= \frac{\partial v^y}{\partial y} L\delta t \\ \delta(\Delta z) &= \frac{\partial v^z}{\partial z} L\delta t\,. \end{aligned} \tag{17.15}$$

The corresponding volume change is

$$\delta(\Delta x\Delta y\Delta z) = \delta(\Delta x)\Delta y\Delta z + \delta(\Delta y)\Delta z\Delta x + \delta(\Delta z)\Delta x\Delta y = \sum_{i=1}^{3} \frac{\partial v^i}{\partial x^i} L^3\delta t \equiv \frac{\partial v^i}{\partial x^i} L^3\delta t \tag{17.16}$$

so that

$$\frac{\partial V}{\partial t} = V \frac{\partial v^i}{\partial x^i} \,. \tag{17.17}$$

Thus eq. (17.14) becomes

$$V_{,0} u^0 + V_{,i} u^i = V \frac{\partial v^i}{\partial x^i} + O(|x^\alpha|) \ ,$$

and since $u^i = v^i + O(|x^\alpha|^2)$ and $u^0 = 1 + O(|x^\alpha|^2)$, we find

$$V_{,0} u^0 + V_{,i} u^i = V \frac{\partial u^\alpha}{\partial x^\alpha} + O(|x^\alpha|) \ . \tag{17.18}$$

By replacing this term in eq. (17.9) we finally find

$$\frac{d}{d\tau}(nV) = 0 \quad \rightarrow \quad n_{,\alpha} V u^\alpha + nV u^\alpha_{,\alpha} = 0 \quad \rightarrow \quad (nu^\alpha)_{,\alpha} \quad \rightarrow \quad (nu^\alpha)_{;\alpha} = 0 \tag{17.19}$$

where we have used the property that in locally inertial frames ordinary and covariant derivative coincide. Since this is a tensorial equation, it must be valid in any reference frame, **Q.E.D.**

17.2.2 The first law of Thermodynamics

Given a fluid with energy density ϵ and entropy per baryon s, and given a fluid element of volume V, formed by a given number of baryons $A = nV$, it will have an energy $E = V\epsilon$, where ϵ is the energy density, and an entropy $S = As$. The I^{st} law of thermodynamics

$$dE = -pdV + TdS \tag{17.20}$$

can then be written as

$$d\left(\frac{A}{n}\epsilon\right) = -pd\left(\frac{A}{n}\right) + Td(As) \,. \tag{17.21}$$

Multiplying by n/A,

$$d\epsilon = \frac{\epsilon + p}{n} dn + nTds \,. \tag{17.22}$$

Given the EOS

$$\epsilon = \epsilon(n, s), \tag{17.23}$$

we find that

$$\left(\frac{\partial \epsilon}{\partial n}\right)_s = \frac{\epsilon + p}{n} \qquad \text{and} \qquad \left(\frac{\partial \epsilon}{\partial s}\right)_n = nT, \tag{17.24}$$

and the pressure and the temperature of the fluid are

$$p(n, s) = n\left(\frac{\partial \epsilon}{\partial n}\right)_s - \epsilon \tag{17.25}$$

$$T(n, s) = \frac{1}{n}\left(\frac{\partial \epsilon}{\partial s}\right)_n \,. \tag{17.26}$$

Thus, given an equation of state, namely a relation between one thermodynamical variable (ϵ in the previous example) and two other variables (n and s), the remaining variables (p and T in the example) can be determined in terms of them.

Another important function which describes the fluid is the *chemical potential* μ, which is the energy per baryon required to create a small extra quantity of fluid, composed by δA baryons, and to insert it in the fluid volume in the same thermodynamical state. The volume variation due to the introduction of the extra baryons is $\delta V = \delta A/n$, consequently

$$\mu = \frac{p\delta V + \epsilon\delta V}{\delta A} = \frac{p+\epsilon}{n},$$

and from eq. (17.24)

$$\mu = \frac{p+\epsilon}{n} = \left(\frac{\partial\epsilon}{\partial n}\right)_s . \tag{17.27}$$

17.2.3 Barotropic equation of state

If the EOS does not depend on temperature or entropy, it is named *barotropic*, and can be written as an equation relating the pressure and the energy density

$$p = p(\epsilon) \qquad \text{or} \qquad \epsilon = \epsilon(p). \tag{17.28}$$

For instance, the EOS of matter in neutron stars a few years after birth can be considered as barotropic, because, as explained in section 17.1, one can assume that matter behaves as a degenerate gas at zero temperature.

For a barotropic EOS, the first law of thermodynamics becomes

$$d\epsilon = \frac{\epsilon + p}{n} dn\,. \tag{17.29}$$

Notice that differentiating the definition of the chemical potential $\mu \equiv \frac{\epsilon+p}{n}$, using (17.29) we find

$$d\mu = -\frac{\epsilon+p}{n^2}\,dn + \frac{d\epsilon + dp}{n} = \frac{dp}{n}; \tag{17.30}$$

therefore for a barotropic EOS we also have

$$dp = n d\mu\,. \tag{17.31}$$

17.2.4 The Stress-Energy tensor of a perfect fluid

In relativity a fluid is said "perfect" if both viscosity and heat flow are absent. We shall now show that the stress-energy tensor of a perfect fluid is

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + pg^{\mu\nu}\,. \tag{17.32}$$

As explained in chapter 7, T^{00} is the energy density; T^{0i} ($i = 1, 2, 3$) is the energy which flows per unit time across the unit surface orthogonal to the axis x^i; T^{ij} is the amount of

i_{th}-component of momentum which flows per unit time across the unit surface orthogonal to the axis x^j.

Furthermore, since the momentum which flows per unit time is a force, T^{ij} can also be interpreted as the i_{th}-component of the *force* per unit surface orthogonal to the axis x^j.

Let us consider a fluid element and the associated LICF. In this frame the fluid is at rest (the velocity of the fluid inside the element is of order $O(|x^\alpha|)$) and the components of the stress-energy tensor are the following

- $T^{00} = \epsilon$, the energy density.
- $T^{0i} = 0$, indeed the fluid element does not exchange energy with its surroundings, because the fluid is at rest and there is no heat flow.
- $T^{ij} = p\delta^{ij}$. Indeed in a perfect fluid no tangential stresses are allowed, which means that the force exerted on the surface orthogonal to the axis x^j must be parallel to the axis x^j, and the force per unit surface is, by definition, the pressure;

Thus, in the chosen rest frame the fluid stress-energy tensor is

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} . \tag{17.33}$$

and since in this frame $u^\mu = (1, 0, 0, 0)$, it can also be written as

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + p\eta^{\mu\nu}. \tag{17.34}$$

Since in a LICF $g^{\mu\nu} \equiv \eta^{\mu\nu}$, we can also write

$$T^{\mu\nu} = (\epsilon + p)u^\mu u^\nu + pg^{\mu\nu}. \tag{17.35}$$

This is a tensorial expression, and the principle of general covariance establishes that it must be valid in any other reference frame. Thus, eq. (17.35) is the **covariant** form for the stress–energy tensor of a perfect fluid in general relativity. Note that according to eq. (17.33), which follows from the assumption that viscosity and heat flow are absent, a comoving observer sees the fluid around him as isotropic.

17.2.5 Conservation laws for the stress-energy tensor

The stress-energy tensor (17.35) satisfies the conservation law (see chapter 7)

$$T^{\mu\nu}{}_{;\nu} = 0 . \tag{17.36}$$

Its contraction with u^μ gives

$$\begin{aligned} u_\mu T^{\mu\nu}{}_{;\nu} &= u_\mu u^\mu u^\nu(\epsilon + p)_{,\nu} + (\epsilon + p)(u_\mu u^\mu u^\nu{}_{;\nu} + u_\mu u^\mu{}_{;\nu} u^\nu) + u^\nu p_{,\nu} \\ &= -u^\nu \epsilon_{,\nu} - (\epsilon + p)u^\nu{}_{;\nu} = 0, \end{aligned} \tag{17.37}$$

where we have used the relation

$$u_\mu (u^\mu)_{;\nu} = \frac{1}{2}(u_\mu u^\mu)_{;\nu} = 0\,. \tag{17.38}$$

Using the baryon number conservation (17.7), eq. (17.37) gives

$$u^\nu \epsilon_{,\nu} = -(\epsilon + p) u^\nu{}_{;\nu} = \frac{\epsilon + p}{n} u^\nu n_{,\nu} \tag{17.39}$$

i.e.

$$\frac{d\epsilon}{d\tau} = \frac{\epsilon + p}{n}\frac{dn}{d\tau}\,; \tag{17.40}$$

on the other hand, from the first law of thermodynamics (17.22),

$$\frac{d\epsilon}{d\tau} = \frac{\epsilon + p}{n}\frac{dn}{d\tau} + nT\frac{ds}{d\tau}, \tag{17.41}$$

and the two equations are compatible only if

$$\frac{ds}{d\tau} = 0\,, \tag{17.42}$$

which means that a fluid element does not exchange heat with its surroundings, as it must be for a perfect fluid.

Thus, the contraction of the stress-energy tensor conservation law with the fluid four-velocity and the baryon number conservation, implies that a perfect fluid is *isentropic*.

To study the space components of (17.36) we define the projector onto the subspace orthogonal to u^μ:

$$P_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu\,. \tag{17.43}$$

It is a projector because $P^2 = P$, and it projects onto the subspace orthogonal to u^μ because $P_{\mu\nu}u^\nu = 0$.

By applying $P_{\mu\nu}$ to eq. (17.36) we find

$$\begin{aligned} P_{\gamma\alpha} T^{\alpha\beta}{}_{;\beta} &= P_{\gamma\alpha}\left\{(\epsilon+p)_{,\beta}u^\alpha u^\beta + (\epsilon+p)(u^\beta u^\alpha{}_{;\beta} + u^\alpha u^\beta{}_{;\beta}) + g^{\alpha\beta}p_{,\beta}\right\} \\ &= (g_{\gamma\alpha} + u_\gamma u_\alpha)(\epsilon+p)u^\beta u^\alpha{}_{;\beta} + P_\gamma{}^\beta p_{,\beta} \\ &= (\epsilon+p)u^\beta u_{\gamma;\beta} + P_\gamma{}^\beta p_{,\beta} = 0 \end{aligned} \tag{17.44}$$

where we have used eq. (17.38). This equation gives

$$P_\gamma{}^\beta p_{,\beta} = -(\epsilon+p)u^\beta u_{\gamma;\beta}\,, \tag{17.45}$$

and says that the pressure gradient projected on the subspace orthogonal to u^μ (that is, the space gradient of the pressure) is equal to the fluid acceleration, $u^\beta u_{\gamma;\beta}$, times the energy density (plus the pressure); this is the relativistic generalization of one of Euler's equation.

17.3 The equations of stellar structure in general relativity

In this section we shall derive the equations which describe the structure of a non rotating star in static equilibrium according to general relativity. Since the spacetime generated by such star is static and spherically symmetric, the appropriate form of the metric is

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \qquad (17.46)$$

In this expression and in the following we shall use geometric units, setting $G = c = 1$. We shall assume that the star is composed by a perfect fluid of stress-energy tensor

$$T^{\alpha\beta} = (\epsilon + p)u^\alpha u^\beta + pg^{\alpha\beta}, \qquad (17.47)$$

where $u^\alpha = \frac{dx^\alpha}{d\tau}$ is the four-velocity of an element of fluid, and p and ϵ are the pressure and the energy-density measured by an observer in a locally inertial frame locally at rest with respect to the fluid as discussed in previous sections.

It should be stressed that ϵ is the relativistic energy density, which reduces to the rest energy density ρc^2 (where ρ is the mass density) in the non relativistic limit.

At this point some considerations are needed about the dimensions of the physical quantities we are dealing with. Since we are working in geometrical units $G = c = 1$, $T_{\mu\nu}$ has the same dimensions as $G_{\mu\nu}$, i.e.

$$[T_{\mu\nu}] = [l^{-2}].$$

Consequently, both ϵ and p are $[l^{-2}]$ quantities. This means that

$$\epsilon = \frac{G}{c^4}\epsilon_{phys}, \qquad \text{and} \qquad p = \frac{G}{c^4}p_{phys}, \qquad (17.48)$$

where ϵ_{phys} and p_{phys} are the energy density and the pressure in physical units, i.e. $[\epsilon_{phys}] = [p_{phys}] = [ml^{-1}t^{-2}]$.

Since by assumption the fluid is at rest, the only non vanishing component of the velocity of the generic fluid element is given by

$$g_{\mu\nu}u^\mu u^\nu = -1 \qquad \to \qquad u^0 = e^{-\nu}, \qquad u_0 = -e^\nu, \qquad (17.49)$$

hence the non vanishing components of the stress-energy tensor are

$$\begin{cases} T_{00} = \epsilon e^{2\nu} & T_{\theta\theta} = r^2 p \\ T_{rr} = pe^{2\lambda} & T_{\varphi\varphi} = \sin^2\theta T_{\theta\theta}. \end{cases} \qquad (17.50)$$

The pressure and energy-density are related by an assigned equation of state.

The equations to solve are

$$\begin{cases} a) \quad G_{\mu\nu} = 8\pi T_{\mu\nu} \\ b) \quad T^{\mu\nu}{}_{;\nu} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}\left(\sqrt{-g}T^{\mu\nu}\right) + \Gamma^\mu_{\lambda\nu}T^{\nu\lambda} = 0 \end{cases}, \qquad (17.51)$$

where

$$\begin{cases} T^{00} = \epsilon e^{-2\nu} & T^{\theta\theta} = \frac{p}{r^2} \\ T^{rr} = p e^{-2\lambda} & T^{\varphi\varphi} = \frac{1}{\sin^2\theta} T^{\theta\theta}. \end{cases} \tag{17.52}$$

It should be noted that eqs. (17.51) a) and b) are not independent. Indeed, as discussed in chapter 8, the divergenceless equation satisfied by the stress-energy tensor is a consequence of the Bianchi identities satisfied by the Riemann tensor. We write explicitly the two equations to make the calculations easier.

In order to write explicitely eq. (17.51b) we need the expression of Christoffel's symbols

$$\begin{aligned} \Gamma^r_{00} &= e^{2(\nu-\lambda)}\nu_{,r} & \Gamma^r_{\theta\theta} &= -e^{-2\lambda} r \\ \Gamma^0_{0r} &= \nu_{,r} & \Gamma^\varphi_{\theta\varphi} &= \frac{\cos\theta}{\sin\theta} \\ \Gamma^r_{rr} &= \lambda_{,r} & \Gamma^r_{\varphi\varphi} &= -e^{-2\lambda} r \sin^2\theta \\ \Gamma^\theta_{r\theta} &= \frac{1}{r} & \Gamma^\theta_{\varphi\varphi} &= -\cos\theta\sin\theta \\ \Gamma^\varphi_{r\varphi} &= \frac{1}{r} & \sqrt{-g} &= r^2 e^{\nu+\lambda}\sin\theta. \end{aligned} \tag{17.53}$$

The only non trivial component of eq. (17.51b) is $\mu = r$ which gives

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\nu}\left(\sqrt{-g}T^{r\nu}\right) + \Gamma^r_{\lambda\nu}T^{\nu\lambda} = 0, \tag{17.54}$$

i.e.

$$\begin{aligned} &\frac{1}{\sqrt{-g}}\frac{\partial}{\partial r}\left(\sqrt{-g}T^{rr}\right) + \Gamma^r_{00}T^{00} + +\Gamma^r_{rr}T^{rr} + \Gamma^r_{\theta\theta}T^{\theta\theta} + \Gamma^r_{\varphi\varphi}T^{\varphi\varphi} = \\ &\frac{e^{-(\nu+\lambda)}}{r^2}\left(r^2 e^{(\nu+\lambda)} p e^{-2\lambda}\right)_{,r} + e^{2(\nu-\lambda)}\nu_{,r}\epsilon e^{-2\nu} + e^{-2\lambda}\lambda_{,r}p - 2\frac{e^{-2\lambda}p}{r} = 0, \end{aligned} \tag{17.55}$$

which becomes

$$\nu_{,r} = -\frac{p_{,r}}{\epsilon + p}. \tag{17.56}$$

Einstein's equations give

$$\begin{aligned} a) \quad & G_{00} = 8\pi T_{00}, \quad \frac{1}{r^2}e^{2\nu}\frac{d}{dr}\left[r\left(1 - e^{-2\lambda}\right)\right] = 8\pi\epsilon e^{2\nu} \\ b) \quad & G_{rr} = 8\pi T_{rr}, \quad -\frac{1}{r^2}e^{2\lambda}\left(1 - e^{-2\lambda}\right) + \frac{2}{r}\nu_{,r} = 8\pi p e^{2\lambda} \\ c) \quad & G_{\theta\theta} = 8\pi T_{\theta\theta}, \quad r^2 e^{-2\lambda}\left[\nu_{,rr}\nu^2_{,r} + \frac{\nu_{,r}}{r} - \nu_{,r}\lambda_{,r} - \frac{\lambda_{,r}}{r}\right] = 8\pi r^2 p. \end{aligned} \tag{17.57}$$

If we put

$$m(r) = \frac{1}{2}r\left(1 - e^{-2\lambda}\right), \qquad \to \qquad e^{-2\lambda} = 1 - \frac{2m(r)}{r}, \tag{17.58}$$

eq. (17.57a) becomes

$$\frac{dm(r)}{dr} = 4\pi r^2\epsilon, \tag{17.59}$$

which is the generalization of the newtonian equation (16.16).

Eq. (17.57b) can be rewritten as

$$\frac{\left(1-e^{-2\lambda}\right)}{r^2} - 2e^{-2\lambda}\frac{\nu_{,r}}{r} = -8\pi p, \tag{17.60}$$

and by using eq. (17.58) it becomes

$$\nu_{,r} = \frac{m(r) + 4\pi r^3 p}{[r\,(r - 2m(r))]}. \tag{17.61}$$

In the newtonian limit the pressure in geometric units is small compared to the energy-density. For example in the case of the Sun, the ratio between the central pressure and density is $\sim 10^{-6}$. In addition $m(r) << r$ and eq. (17.61) reduces to

$$\nu_{,r} = \frac{m(r)}{r^2}. \tag{17.62}$$

Remembering that in this limit $e^{2\nu} \to 1 + \frac{2\Phi}{c^2}$ where Φ is the newtonian potential and $\Phi_{,r} = \frac{m(r)}{r^2}$, eq. (17.62) simply says that the gravitational force is that of the mass enclosed within a sphere of radius r. From eq. (17.61) we see that in general relativity there is the additional contribution, $4\pi r^3 p$, which is due to the pressure. This should not be surprising, because dimensionally p is an energy density, thus the term $4\pi r^3 p$ acts as an *effective mass*. This means that the active mass which attracts the mass shell between r and $r+dr$ is due to both contributions, and the pressure which should contrast gravity, to some extent enhance its effects. This phenomenon is called *regeneration of the pressure*.

Eqs. (17.56) and (17.61) can be combined, and the final set of equations, known as the Oppenheimer-Volkoff equations (TOV equations), is

$$\begin{cases} \dfrac{dm(r)}{dr} = 4\pi r^2 \epsilon \\[2ex] \dfrac{dp}{dr} = -\dfrac{(\epsilon + p)[m(r) + 4\pi r^3 p]}{r[r - 2m(r)]}. \end{cases} \tag{17.63}$$

17.3.1 The boundary conditions

To integrate this system we need

1. to choose an equation of state connecting the pressure and the energy density
2. to impose that $m(r=0) = 0$.

The reason for the condition (2) is the following. Take a tiny sphere of radius x. The circumference will be $2\pi x$ and the proper radius

$$r = \int_0^x e^{\lambda} dr \simeq e^{\lambda} x, \tag{17.64}$$

hence their ratio is $2\pi e^{-\lambda}$. Since the spacetime is locally flat the ratio between the circumference of an infinitesimal sphere and the radius must be 2π. This implies that as $r \to 0$ e^{λ} must tend to 1. Since

$$e^{2\lambda} = \frac{1}{1 - \frac{2m(r)}{r}}, \tag{17.65}$$

it follows that $m(r)$ must tend to zero faster then r.

For any assigned equation of state $p = p(\epsilon)$, we have a one-parameter family of solutions identified by the value of the energy density at $r = 0$, i.e. $\epsilon(r = 0) = \epsilon_0$. Once $m(r)$, $p(r)$, and $\epsilon(r)$ have been determined by numerical integration, $e^{2\lambda}$ follows from eq. (17.65) and $\nu(r)$ can be found by integrating eq. (17.56)

$$\nu = \int -\frac{p_{,r}}{(\epsilon + p)} dr + \nu_0, \tag{17.66}$$

where ν_0 is a constant to be determined. The solution of eqs. (17.63), together with (17.65) and (17.66), describes the gravitational field and the distribution of pressure and energy density *inside the star.*

Outside the star $p = \epsilon = 0$ and Einstein's equations reduce to those for a vacuum, static, spherically symmetric spacetime whose unique solution is, by Birkhoff's theorem, the Schwarzschild metric. Thus, the metric computed in the interior of the star must reduce to the Schwarzschild metric when $r = R$, and by imposing this condition we find the constant ν_0 of eq. (17.66):

$$e^{2\nu(r=R)} \equiv e^{2\nu_0} \cdot e^{2\int_0^R -\frac{p_{,r}}{(\epsilon+p)} dr} = 1 - \frac{2M(R)}{R} \qquad \to \qquad e^{2\nu_0} = \frac{1 - \frac{2M(R)}{R}}{e^{2\int_0^R -\frac{p_{,r}}{(\epsilon+p)} dr}}, \tag{17.67}$$

and the constant appearing in eq. (17.66) can determined. The quantity

$$M(R) = 4\pi \int_0^R r^2 \epsilon \; dr, \tag{17.68}$$

has the same form as in newtonian theory and is the total mass-energy inside the radius R. This interpretation will be further clarified in the next chapter on the far field limit of isolated objects, where we will show that $M(R)$ can be obtained as an integral over a large space volume of the conserved quantity $(-g)\,(T^{00} + t^{00})$.

We can split $M(R)$ in three contributions

$$M(R) = E_R + E_{int} + E_B;$$

E_R is the rest mass energy of the star, i.e. the integral over the proper volume element

$$dV_{prop} = \sqrt{g^{(3)}} \; dr \; d\theta \; d\varphi = e^{\lambda(r)} r^2 \sin\theta \; dr \; d\theta \; d\varphi$$

of the nucleons rest mass density $m_N n$, where m_N is the average mass of the baryon species present in the star

$$E_R = \int_V m_N n \; dV_{prop} = \int_0^R e^{\lambda(r)} m_N n \; r^2 \int_0^{\pi} \sin(\theta) \; d\theta \int_0^{2\pi} d\varphi = 4\pi \int_0^R \frac{m_N n \; r^2}{\sqrt{1 - \frac{2m(r)}{r}}}; \tag{17.69}$$

E_{int} is the internal energy (thermal, compressional,etc.), given by

$$E_{int} = \int_V [\epsilon - m_N n]\, dV_{prop} = 4\pi \int_0^R \frac{[\epsilon - m_N n] r^2}{\sqrt{1 - \frac{2m(r)}{r}}}. \tag{17.70}$$

The last contributions to the total mass-energy is the gravitational potential energy, i.e. the binding energy E_B, given by

$$E_B = M(R) - E_R - E_{int} = 4\pi \int_0^R r^2 \epsilon [1 - e^{\lambda}]\, dr = 4\pi \int_0^R r^2 \epsilon \left[1 - \frac{1}{\sqrt{1 - \frac{2m(r)}{r}}}\right]. \tag{17.71}$$

It is easy to see that $E_B < 0$ as required for a bounded system.

A note on the chemical potential

Let us consider a spherical star with a barotropic equation of state $p = p(\epsilon)$; combining eqs. (17.27) and (17.31) we find

$$dp = n d\mu = \frac{\epsilon + p}{\mu} d\mu \,. \tag{17.72}$$

By integrating this equation, and using eq. (17.66) we find

$$\int_{\mu(r)}^{\mu(r')} \frac{d\mu}{\mu} = \int_{p(r)}^{p(r')} \frac{dp}{\epsilon + p} = -\int_{\nu(r)}^{\nu(r')} d\nu = \nu(r) - \nu(r'), \tag{17.73}$$

i.e.

$$\mu(r) e^{\nu(r)} = \mu(r') e^{\nu(r')} = constant \,. \tag{17.74}$$

This equation says that the chemical potential, corrected by the redshift factor e^{ν}, at any depth in the star is a constant. In particular, for any $r < R$ (R stellar radius)

$$\mu(r) e^{\nu(r)} = \left(1 - \frac{2M}{R}\right)^{1/2} \mu(R) \,. \tag{17.75}$$

17.4 The Schwarzschild solution for a homogeneous star

An analytic solution of the equations of stellar structure (17.63) can be obtained by considering the very simple equation of state:

$$\epsilon = const.$$

This solution was found by K. Schwarzschild in 1916 and this is the only exact solution of eqs. (17.63) found up to the present time.

Although homogeneous stars are unrealistic (the speed of sound $v = \left(\frac{dp}{d\epsilon}\right) \to \infty$,) they can be used as a good approximation for the core of very dense stars, and the interior Schwarzschild solution has been used as a simplified model in a variety of situations to study the effects of gravity in a regime as strong as it can ever become under the condition of hydrostatic equilibrium.

If $\epsilon = const$

$$m(r) = \frac{4}{3}\pi r^3 \, \epsilon, \tag{17.76}$$

and from eq. (17.58) one of the metric functions is immediately found

$$e^{2\lambda} = \left(1 - \frac{2m(r)}{r}\right)^{-1} \qquad \to \qquad e^{2\lambda} = \left(1 - \frac{8}{3}\pi\epsilon r^2\right)^{-1}. \tag{17.77}$$

The Oppenheimer-Volkoff equations reduce to

$$\frac{dp}{dr} = -\frac{4}{3}\pi r \frac{(\epsilon + p)(\epsilon + 3p)}{1 - \frac{8\pi}{3} r^2 \epsilon}, \tag{17.78}$$

that can be integrated to find the pressure; the integration is performed between r and the radius of the star $r = R$, where the pressure vanishes:

$$\log\left(\frac{\epsilon + 3p}{\epsilon + p}\right)\Bigg|_{p(r)}^{0} = \frac{1}{2}\log\left(1 - \frac{8\pi}{3} r^2 \epsilon\right)\Bigg|_{r}^{R}, \tag{17.79}$$

which gives

$$p = \epsilon \frac{(y - y_1)}{(3y_1 - y)}, \tag{17.80}$$

where

$$y^2 = 1 - \frac{8\pi}{3} r^2 \epsilon = 1 - \frac{2m(r)}{r}, \qquad \text{and} \qquad y_1^2 = y^2(R) = 1 - \frac{2M}{R}, \tag{17.81}$$

and $M \equiv M(R)$.

It is interesting to note that if we put $r = 0$ in eq. (17.80) we find

$$p(r = 0) = p_0 = \epsilon \frac{1 - \sqrt{1 - \frac{2M}{R}}}{3\sqrt{1 - \frac{2M}{R}} - 1}. \tag{17.82}$$

If the denominator of this expression is zero the central pressure becomes infinite, and negative if it is smaller than zero. Thus, homogeneous stars can exist only if

$$3\sqrt{1-\frac{2M}{R}}-1>0 \qquad \to \qquad \frac{M}{R}<\frac{4}{9}, \tag{17.83}$$

or, equivalently,

$$R>\frac{9}{4}M\ . \tag{17.84}$$

This equation sets a lower limit on the radius that a star of a given mass can have, provided $\epsilon = const$. However, in the next section we will show that this result holds for a generic equation of state.

The radius of the star can be found by integrating eq. (17.78) from $r=0$ (where $p=p_0$) and R

$$-\log\left(\frac{\epsilon+3p}{\epsilon+p}\right)=\frac{1}{2}\log\left(1-\frac{8\pi}{3}R^2\epsilon\right) \qquad \to \qquad \log\left(\frac{\epsilon+3p}{\epsilon+p}\right)^{-1}=\log\sqrt{\left(1-\frac{2M}{R}\right)} \tag{17.85}$$

from which we find

$$R=\frac{2M}{\left[1-\frac{(\epsilon+p_0)^2}{(\epsilon+3p_0)^2}\right]}. \tag{17.86}$$

Thus for any assigned value of ϵ and of p_0 we have a configuration of radius R given by (17.86).

To complete the solution we need to find the metric function $\nu(r)$, which can be determined from eq. (17.66)

$$\nu-\nu_0=-\int_0^r \frac{p_{,r}}{[\epsilon+p(r)]}dr=-\int_1^y \frac{p_{,y}}{[\epsilon+p(y)]}dy; \tag{17.87}$$

since

$$p_{,y}=\epsilon\left[\frac{2y_1}{(3y_1-y)^2}\right], \qquad (\epsilon+p)=\epsilon\left[\frac{2y_1}{(3y_1-y)}\right],$$

we find

$$\nu=\nu_0-\int_1^y \frac{dy}{(3y_1-y)} \qquad \to \qquad e^{2\nu}=e^{2\nu_0}\left(\frac{3y_1-y}{3y_1-1}\right)^2. \tag{17.88}$$

At the boundary of the star $y(R)=y_1$ and

$$e^{2\nu(R)}=e^{2\nu_0}\left(\frac{4y_1^2}{(3y_1-1)^2}\right); \tag{17.89}$$

on the other hand we know that the metric must reduce to the Schwarzschild metric, therefore it must also be

$$e^{2\nu(R)}=1-\frac{2M}{R}\equiv y_1^2, \tag{17.90}$$

and by equating eq. (17.89) and (17.90) we find the value of the integration constant ν_0

$$e^{2\nu_0}=\frac{(3y_1-1)^2}{4},$$

and the solution for $\nu(r)$ is completely determined

$$e^{2\nu(r)} = \frac{(3y_1 - y)^2}{4}. \tag{17.91}$$

17.5 Relativistic polytropes

In this section we shall generalize the Lane-Emden equation in general relativity. Following what we did in section 12.4 for the newtonian case (see eqs. 16.38), we shall solve the relativistic equations of stellar structure (17.63) assuming that the equation of state of the matter inside the star is of a polytropic form, i.e.

$$\begin{cases} \dfrac{dm(r)}{dr} = 4\pi r^2 \epsilon \\ \\ \dfrac{dp}{dr} = -\dfrac{(\epsilon + p)[m(r) + 4\pi r^3 p]}{r[r - 2m(r)]} \\ \\ p = K\epsilon^{\gamma}, \end{cases} \tag{17.92}$$

where we remind that ϵ is the relativistic energy density, and we shall also assume that ϵ and p are expressible in the manner (cfr. eq. (16.40))

$$\begin{cases} \gamma = 1 + \dfrac{1}{n}, \\ \\ \epsilon = \epsilon_0\ \Theta^n(r) \\ \\ p = K\ \rho_0^{1+\frac{1}{n}}\ \Theta^{(n+1)}(r) = p_0\ \Theta^{(n+1)}(r), \qquad p_0 = K\ \rho_0^{1+\frac{1}{n}}, \end{cases} \tag{17.93}$$

With these substitutions eqs. (17.92) become

$$\begin{cases} \dfrac{dm(r)}{dr} = 4\pi\epsilon_0 r^2 \Theta^n(r) \\ \\ \dfrac{d\Theta(r)}{dr} = -\left[\dfrac{\epsilon_0 + p_0\Theta}{p_0(n+1)}\right] \dfrac{m(r) + 4\pi r^3 p_0\ \Theta^{(n+1)}}{r[r - 2m(r)]}. \end{cases} \tag{17.94}$$

By putting

$$\alpha_0 = \frac{\epsilon_0}{p_0}, \tag{17.95}$$

these equations become

$$\begin{cases} \dfrac{dm(r)}{dr} = 4\pi\epsilon_0 r^2 \Theta^n(r) \\ \\ \dfrac{d\Theta(r)}{dr} = -\left[\dfrac{\alpha_0 + \Theta}{(n+1)}\right] \dfrac{m(r) + 4\pi r^3\ \frac{\epsilon_0}{\alpha_0}\ \Theta^{(n+1)}]}{r[r - 2m(r)]}. \end{cases} \tag{17.96}$$

As explained in section 12.7, both ϵ and p have dimensions $[l^{-2}]$, therefore the quantity $\sqrt{\epsilon_0}$ has dimension $[l^{-1}]$ and we can use it to rescale the radial coordinate as follows. We put

$$\xi = r\,\sqrt{\epsilon_0}, \qquad \text{and} \qquad \mathcal{M} = \sqrt{\epsilon_0}\; m \tag{17.97}$$

and rewrite eqs. (17.96) in terms of the new variables (note that ξ and $\mathcal{M}$ are dimensionless quantities)

$$\begin{cases} \dfrac{d\mathcal{M}(\xi)}{d\xi} = 4\pi\xi^2\Theta^n(\xi) \\ \\ \dfrac{d\Theta(\xi)}{d\xi} = -\left[\dfrac{\alpha_0+\Theta}{(n+1)}\right]\dfrac{\left[\mathcal{M}(\xi)+4\pi\xi^3\,\frac{1}{\alpha_0}\,\Theta^{(n+1)}\right]}{\xi\left[\xi-2\mathcal{M}(\xi)\right]}. \end{cases} \tag{17.98}$$

We may, at this point, multiply the second equation by ξ^2, differentiate with respect to ξ and substituting $\frac{d\mathcal{M}(\xi)}{d\xi}$ into the resulting equation find a second order differential equation for Θ in the Lane-Emden form (cfr. 16.43); however, the equation we would get is much more complicated than eq. (16.43), and it is much better to work with the system of two first order eqs. (17.98).

Another important difference with the equation of newtonian polytropes should be stressed. In the newtonian case if we assign the value of the polytropic index n and integrate the Lane-Emden equation finding $\Theta(\xi)$ up to the stellar radius ξ_1, from this solution we can construct a family of solutions by assigning the value of K and of the central density ρ_0; no further integrations are needed and, for instance, the radius and the mass of the star can be found by eqs. (16.48) and (16.49). The situation changes in the relativistic equations, because to solve eqs. (17.98) we need to assign both n **and** α_0, i.e. the ratio between the energy density and the pressure at $\xi = 0$.

In order to integrate the structure equations numerically, we need to Taylor expand the functions $\Theta(\xi)$ and $\mathcal{M}(\xi)$ near the origin as in (16.55); from the newtonian expansion we know that only even powers of Θ are needed and therefore $\mathcal{M}(\xi)$ will be expanded in odd powers of ξ (cfr. the first eq. 17.98), therefore we shall put

$$\begin{aligned} \Theta(\xi) &\sim 1 + \Theta_2\,\xi^2 + \Theta_4\,\xi^4 + O(\xi^6), \\ \mathcal{M}(\xi) &\sim m_3\xi^3 + m_5\xi^5 + O(\xi^7). \end{aligned}$$

By inserting this expansions in eqs. (17.98) we find

$$\begin{aligned} &3m_3\xi^2 + 5m_5\xi^4 = 4\pi\xi^2 + 4\pi n\;\Theta_2\;\xi^4 \\ &2\;\Theta_2\;\xi + 4\;\Theta_4\;\xi^3 = -\frac{1}{n+1}\left\{\left(m_3 + \frac{4\pi}{\alpha_0}\right)\left[(1+\alpha_0)\xi + \Theta_2\;\xi^3\right]\right\} \end{aligned}$$

and by equating the coefficients of the same power of ξ we find

$$\begin{aligned} &m_3 - \tfrac{4\pi}{3} = 0 &&m_5 - \frac{4\pi n\Theta_2}{5} = 0 \\ &\Theta_2 + \tfrac{1+\alpha_0}{2(n+1)}\left(m_3 + \tfrac{4\pi}{\alpha_0}\right) = 0 &&\Theta_4 + \frac{\Theta_2}{2(n+1)}\left(m_3 + \frac{4\pi}{\alpha_0}\right) = 0 \end{aligned} \tag{17.99}$$

from which we find

$$m_3 = \tfrac{4\pi}{3}, \qquad \Theta_2 = -2\pi\frac{(1+\alpha_0)(3+\alpha_0)}{3\alpha_0(n+1)}$$
$$m_5 = \tfrac{4\pi n\Theta_2}{5} \qquad \Theta_4 = -\frac{\Theta_2}{2(n+1)}\left(m_3 + \frac{4\pi}{\alpha_0}\right).$$

With these initial conditions we can integrate the structure equations and find the value of ξ where the function Θ vanishes so that the pressure vanishes and we are sure we have reached the boundary of the star. Be ξ_1 such value and $\Theta'_1 = \Theta'(\xi_1)$. From the second eq. (17.98) we find

$$\Theta'_1 = \left[\frac{\alpha_0}{(n+1)}\right]\frac{\mathcal{M}(\xi_1)}{\xi_1[\xi_1 - 2\mathcal{M}(\xi_1)]},$$

from which we find the mass of the star

$$\mathcal{M}(\xi_1) = \frac{(n+1)\xi_1^2|\Theta'_1|}{\alpha_0 + 2\xi_1(n+1)|\Theta'_1|}. \tag{17.100}$$

Once we know the function $\Theta(\xi)$, from eqs. (17.93) we know how the energy density and the pressure are distributed inside the star and we can compute the function $\mathcal{M}(\xi)$

$$\mathcal{M}(\xi) = \int_0^{\xi} 4\pi\epsilon_0{\xi'}^2\Theta^n(\xi')\, d\xi',$$

and the metric function $e^{2\lambda}$

$$e^{2\lambda} = \frac{1}{1 - \frac{2\mathcal{M}(\xi)}{\xi}}.$$

The remaining metric function $e^{2\nu}$ can be found from eq. (17.66) which now becomes

$$\nu = \int_0^{\xi} -\frac{p_{,\xi}}{(\epsilon+p)}d\xi + \nu_0 = \int_0^{\xi} -\frac{(n+1)\frac{d\Theta}{d\xi}}{\alpha_0+\Theta}d\xi + \nu_0 = ln\left[\frac{\alpha_0+1}{\alpha_0+\Theta(\xi)}\right]^{(n+1)} + \nu_0, \tag{17.101}$$

At the surface of the star the metric must reduce to the Schwarzschild metric and therefore

$$e^{2\nu_0}\cdot\left[\frac{(\alpha_0+1)}{(\alpha_0+\Theta)}\right]^{2(n+1)} = 1 - \frac{2\mathcal{M}(\xi)}{\xi}$$

which, using eq. (17.100), gives

$$e^{2\nu_0} = \left[\frac{\alpha_0}{\alpha_0+1}\right]^{2(n+1)}\cdot\frac{\alpha_0}{\alpha_0 + 2\xi_1(n+1)|\Theta'_1|} \tag{17.102}$$

Thus,

$$e^{2\nu(\xi)} = \left[\frac{\alpha_0}{\alpha_0+1}\right]^{2(n+1)}\frac{\alpha_0}{\alpha_0 + 2\xi_1(n+1)|\Theta'_1|}\cdot\left[\frac{\alpha_0+1}{\alpha_0+\Theta(\xi)}\right]^{2(n+1)} \tag{17.103}$$

and the solution is finally complete.

17.6 Buchdal's theorem

A theorem proved by Buchdal in 1959 establishes that the result obtained in the section 17.4 about the maximum value that the ratio M/R in a star of constant energy-density can reach, i.e. $M/R < 4/9$, is much more general. The theorem is based on the only assumption that the star is static, and that the energy density is positive, and monotonically decreasing function of the radial coordinate, i.e.

$$\epsilon \geq 0, \qquad \frac{d\epsilon}{dr} \leq 0.$$

No assumption is made on the equation of state that relates ϵ and the pressure p. The relevant equations are

$$\begin{cases} G_{rr} = 8\pi T_{rr} \\ G_{\theta\theta} = 8\pi T_{\theta\theta} \end{cases} \qquad \begin{cases} (1) \quad -\frac{e^{2\lambda}}{r^2}\left[1 - e^{-2\lambda}\right] + \frac{2}{r}\nu_{,r} = 8\pi p e^{2\lambda}, \\ (2) \quad r^2 e^{-2\lambda}\left[\nu_{,rr} + \nu_{,r}^2 + \frac{\nu_{,r}}{r} - \nu_{,r}\lambda_{,r} - \frac{\lambda_{,r}}{r}\right] = 8\pi r^2\, p, \end{cases} \tag{17.104}$$

By taking the following combination of eqs. (17.104)

$$r^2 e^{-2\lambda} EQ.(1) - EQ.(2) = 0, \tag{17.105}$$

we find

$$\frac{d}{dr}\left[\frac{e^{-\lambda}}{r}\frac{d}{dr}\left(e^{\nu}\right)\right] = e^{\nu+\lambda}\frac{d}{dr}\left[\frac{m(r)}{r^3}\right], \tag{17.106}$$

where, as usual, $m(r) = 4\pi \int_0^r \epsilon r'^2 dr'$. For any r we can always define a density $\overline{\epsilon_r}$ such that

$$m(r) = \frac{4}{3}\pi \overline{\epsilon_r} r^3, \tag{17.107}$$

and since ϵ is a monotonically decreasing function of r, $\overline{\epsilon_r}$, and consequently $\frac{m(r)}{r^3}$, are also monotonically decreasing. Hence

$$\frac{d}{dr}\left[\frac{e^{-\lambda}}{r}\frac{d\left(e^{\nu}\right)}{dr}\right] \leq 0. \tag{17.108}$$

It should be noted that the minimum value of $\overline{\epsilon_r}$ is attained at the boundary, where

$$M = \frac{4}{3}\pi \overline{\epsilon_R} R^3, \tag{17.109}$$

where $\overline{\epsilon_R} = \epsilon_{min}$. Inside the star $\overline{\epsilon_r} \geq \epsilon_{min}$, and consequently

$$\frac{4}{3}\pi \overline{\epsilon_r} R^3 \geq \frac{4}{3}\pi \epsilon_{min} R^3, \qquad \to \qquad m(r) \geq \frac{M}{R^3} r^3, \tag{17.110}$$

i.e. $m(r)$ is always bigger than what it would be if the density would be constant and equal to ϵ_{min}. From eq. (17.108) it follows that

$$\frac{e^{-\lambda}}{r}\frac{d\left(e^{\nu}\right)}{dr} \geq \frac{e^{-\lambda}}{r}\frac{d\left(e^{\nu}\right)}{dr}\Big|_{r=R}. \tag{17.111}$$

When $r = R$ the metric reduces to the Schwarzschild metric, therefore $e^{2\nu}\Big|_{r=R} = e^{-2\lambda}\Big|_{r=R} = 1 - \frac{2M}{R}$ and eq. (17.111) gives

$$\frac{e^{-\lambda}}{r}\frac{d\left(e^{\nu}\right)}{dr} \geq \frac{M}{R^3} \qquad \to \qquad \frac{de^{\nu}}{dr} \geq re^{\lambda}\frac{M}{R^3}. \tag{17.112}$$

By integrating eq. (17.112) between 0 and R we find

$$e^{\nu}(R) - e^{\nu}(0) \geq \frac{M}{R^3}\int_0^R re^{\lambda}dr, \tag{17.113}$$

and since $e^{-2\lambda} = 1 - \frac{2m(r)}{r}$

$$e^{\nu}(0) \leq \sqrt{1 - \frac{2M}{R}} - \frac{M}{R^3}\int_0^R \frac{rdr}{\sqrt{1 - \frac{2m(r)}{r}}}. \tag{17.114}$$

We want to establish an upper boundary for $e^{\nu}(0)$, and therefore we need to determine when the right hand side of eq. (17.114) attains its maximum value. From eq. (17.110) we know that $m(r) \geq \frac{M}{R^3}r^3$, and consequently

$$\sqrt{1 - \frac{2m(r)}{r}} \leq \sqrt{1 - \frac{2M}{R^3}r^2}, \qquad \to \qquad \int_0^R \frac{rdr}{\sqrt{1 - \frac{2m(r)}{r}}} \geq \int_0^R \frac{rdr}{\sqrt{1 - \frac{2M}{R^3}r^2}}. \tag{17.115}$$

Thus the maximum value of the right hand side of eq. (17.114) is

$$R.H.S._{max} = \sqrt{1 - \frac{2M}{R}} - \frac{M}{R^3}\int_0^R \frac{rdr}{\sqrt{1 - \frac{2M}{R^3}r^2}} = \frac{3}{2}\sqrt{1 - \frac{2M}{R}} - \frac{1}{2}, \tag{17.116}$$

and consequently

$$e^{\nu}(0) \ \leq \ \frac{3}{2}\sqrt{1 - \frac{2M}{R}} - \frac{1}{2}. \tag{17.117}$$

We shall now use the second hypothesis of the theorem, i.e. the condition that the metric is static. A static spacetime admits a timelike Killing vector, which must remain timelike in the interior of the star, i.e.

$$\vec{\xi} \cdot \vec{\xi} = g_{00}(\xi^0)^2 < 0 \qquad \to \qquad g_{00} = -e^{2\nu} < 0 \qquad \to \qquad e^{2\nu} > 0. \tag{17.118}$$

It follows from eq. (17.117) that

$$\frac{3}{2}\sqrt{1 - \frac{2M}{R}} - \frac{1}{2} > 0, \tag{17.119}$$

and finally

$$\frac{M}{R} < \frac{4}{9}. \tag{17.120}$$

Q.E.D.

It should be noted that, since $\frac{M}{R} < \frac{4}{9}$, it follows that $R > \frac{9}{4}M$, and since the Schwarzschild radius is $R_S = 2M$ this means that a star cannot have radius smaller than the Schwarzschild radius.

17.7 A necessary condition for the stability of a compact star

A solution of the TOV equations (17.63) which satisfies the appropriate boundary conditions discussed in section 17.3.1, describes a stellar configuration in hydrostatic equilibrium. This equilibrium can, in principle, be either stable or unstable. In this section we will study the conditions for stability.

Let us consider a sequence of equilibrium configurations obtained by integrating the TOV equations for an assigned EOS, with different values of the central energy density ϵ_0. The gravitational mass will thus be a function of ϵ_0: $M = M(\epsilon_0)$.

Let us consider the profile $M(\epsilon_0)$ given in figure 17.4. Each point of this curve identifies an equilibrium configuration. Given a star in the equilibrium configuration A if a small radial

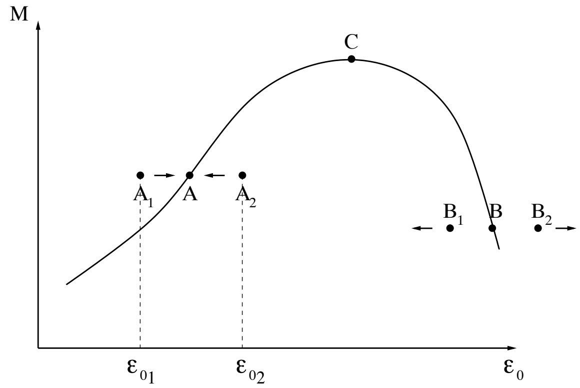

Figure 17.4: The mass of equilibrium stellar configurations is plotted versus the central energy density, near a relative maximum.

perturbation *reduces* its central energy density to a value, say, ϵ_{01}, the new (non-equilibrium) configuration will be represented by point A_1 (because the mass of the star does not change). Point A_1 is *above* the curve, therefore the perturbed star has a mass which is *larger* than the mass that the equilibrium configuration corresponding to ϵ_{01} would have. Consequently, the star is off equilibrium because gravity exceeds pressure, and the star will contract, so that its central density increases and it can return to the equilibrium configuration A.

In a similar way, if a perturbation *increases* the central energy density to ϵ_{02}, the new configuration is represented by a point A_2 *below* the curve. The star in A_2 has mass *smaller* than that of the equilibrium configuration corresponding to ϵ_{02}. In this case gravity is weaker than pressure, and the star will expand to return to the equilibrium configuration. Thus, the equilibrium in A is stable.

We can conclude that if A is a *stable* equilibrium configuration, in A

$$\frac{dM}{d\epsilon_0} > 0\,, \tag{17.121}$$

Conversely, a similar discussion about the point B where

$$\frac{dM}{d\epsilon_0} < 0\,, \tag{17.122}$$

shows that a displacement to B_1, brings the star to a configuration where gravity is weaker than pressure, so that the star expands further reducing the central density. Similarly, a displacement to B_2 brings the star to a configuration where gravity exceeds pressure, so that the star contracts, and the central density further increases: the equilibrium in B is unstable.

In figure 17.4, the branch on the left of the maximum C corresponds to stable configurations, that on the right to unstable configurations. The point C is the configuration of *maximum mass*.

An example is the case of Newtonian polytropes: the function $M(\epsilon_0)$, given in eq.(16.49), which we rewrite here for simplicity

$$M = 4\pi\ \xi_1^2\ |\Theta'(\xi_1)|\ \left[\frac{(n+1)K}{4\pi G}\right]^{\frac{3}{2}} \cdot \rho_0^{\frac{3-n}{2n}} \tag{17.123}$$

shows that M is an increasing function of the central density ρ_0 for $n < 3$, it is stationary for $n = 3$, and decreasing for $n > 3$; therefore the star is stable only if $n < 3$.

For a realistic equation of state, the curve $M(\epsilon_0)$ has a profile similar to that shown in figure 17.5. The stable branch on the left of point A represents white dwarf configurations,

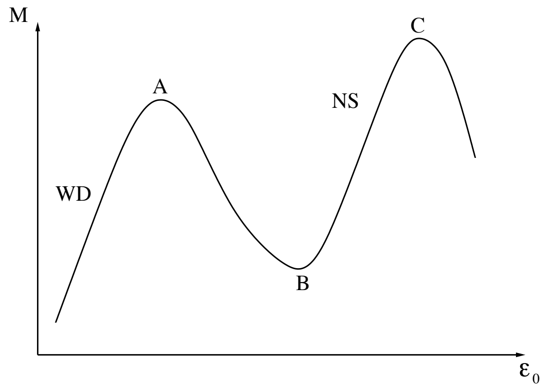

Figure 17.5: Masses of equilibrium stellar configurations vs. central densities. The stable branches corresponding to white dwarfs and neutron stars are explicitly shown.

while the stable branch BC represents neutron star configurations.

If we consider the stellar mass as a function of the radius, we find that since

$$\frac{dM}{dR} = \frac{dM}{d\epsilon_0} \cdot \frac{d\epsilon_0}{dR}\,, \tag{17.124}$$

the stability criterion (17.121) is satisfied if

a) $dR/d\epsilon_0 > 0$ **and** $dM/dR > 0$
or if
b) $dR/d\epsilon_0 < 0$ **and** $dM/dR < 0$.

In general, both for white dwarfs and for neutron stars the radius of the star *decreases* as the central density increases, therefore the stable branches of the function $M(R)$ are those for which

$$\frac{dM}{dR} < 0\,. \tag{17.125}$$

17.7.1 Is the condition $\frac{dM}{d\epsilon_0} > 0$ sufficient to say that a star is stable?

The question in the heading of this subsection can be rephrased as follows:
if $\frac{dM}{d\epsilon_0} > 0$, can we say that the star is stable?
The answer is No, and the reason can be understood by considering the theory of radial perturbations of stars. Since this interesting development is outside the scopes of this book, we shall just sketch the main results of the theory and give the basic notions to understand them.

A star has an infinite set of radial proper oscillation modes, labelled by an index $n = 0, 1, 2, \ldots$; when the star oscillates in a mode, each fluid element is displaced from the equilibrium position by a radial displacement $\xi(t, r)$. For the n_{th}- mode ξ has the form

$$\xi_n(r, t) = u_n(r)e^{\mathrm{i}\omega_n t} \tag{17.126}$$

where ω_n is the mode frequency and $u(r)$ its amplitude. The mode number n corresponds to the number of nodes that $u(r)$ has inside the star: $n = 0$ for zero nodes, $n = 1$ for 1 node etc. The mode frequencies are ordered:

$$\omega_0^2 < \omega_1^2 < \omega_2^2 < \ldots\,, \tag{17.127}$$

and the mode corresponding to ω_0 is said the *fundamental mode*.

If $\omega_n^2 > 0$, the fluid element oscillates about the equilibrium position and the mode is stable; conversely, if $\omega_n^2 < 0$ the radial displacement grows exponentially and the mode is unstable.

A stable fundamental mode corresponds to a global oscillation of the star, which is expanding or contracting all at the same time; indeed, this is also called the "breathing mode". When the star contracts the central density increases and $\xi(t, r) < 0$ throughout the star, when it expands the central density decreases and $\xi(t, r) > 0$. The previous discussion about stability clearly applies to this case.

However the star may oscillate in a different mode with $n > 0$. Since in this case $u(r)$ has one node inside the star, we may have a situation in which near the origin $\xi(t, r) < 0$ and the central density increases, but in some other region of the star $\xi(t, r) > 0$ and in that region the density would decrease. Thus, the previous discussion about stability would not be applicable, and we need to appeal to the theory of radial pulsation to understand what is going on. The theory states the following: suppose we compute a sequence of stellar models (with the same EOS) differing for the value of ϵ_0, and for each model we compute the mass and the frequency of the various radial modes. Knowing $M(\epsilon_0)$ along the sequence, we can compute $\frac{dM}{d\epsilon_0}$. If for some value of ϵ_0 we find that there is an extremal point, i.e.

$$\frac{dM}{d\epsilon_0} = 0\,, \tag{17.128}$$

then for that same ϵ_c the square of the frequency of one of the modes must cross the real axis and change sign, therefore in that point

$$\omega^2_{i,\epsilon_c} = 0. \tag{17.129}$$

This means the the i_{th}-mode becomes unstable.

In general, the $n = 0$ mode (which is the one with lowest frequency) is the first to become unstable.

Now suppose that we have the curve shown in figure 17.6 and suppose that for $\epsilon_0 < \epsilon_A$ the fundamental mode has frequency such that $\omega_0^2 > 0$, i.e. it is stable. A is an extremal

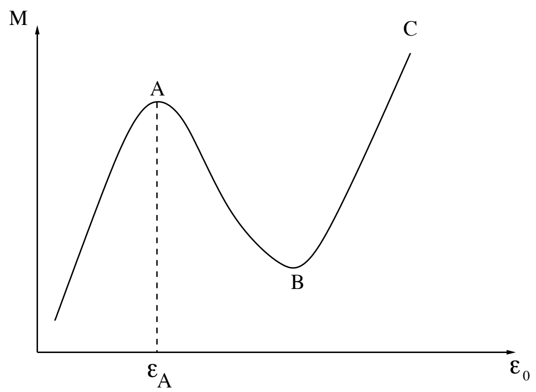

Figure 17.6: Masses of equilibrium stellar configurations vs. central densities. Though in the BC branch $\frac{dM}{d\epsilon_0} > 0$, that branch may correspond to unstable configuration, as explained in the text.

point, therefore in A $\omega_0^2 = 0$, and all configurations belonging to the branch AB will be unstable because their fundamental mode will have $\omega_0^2 < 0$.
If we increase the density we reach the second extremal point B. Here two things may happen:

1) ω_0^2 changes sign again becoming positive. In this case the star corresponding to B and all configurations of the branch BC would be stable. This is the situation we have described in the previous section.
or
2) ω_0^2 remains negative (i.e. the fundamental mode remains unstable) **and** the frequency of the $n = 1$ radial mode changes sign, so that also the $n = 1$ mode become unstable. In this case all configurations on the branch BC would be unstable.

This example clearly illustrates that the fact that $\frac{dM}{d\epsilon_0} > 0$ does not provide a sufficient condition for stability.