Chapter 16

Gravitational waves from rotating compact stars

In this section we shall show that a rotating star emits gravitational waves only if its shape deviates from axial symmetry.

16.1 Stars rigidly rotating around a symmetry axis

Consider an ellipsoid of uniform density ρ. Its quadrupole moment is

$$q_{ij} = \int_V \rho \; x_i x_j \; dx^3, \qquad i = 1, 3$$

and it is related to the inertia tensor

$$I_{ij} = \int_V \rho \left(r^2 \delta_{ij} - x_i x_j \right) \; dx^3$$

by the equation

$$q_{ij} = -I_{ij} + \delta_{ij} \, q \, ,$$

where $q \equiv q_m{}^m$. Consequently, the reduced quadrupole moment can be written as

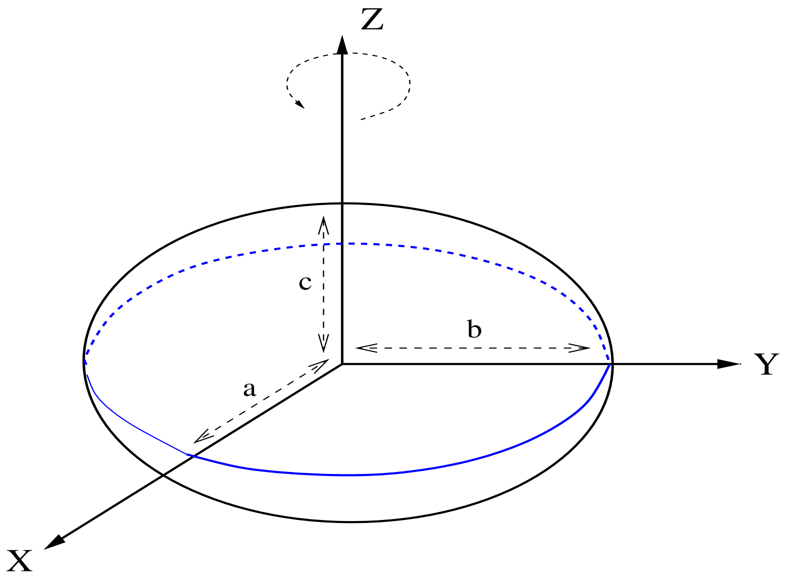
$$Q_{ij} = q_{ij} - \frac{1}{3} \delta_{ij} \, q = - \left(I_{ij} - \frac{1}{3} \delta_{ij} \, I \right) \, .$$

Let us first consider a non rotating ellipsoid, with semiaxes a, b, c, volume $V = \frac{4}{3} \pi abc$, and equation:

$$\left(\frac{x_1}{a} \right)^2 + \left(\frac{x_2}{b} \right)^2 + \left(\frac{x_3}{c} \right)^2 = 1 \, .$$

The inertia tensor is

$$I_{ij} = \frac{M}{5} \begin{pmatrix} b^2 + c^2 & 0 & 0 \\ 0 & c^2 + a^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{pmatrix} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \, ,$$



where I_1, I_2, I_3 are the principal moments of inertia.

Let us now consider an ellipsoid which rotates around one of its principal axes, for instance x^3, with angular velocity $(0, 0, \Omega)$. What is its inertia tensor in this case?
Be $\{x_i\}$ the coordinates of the inertial frame, and $\{x'_i\}$ the coordinates of a co-rotating frame. Then,

$$x_i = R_{ij} x'_j,$$

where R_{ij} is the rotation matrix

$$R_{ij} = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{with} \quad \varphi = \Omega t\,.$$

For instance, a point at rest in the co-rotating frame, with coordinates $x'_i = (1, 0, 0)$, has, in the inertial frame, coordinates $x_i = (\cos\Omega t, \sin\Omega t, 0)$, i.e. it rotates in the $x-y$ plane with angular velocity Ω.

Since in the co-rotating frame $\{x'_i\}$

$$I'_{ij} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix},$$

in the inertial frame $\{x_i\}$ it will be

$$\begin{aligned} I_{ij} &= R_{ik} R_{jl} I'_{kl} = (R I' R^T)_{ij} \\ &= \begin{pmatrix} I_1 \cos^2\varphi + I_2 \sin^2\varphi & -\sin\varphi\cos\varphi(I_2 - I_1) & 0 \\ -\sin\varphi\cos\varphi(I_2 - I_1) & I_1 \sin^2\varphi + I_2 \cos^2\varphi & 0 \\ 0 & 0 & I_3 \end{pmatrix}. \end{aligned}$$

It is easy to check that $\mathrm{Tr}\, I = I_1 + I_2 + I_3 = constant.$
The quadrupole moment therefore is

$$Q_{ij} = -\left(I_{ij} - \frac{1}{3}\delta_{ij}\mathrm{Tr}\, I\right) = -I_{ij} + constant$$

Using $\cos 2\varphi = 2\cos^2\varphi - 1$, etc., the quadrupole moment can be written as

$$Q_{ij} = \frac{I_2 - I_1}{2}\begin{pmatrix} \cos 2\varphi & \sin 2\varphi & 0 \\ \sin 2\varphi & -\cos 2\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix} + constant$$

Since

$$I_1 = \frac{M}{5}(b^2 + c^2), \qquad \text{and} \qquad I_2 = \frac{M}{5}(c^2 + a^2),$$

if a, b are equal, the quadrupole moment is constant and **no gravitational wave is emitted**.

This is a generic result: an axisymmetric object rigidly rotating around its symmetry axis does not radiate gravitational waves.

In realistic cases, $a \neq b$, and $I_1 \neq I_2$; however the difference is expected to be extremely small. It is convenient to express the quadrupole moment of the star in terms of a dimensionless parameter ϵ, the *oblateness*, which expresses the deviation from axisymmetry

$$\epsilon \equiv \frac{a - b}{(a + b)/2}\,.$$

It is easy to show that

$$\frac{I_2 - I_1}{I_3} = \epsilon + O(\epsilon^3)\,.$$

Indeed,

$$a - b = \frac{1}{2}\epsilon(a + b), \qquad (16.1)$$

thus

$$\frac{I_2 - I_1}{I_3} = \frac{a^2 - b^2}{a^2 + b^2} = \frac{(a + b)(a - b)}{a^2 + b^2} = \frac{1}{2}\epsilon\frac{(a + b)^2}{a^2 + b^2}\,. \qquad (16.2)$$

On the other hand, from (16.1) we have

$$(a - b)^2 = O(\epsilon^2) = a^2 + b^2 - 2ab, \qquad (16.3)$$

therefore

$$2ab = a^2 + b^2 + O(\epsilon^2) \qquad (16.4)$$

and

$$\frac{I_2 - I_1}{I_3} = \frac{1}{2}\epsilon\frac{a^2 + b^2 + 2ab}{a^2 + b^2} = \epsilon + O(\epsilon^3)\,. \qquad (16.5)$$

Consequently

$$Q_{ij} = \frac{\epsilon\, I_3}{2}\begin{pmatrix} \cos 2\varphi & \sin 2\varphi & 0 \\ \sin 2\varphi & -\cos 2\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix} + constant. \qquad (16.6)$$

Since $\varphi = \Omega t$, eq. (16.6) shows that GW are emitted at twice the rotation frequency. From eq. (14.40) and (14.44), the waveform is

$$h^{\mathbf{TT}}_{jk}(t,r) = \frac{2G}{rc^4}\,\mathcal{P}_{jklm}\,\left[\frac{d^2}{dt^2}\,Q_{lm}(t-\frac{r}{c})\right],$$

i.e., using eq. (16.6),

$$h^{\mathbf{TT}}_{ij} = h_0\left[\mathcal{P}\begin{pmatrix} -\cos 2\varphi_{ret} & -\sin 2\varphi_{ret} & 0 \\ -\sin 2\varphi_{ret} & \cos 2\varphi_{ret} & 0 \\ 0 & 0 & 0 \end{pmatrix}\right], \qquad \varphi_{ret} = \Omega\left(t - \frac{r}{c}\right), \tag{16.7}$$

where

$$h_0 = \frac{4G\,\Omega^2}{c^4 r}\,I_3\,\epsilon = \frac{16\pi^2 G}{c^4\,r\,T^2}\,I_3\,\epsilon\,, \tag{16.8}$$

where T is the rotation period; the term in square brackets in eq. (16.7) depends on the direction of the observer relative to the star axes. Eq. (16.7) shows that a triaxial star rotating around a principal axis emits gravitational waves at twice the rotation frequency

$$\nu_{GW} = 2\nu_{rot}\,. \tag{16.9}$$

Fastly rotating neutron stars have rotation period of the order of a few ms; a typical value of a neutron star moment of inertia is $\sim 10^{38} Kg\,m^2$. For a galactic source the distance from Earth is of a few kpc, thus, if we assume an oblateness as small as $\sim 10^{-6}$ we find

$$\frac{16\pi^2 G}{c^4\,r\,T^2}\,I_3\,\epsilon = \frac{16\pi^2 G}{c^4}\cdot(1ms)^{-2}\cdot(1Kpc)^{-1}\cdot(10^{38}Kg\,m^2)\cdot(10^{-6}) = 4.21\cdot 10^{-24}\,.$$

This calculation indicates that the wave amplitude can be normalized as follows

$$h_0 = 4.21\cdot 10^{-24}\left[\frac{ms}{T}\right]^2\left[\frac{Kpc}{r}\right]\left[\frac{I_3}{10^{38}Kg\,m^2}\right]\left[\frac{\epsilon}{10^{-6}}\right]\,. \tag{16.10}$$

The rotation period and the star distance can be measured; the moment of inertia can be estimated, if we choose an equation of state among those proposed in the literature to model matter in the neutron star interior; conversely, ϵ is unknown. However, we shall now show how astronomical observations allow to set an upper limit on this parameter. It is known that the rotation period of observed pulsars increases with time, i.e. the star rotational energy decreases. Pulsars slow down mainly because, having a time varying magnetic dipole moment, they radiate electromagnetic waves. A further braking mechanism is provided by gravitational wave emission. We shall now assume that the pulsar radiates its rotational energy entirely in gravitational waves and, using this very strong assumption and the expression of the gravitational luminosity (14.102) in terms of the source quadrupole moment (16.6), we shall show how to estimate the pulsar oblateness. This estimate will be an upper bound for ϵ because we know that only a small fraction of the pulsar energy is dissipated in gravitational waves.

From eq (16.6) we find

$$\dddot{Q}_{kn} = 4\Omega^3 \epsilon I_3 \begin{pmatrix} \sin 2\phi & -\cos 2\phi & 0 \\ -\cos 2\phi & \sin 2\phi & 0 \\ 0 & 0 & 0 \end{pmatrix} ; \tag{16.11}$$

by replacing this expression in (14.102) we find

$$L_{GW} = \frac{32G}{5c^5}\Omega^6 \epsilon^2 I_3^2 . \tag{16.12}$$

The rotational energy, in the Newtonian approximation, is

$$E_{rot} = \frac{1}{2} I_3 \Omega^2 , \tag{16.13}$$

and its time derivative

$$\dot{E}_{rot} = I_3 \Omega \dot{\Omega} . \tag{16.14}$$

Since $L_{GW} \leq -\dot{E}_{rot}$ (with equality if the spin-down is entirely due to gravitational emission), then

$$\frac{32G}{5c^5}(2\pi\nu)^6 \epsilon^2 I_3^2 \leq I_3 (2\pi)^2 \nu |\dot{\nu}| \tag{16.15}$$

(note that $|\dot{\nu}| = -\dot{\nu}$), therefore the spin-down limit on ϵ gives

$$\epsilon \leq \epsilon_{sd} = \left(\frac{5c^5 |\dot{\nu}|}{512\pi^4 G \nu^5 I_3} \right)^{1/2} . \tag{16.16}$$

For instance, in the case of the Crab pulsar, for which $\nu = 30$ Hz and $r = 2$ kpc, if we assume that the momentum of inertia is $I_3 = 10^{38}$ kg m^2, eq. (16.16) gives

$$\epsilon_{sd} = 7.5 \cdot 10^{-4} . \tag{16.17}$$

This calculation has been done for a number of known pulsars (A. Giazotto, S. Bonazzola and E. Gourgoulhon, *On gravitational waves emitted by an ensenble of rotating neutron stars*, Phys. Rev. **D55**, 2014, 1997) and the results are shown in Table 16.1.
As we said, these numbers are only upper bounds. Recent studies which take into account the maximum strain that the crust of a neutron star can support without breaking set a further constraint on ϵ

$$\epsilon \lesssim 5 \cdot 10^{-6} \tag{16.18}$$

(G. Ushomirsky, C. Cutler and L. Bildsten, *Deformations of accreting neutron star crusts and gravitational wave emission*, Mon. Not. Roy. Astron. Soc. **319**, 902, 2000).

The data collected during the past few years by the first generation of interferometric gravitational antennas VIRGO and LIGO are being analyzed; altough waves have not been detected yet, available data allow us to set more stringent constraints on the oblateness of some known pulsars. For instance, the present detectors sensitivity would allow to detect the gravitational signal emitted by the Crab pulsar if its amplitude would exceed $h_0 = 2.0 \cdot 10^{-25}$; since no signal has been detected, it means that

$$h_0 < 2.0 \cdot 10^{-25} , \tag{16.19}$$

Table 16.1: *Upper limits for the oblateness of an ensemble of known pulsars, obtained from spin-down measurements.*

name	ν_{GW} (Hz)	ϵ_{sd}
Vela	22	$1.8 \cdot 10^{-3}$
Crab	60	$7.5 \cdot 10^{-4}$
Geminga	8.4	$2.3 \cdot 10^{-3}$
PSR B 1509-68	13.2	$1.4 \cdot 10^{-2}$
PSR B 1706-44	20	$1.9 \cdot 10^{-3}$
PSR B 1957+20	1242	$1.6 \cdot 10^{-9}$
PSR J 0437-4715	348	$2.9 \cdot 10^{-8}$

and using eq. (16.10), this equation implies that the Crab oblateness satisfies the following constraint

$$\epsilon \leq 1.1 \cdot 10^{-4} \,. \tag{16.20}$$

This limit is *more restrictive* than the spin-down limit (16.17), even though it is larger than the theoretical value arising from the maximal strain sustainable by the crust, (16.18). However, this result is very important, since data analysis from LIGO/VIRGO tells us something which we did not know from astrophysical observation (Abbot *et al.*, Astrophysical Journal, 713, 671, 2010, "Searches for gravitational waves from known pulsars with S5 LIGO data").

16.2 Wobbling stars

Let us now consider the case in which the star rotates about an axis which forms an angle with one of the principal axes, say, I_3. The angle between the two axes is called "wobble angle". In this case, the angular velocity precedes around I_3 (see figure 16.1). For simplicity, let us assume that I_3 is a symmetry axis of the ellipsoid, i.e.

$$a = b \qquad \rightarrow \qquad I_1 = I_2 \,,$$

and that the wobble angle θ is small, i.e. $\theta \ll 1$. Be $\{x_i'\}$ the coordinates of the co-rotating frame $\mathbf{O}'$ and $\{x_i\}$ those of the inertial frame $\mathbf{O}$. As usual $x_i = R_{ij}x_j'$, where R_{ij} is the rotation matrix.

The transformation from $\mathbf{O}'$ to $\mathbf{O}$ is the composition of two rotations:

- A rotation of $\mathbf{O}'$ around the x' axis by a small angle θ (constant); the new frame $\mathbf{O}''$ has the z'' axis coincident with the rotation axis. The corresponding rotation matrix is

$$R_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & -\theta & 1 \end{pmatrix} + O(\theta^2) \,. \tag{16.21}$$

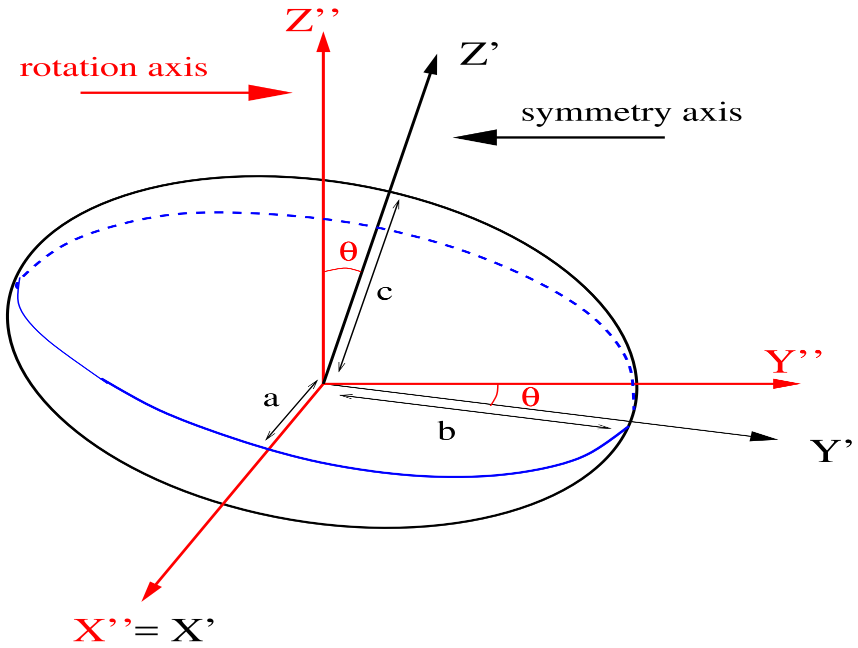


Figure 16.1: From $\mathbf{O}'$ to $\mathbf{O}''$

- A time dependent rotation around the z'' axis, by an angle $\varphi = \Omega t$; the corresponding rotation matrix is

$$R_z = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{16.22}$$

  After this rotation, the symmetry axis of the ellipsoid precedes around the z axis, with angular velocity Ω.

The rotation matrix from $\mathbf{O}'$ to $\mathbf{O}$ therefore is

$$\begin{aligned} R &= R_z R_x = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta \\ 0 & -\theta & 1 \end{pmatrix} + O(\theta^2) \\ &= \begin{pmatrix} \cos\varphi & -\sin\varphi & -\theta\sin\varphi \\ \sin\varphi & \cos\varphi & \theta\cos\varphi \\ 0 & -\theta & 1 \end{pmatrix} + O(\theta^2) . \end{aligned} \tag{16.23}$$

Since in the co-rotating frame $\mathbf{O}'$

$$I'_{ij} = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & I_3 \end{pmatrix} ,$$

in the inertial frame $\mathbf{O}$ it will be

$$I_{ij} = R_{ik} R_{jl} I'_{kl} = (R I' R^T)_{ij} \tag{16.24}$$

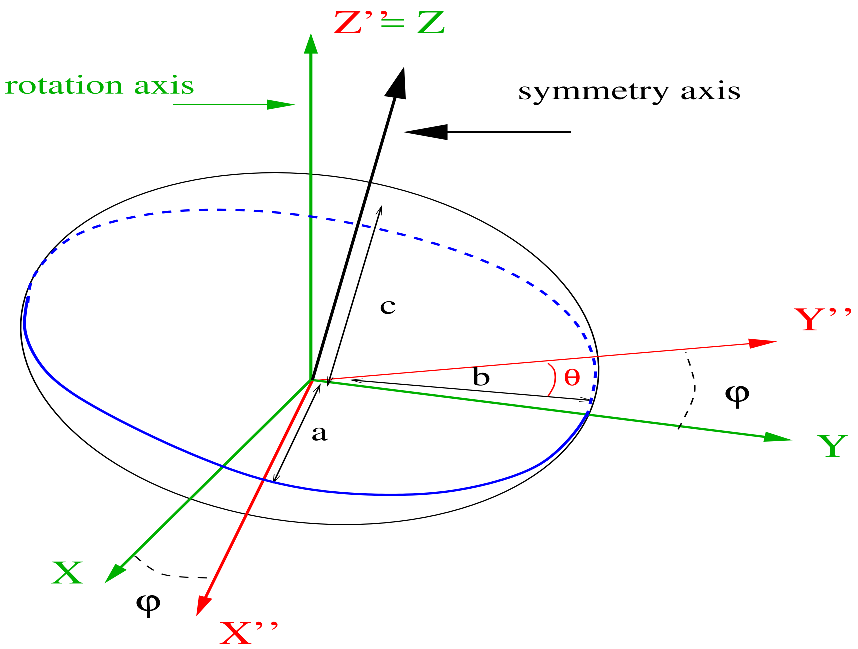


Figure 16.2: From $\mathbf{O}''$ to $\mathbf{O}$

$$= \begin{pmatrix} I_1 & 0 & (I_1 - I_3)\theta \ \sin\varphi \\ 0 & I_1 & -(I_1 - I_3)\theta \ \cos\varphi \\ (I_1 - I_3)\theta \ \sin\varphi & -(I_1 - I_3)\theta \ \cos\varphi & I_3 \end{pmatrix} + O(\theta^2)\,. \tag{16.25}$$

The quadrupole moment can then be written as

$$Q_{ij} = -I_{ij} + const. = (I_1 - I_3)\ \theta \begin{pmatrix} 0 & 0 & -\sin\varphi \\ 0 & 0 & \cos\varphi \\ -\sin\varphi & \cos\varphi & 0 \end{pmatrix} + const + O(\theta^2)\,, \tag{16.26}$$

and the wave amplitude therefore is

$$h^{\mathbf{TT}}_{jk}(t, r) = \frac{2G}{rc^4}\ \mathcal{P}_{jklm}\ \left[\frac{d^2}{dt^2}\ Q^{lm}(t - \frac{r}{c})\right],$$

i.e., using eq. (16.26),

$$h^{\mathbf{TT}}_{ij} = h_0 \left[\mathcal{P} \begin{pmatrix} 0 & 0 & \sin\varphi_{ret} \\ 0 & 0 & -\cos\varphi_{ret} \\ \sin\varphi_{ret} & -\cos\varphi_{ret} & 0 \end{pmatrix}\right], \qquad \varphi_{ret} = \Omega\left(t - \frac{r}{c}\right), \tag{16.27}$$

where

$$h_0 = \frac{2G\ \Omega^2}{c^4 r}\ (I_1 - I_3)\ \theta = \frac{8\pi^2 G}{c^4\ r\ T^2}\ (I_1 - I_3)\ \theta\ . \tag{16.28}$$

From eq. (16.27) we see that when the star rotates around an axis which does not coincide with a principal axis gravitational waves are emitted at the rotation frequency

$$\nu^{GW} = \nu_{rot}\,.$$

As the oblateness, the wobble angle is an unknown parameter.