Chapter 17

Einstein's equations and variational principles

In this chapter we shall show that Einstein's equations can be derived using a variational approach. We shall brifly remind how the action principle can be applied in special relativity to derive Euler-Lagrange's equations for a given field, and then generalize the procedure in presence of gravity.

17.0.1 Action principle in special relativity

Let us consider a collection of tensor fields in special relativity

$$\left\{\mathbf{\Phi}^{(A)}(x)\right\}_{A=1,\dots}, \tag{17.1}$$

where x denotes the point of coordinates $\{x^\mu\}$. We shall use symbols in boldface to denote a generic tensor.

The *action* is a functional of these fields and of their first derivatives, written as an integral of a Lagrangian density over the 4-dimensional volume:

$$I = \int d^4x\ \mathcal{L}\left(\mathbf{\Phi}^{(1)},\dots,\mathbf{\Phi}^{(A)},\dots,\frac{\partial\mathbf{\Phi}^{(1)}}{\partial x^\mu},\dots,\frac{\partial\mathbf{\Phi}^{(A)}}{\partial x^\mu},\dots\right). \tag{17.2}$$

All field variations $\delta\mathbf{\Phi}$ are assumed to vanish on the boundary of the integration volume or asymptotically, if the volume is infinite.

Let us consider the variation of the action with respect to a given field $\mathbf{\Phi}^{(A)}$

$$\begin{aligned}\delta I &= \int d^4x\left(\frac{\partial\mathcal{L}}{\partial\mathbf{\Phi}^{(A)}}\delta\mathbf{\Phi}^{(A)} + \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\mathbf{\Phi}^{(A)}}{\partial x^\mu}\right)}\delta\frac{\partial\mathbf{\Phi}^{(A)}}{\partial x^\mu}\right)\\ &= \int d^4x\left(\frac{\partial\mathcal{L}}{\partial\mathbf{\Phi}^{(A)}}\delta\mathbf{\Phi}^{(A)} + \frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\mathbf{\Phi}^{(A)}}{\partial x^\mu}\right)}\frac{\partial(\delta\mathbf{\Phi}^{(A)})}{\partial x^\mu}\right). \end{aligned} \tag{17.3}$$

(Note that the operations of variation and differentiation commute). The last term of this equation can be integrated by parts

$$\int d^4x \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \mathbf{\Phi}^{(A)}}{\partial x^\mu}\right)} \frac{\partial(\delta \mathbf{\Phi}^{(A)})}{\partial x^\mu} = \int d^4x \frac{\partial}{\partial x^\mu}\left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \mathbf{\Phi}^{(A)}}{\partial x^\mu}\right)}\delta\mathbf{\Phi}^{(A)}\right] - \int d^4x \frac{\partial}{\partial x^\mu}\left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \mathbf{\Phi}^{(A)}}{\partial x^\mu}\right)}\right]\delta\mathbf{\Phi}^{(A)} \tag{17.4}$$

By Gauss' theorem, the volume integral of the 4-divergence of $\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \mathbf{\Phi}^{(A)}}{\partial x^\mu}\right)}\delta\mathbf{\Phi}^{(A)}$ is equal to the integral of this quantity over the volume boundary; since $\delta\mathbf{\Phi}^{(A)}$ vanishes on the boundary of the integration volume, the first integral on the RHS of eq. (17.4) vanishes and eq. (17.3) becomes

$$\delta I = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \mathbf{\Phi}^{(A)}} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \mathbf{\Phi}^{(A)}}{\partial x^\mu}\right)}\right)\delta\mathbf{\Phi}^{(A)}\,. \tag{17.5}$$

The equation of motion for the considered field are then found by imposing the stationarity of δI with respect to it:

$$\delta I = 0, \qquad \forall\ \delta\mathbf{\Phi}^{(A)}\,,$$

and since the integral (17.5) has to vanish *for every* $\delta\mathbf{\Phi}^{(A)}(x)$, it follows that

$$\frac{\partial \mathcal{L}}{\partial \mathbf{\Phi}^{(A)}} - \frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \mathbf{\Phi}^{(A)}}{\partial x^\mu}\right)} = 0 \tag{17.6}$$

which are the Euler-Lagrange equations for the field $\mathbf{\Phi}^{(A)}$.

17.0.2 Action principle in general relativity

In general relativity, in addition to the fields $\left\{\mathbf{\Phi}^{(A)}\right\}$, there is the metric tensor field

$$\mathbf{g}(x) = (g_{\mu\nu}(x))\ , \tag{17.7}$$

which describes the gravitational field whose action is the *Einstein-Hilbert* action

$$I^{E-H} = \frac{c^3}{16\pi G}\int d^4x\ \sqrt{-g}\ R\,. \tag{17.8}$$

Due to the strong equivalence principle, in a locally inertial frame the dynamics of all fields $\left\{\mathbf{\Phi}^{(A)}\right\}$ except gravity is described by the action (17.2). Therefore, according to the principal of general covariance, in a general frame the action, which is a scalar, retains the same form provided $\eta_{\mu\nu} \to g_{\mu\nu}$, the partial derivatives $\frac{\partial}{\partial x^\mu}$ are replaced by covariant derivatives ∇_μ, and the integration volume element d^4x is replaced by the covariant volume element $\sqrt{-g}d^4x$.

With these replacements, we shall now show that the results of the previous section (in particular, the derivation of the Euler-Lagrange equations) remain valid. The total action is

$$I = I^{E-H} + I^{fields} \tag{17.9}$$

with

$$\begin{aligned} I^{fields} = \int d^4x \, \sqrt{-g}\mathcal{L}^{fields} \Big(&\mathbf{\Phi}^{(1)},, \dots, \mathbf{\Phi}^{(A)}, \dots, \\ &\nabla_\mu \mathbf{\Phi}^{(1)}, \dots, \nabla_\mu \mathbf{\Phi}^{(A)}, \dots, \mathbf{g}\Big) \, . \end{aligned} \tag{17.10}$$

Note that now the Lagrangian density $\mathcal{L}^{fields}$ depends explicitely on $\mathbf{g}$ because we have replaced $\eta_{\mu\nu}$ by $g_{\mu\nu}$ and $\frac{\partial}{\partial x^\mu}$ by ∇_μ.

As in special relativity, the equations for a given field $\mathbf{\Phi}^{(A)}$ are found by varying the action with respect to that field, and since Einstein-Hilbert's action does not depend on $\mathbf{\Phi}$, we find

$$\begin{aligned} \delta I \equiv \delta I^{fields} &= \int d^4x \, \sqrt{-g} \left(\frac{\partial \mathcal{L}^{fields}}{\partial \mathbf{\Phi}^{(A)}} \delta \mathbf{\Phi}^{(A)} + \frac{\partial \mathcal{L}^{fields}}{\partial \left(\nabla_\mu \mathbf{\Phi}^{(A)}\right)} \delta \nabla_\mu \mathbf{\Phi}^{(A)} \right) \\ &= \int d^4x \, \sqrt{-g} \left(\frac{\partial \mathcal{L}^{fields}}{\partial \mathbf{\Phi}^{(A)}} \delta \mathbf{\Phi}^{(A)} + \frac{\partial \mathcal{L}^{fields}}{\partial \left(\nabla_\mu \mathbf{\Phi}^{(A)}\right)} \nabla_\mu \delta \mathbf{\Phi}^{(A)} \right) \end{aligned} \tag{17.11}$$

where we have used the property $\delta \, \nabla_\mu = \nabla_\mu \, \delta$. The last term in eq. (17.11) can be integrated by parts

$$\begin{aligned} \int d^4x \, \sqrt{-g} \, \frac{\partial \mathcal{L}^{fields}}{\partial \left(\nabla_\mu \mathbf{\Phi}^{(A)}\right)} \nabla_\mu \delta \mathbf{\Phi}^{(A)} &= \int d^4x \, \sqrt{-g} \, \nabla_\mu \left[\frac{\partial \mathcal{L}^{fields}}{\partial \left(\nabla_\mu \mathbf{\Phi}^{(A)}\right)} \delta \mathbf{\Phi}^{(A)} \right] \\ &- \int d^4x \, \sqrt{-g} \, \nabla_\mu \left[\frac{\partial \mathcal{L}^{fields}}{\partial \left(\nabla_\mu \mathbf{\Phi}^{(A)}\right)} \right] \delta \mathbf{\Phi}^{(A)} \, . \end{aligned} \tag{17.12}$$

In order to show that the first integral on the RHS vanishes, we need to generalize Gauss' theorem in curved spacetime.

17.0.3 Gauss' theorem in curved space

In this section we shall enunciate Gauss' theorem in curved space.
Preliminary definitions:
- Given a manifold $\mathcal{M}$ described by coordinates $\{x^\mu\}$, and a metric $g_{\mu\nu}$ on $\mathcal{M}$.
- Given a submanifold $\mathcal{N} \subset \mathcal{M}$ described by coordinates $\{y^i\}$, such that on $\mathcal{N}$ $x^\mu = x^\mu(y^i)$.

We define the metric *induced* on $\mathcal{N}$ from $\mathcal{M}$ as

$$\gamma_{ij} \equiv \frac{\partial x^\mu}{\partial y^i} \frac{\partial x^\nu}{\partial y^j} g_{\mu\nu} \, . \tag{17.13}$$

We can now generalize Gauss' theorem to curved space:
- Be Ω an n-dimensional volume described by coordinates $\{x^\mu\}_{\mu=0,\dots,n-1}$, and $g_{\mu\nu}$ the metric

on Ω.
- Be $\partial\Omega$ the boundary of Ω, described by coordinates $\{y^j\}_{j=0,\dots,n-2}$ with normal vector n_μ (having $|n_\mu n^\mu| = 1$ for a timelike or spacelike surface); be γ_{ij} the metric induced on $\partial\Omega$ from $g_{\mu\nu}$.
- Given a vector field V^μ defined in Ω, then

$$\int_\Omega d^4x\ \sqrt{-g}\ \nabla_\mu V^\mu = \int_{\partial\Omega} d^3y\ \sqrt{-\gamma}\ V^\mu n_\mu\,. \tag{17.14}$$

If we define the surface integration element as

$$dS_\mu \equiv \sqrt{-\gamma} n_\mu d^3y\,, \tag{17.15}$$

Gauss' theorem can also be written as

$$\int_\Omega d^4x\ \sqrt{-g}\ \nabla_\mu V^\mu = \int_{\partial\Omega} V^\mu dS_\mu\,. \tag{17.16}$$

In particular, if one considers an infinite volume, and if V^μ vanishes asymptotically, then the volume integral of $\nabla_\mu V^\mu$ vanishes.
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Using Gauss' theorem generalized to curved spacetime, and the condition that $\delta\mathbf{\Phi}^{(A)} = 0$ on the volume boundary, it is easy to see that eq. (17.12) reduces to

$$\int d^4x\ \sqrt{-g}\ \frac{\partial\mathcal{L}^{fields}}{\partial\left(\nabla_\mu\mathbf{\Phi}^{(A)}\right)}\nabla_\mu\delta\mathbf{\Phi}^{(A)} = -\int d^4x\ \sqrt{-g}\ \nabla_\mu\left[\frac{\partial\mathcal{L}^{fields}}{\partial\left(\nabla_\mu\mathbf{\Phi}^{(A)}\right)}\right]\delta\mathbf{\Phi}^{(A)}\ . \tag{17.17}$$

and consequently eq. (17.11) becomes

$$\delta I \ \equiv\ \delta I^{fields} = \int d^4x\ \sqrt{-g}\left(\frac{\partial\mathcal{L}}{\partial\mathbf{\Phi}^{(A)}} - \nabla_\mu\frac{\partial\mathcal{L}}{\partial\left(\nabla_\mu\mathbf{\Phi}^{(A)}\right)}\right)\delta\mathbf{\Phi}^{(A)}\ . \tag{17.18}$$

Finally, by imposing

$$\delta I = 0, \qquad \forall\ \delta\mathbf{\Phi}^{(A)}\,,$$

we find the Euler-Lagrange equations for the field $\mathbf{\Phi}^{(A)}$, generalized to curved spacetime:

$$\frac{\partial\mathcal{L}}{\partial\mathbf{\Phi}^{(A)}} - \nabla_\mu\frac{\partial\mathcal{L}}{\partial\left(\nabla_\mu\mathbf{\Phi}^{(A)}\right)} = 0\ . \tag{17.19}$$

17.1 Einstein's equations in vacuum

We shall now derive Einstein's equations in vacuum, by varying the *Einstein-Hilbert* action

$$I^{E-H} = \frac{c^3}{16\pi G}\int d^4x\ \sqrt{-g}\ R \tag{17.20}$$

with respect to the metric tensor:

$$\delta I^{E-H} = \frac{c^3}{16\pi G}\int d^4x\ \delta(\sqrt{-g}\ R) = \frac{c^3}{16\pi G}\int d^4x\ \left[\delta(\sqrt{-g})\ R + \sqrt{-g}\ \delta R\right]\ . \tag{17.21}$$

17.1.1 Evaluation of $\delta(\sqrt{-g})$

$$\delta g = \frac{\partial g}{\partial g_{\mu\nu}} \delta g_{\mu\nu} \,. \tag{17.22}$$

The determinant g is a polynomial in $g_{\mu\nu}$, i.e.

$$g = g(g_{\mu\nu}) \,. \tag{17.23}$$

We remind that g is given by the following formula

$$g = \sum_\nu g_{\mu\nu} M_{\mu\nu} (-1)^{\mu+\nu} \quad \text{(no sum over } \mu) \tag{17.24}$$

where μ is fixed, $M_{\mu\nu}$ is the minor μ, ν, i.e. the determinant of the matrix obtained by cutting the row μ and the column ν from the matrix $g_{\mu\nu}$. Thus, by differentiating g with respect to $g_{\mu\nu}$ we find

$$\frac{\partial g}{\partial g_{\mu\nu}} = (-1)^{\mu+\nu} M_{\mu\nu} \tag{17.25}$$

(no sum on μ and on ν).

Since the components of $g^{\mu\nu}$, the matrix inverse to $g_{\mu\nu}$, are given by

$$g^{\mu\nu} = \frac{1}{g} M_{\mu\nu} (-1)^{\mu+\nu} \,, \tag{17.26}$$

eq. (17.25) becomes

$$\frac{\partial g}{\partial g_{\mu\nu}} = g g^{\mu\nu} \,. \tag{17.27}$$

Thus

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu} \,. \tag{17.28}$$

Furthermore, since

$$\begin{aligned} \delta(g_{\mu\nu} g^{\mu\nu}) &= 0 \\ &= \delta g_{\mu\nu} g^{\mu\nu} + g_{\mu\nu} \delta g^{\mu\nu} \,, \end{aligned} \tag{17.29}$$

equation (17.28) becomes

$$\delta g = -g g_{\mu\nu} \delta g^{\mu\nu} \,, \tag{17.30}$$

and

$$\delta(\sqrt{-g}) = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu} \,. \tag{17.31}$$

Using this result, eq. (17.21) can be written as

$$\delta I^{E-H} = \frac{c^3}{16\pi G} \int d^4x \, \left[-\frac{1}{2} \sqrt{-g} \; g_{\mu\nu} \delta g^{\mu\nu} \; R + \sqrt{-g} \; \delta R \right] \,. \tag{17.32}$$

17.1.2 Evaluation of δR

In order to evaluate δR, we need to prove the *Palatini identity*:

$$\delta R_{\mu\nu} = (\delta \Gamma^\lambda_{\mu\nu})_{;\lambda} - (\delta \Gamma^\lambda_{\mu\lambda})_{;\nu} \,. \tag{17.33}$$

PROOF

By varying the Ricci tensor:

$$R_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\mu\lambda,\nu} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{\alpha\lambda} - \Gamma^{\lambda}_{\alpha\nu}\Gamma^{\alpha}_{\mu\lambda}\,, \tag{17.34}$$

we find

$$\begin{aligned} \delta R_{\mu\nu} &= \delta\Gamma^{\lambda}_{\mu\nu,\lambda} - \delta\Gamma^{\lambda}_{\mu\lambda,\nu} + \delta\Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{\alpha\lambda} \\ &- \delta\Gamma^{\lambda}_{\alpha\nu}\Gamma^{\alpha}_{\mu\lambda} + \Gamma^{\alpha}_{\mu\nu}\delta\Gamma^{\lambda}_{\alpha\lambda} - \Gamma^{\lambda}_{\alpha\nu}\delta\Gamma^{\alpha}_{\mu\lambda}\,. \end{aligned} \tag{17.35}$$

To evaluate $\delta\Gamma^{\lambda}_{\mu\nu}$ we define

$$\Gamma_{\mu\nu\,\delta} \equiv g_{\delta\lambda}\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}\left(g_{\mu\delta,\nu} + g_{\nu\delta,\mu} - g_{\mu\nu\,,\delta}\right)\ , \tag{17.36}$$

and write $\delta\Gamma^{\lambda}_{\mu\nu}$ as follows

$$\begin{aligned} \delta\Gamma^{\lambda}_{\mu\nu} = \delta\left[g^{\lambda\delta}\Gamma_{\mu\nu\,\delta}\right] &= \delta g^{\lambda\delta}\Gamma_{\mu\nu\,\delta} + g^{\lambda\delta}\delta\Gamma_{\mu\nu\,\delta} \\ &= -g^{\rho\lambda}g^{\sigma\delta}\delta g_{\rho\sigma}\Gamma_{\mu\nu\,\delta} + g^{\lambda\rho}\delta\Gamma_{\mu\nu\,\rho} \\ &= -g^{\lambda\rho}\delta g_{\rho\sigma}\Gamma^{\sigma}_{\mu\nu} + g^{\lambda\rho}\frac{1}{2}\left[\delta g_{\mu\rho,\nu} + \delta g_{\nu\rho,\mu} - \delta g_{\mu\nu,\rho}\right] \\ &= \frac{1}{2}g^{\lambda\rho}\left[\delta g_{\mu\rho,\nu} + \delta g_{\nu\rho,\mu} - \delta g_{\mu\nu,\rho} - 2\Gamma^{\sigma}_{\mu\nu}\delta g_{\rho\sigma}\right]\ , \end{aligned} \tag{17.37}$$

where we have used eq. (**??**). Eq. (17.37) can be rearranged as follows

$$\begin{aligned} \delta\Gamma^{\lambda}_{\mu\nu} &= \frac{1}{2}g^{\lambda\rho}\Big[\left(\delta g_{\mu\rho,\nu} - \Gamma^{\alpha}_{\mu\nu}\delta g_{\alpha\rho} - \Gamma^{\alpha}_{\nu\rho}\delta g_{\alpha\mu}\right) + \left(\delta g_{\nu\rho,\mu} - \Gamma^{\alpha}_{\nu\mu}\delta g_{\alpha\rho} - \Gamma^{\alpha}_{\rho\mu}\delta g_{\alpha\nu}\right) \\ &\quad - \left(\delta g_{\mu\nu,\rho} - \Gamma^{\alpha}_{\mu\rho}\delta g_{\alpha\nu} - \Gamma^{\alpha}_{\nu\rho}\delta g_{\alpha\mu}\right)\Big] \\ &= \frac{1}{2}g^{\lambda\rho}\left[\delta g_{\mu\rho;\nu} + \delta g_{\nu\rho;\mu} - \delta g_{\mu\nu;\rho}\right]. \end{aligned} \tag{17.38}$$

Since $\delta g_{\mu\nu}$ is a tensor, from eq. (17.38) it follows that $\delta\Gamma^{\lambda}_{\mu\nu}$ is also a tensor. Therefore, the quantity

$$(\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda} - (\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu}$$

is a tensor and can be evaluated with the usual rules of covariant differentiation:

$$\begin{aligned} \delta R_{\mu\nu} &= (\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda} - (\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu} \\ &= \delta\Gamma^{\lambda}_{\mu\nu,\lambda} - \delta\Gamma^{\lambda}_{\mu\lambda,\nu} + \delta\Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{\alpha\lambda} \\ &- \delta\Gamma^{\lambda}_{\alpha\nu}\Gamma^{\alpha}_{\mu\lambda} + \Gamma^{\alpha}_{\mu\nu}\delta\Gamma^{\lambda}_{\alpha\lambda} - \Gamma^{\lambda}_{\alpha\nu}\delta\Gamma^{\alpha}_{\mu\lambda}\,. \end{aligned} \tag{17.39}$$

A comparison of this equation with eq. (17.35)) shows that

$$\delta R_{\mu\nu} = (\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda} - (\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu}\,.$$

QED.
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δR can now be found using the Palatini identity as follows.

$$\delta R = \delta(g^{\mu\nu}R_{\mu\nu}) = \delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}.$$

The last term gives

$$\begin{aligned} g^{\mu\nu}\delta R_{\mu\nu} &= g^{\mu\nu}\left[(\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda} - (\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu}\right] = (g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu})_{;\lambda} - (g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\lambda})_{;\nu} \\ &= \left(g^{\mu\nu}\delta\Gamma^{\alpha}_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^{\lambda}_{\mu\lambda}\right)_{;\alpha}; \end{aligned} \tag{17.40}$$

(remember that $g_{\mu\nu;\alpha} = 0$). The term $\left(g^{\mu\nu}\delta\Gamma^{\alpha}_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^{\lambda}_{\mu\lambda}\right)_{;\alpha}$ is the covariant divergence of a vector; therefore by Gauss' theorem it vanishes when integrated over the 4-volume $\sqrt{-g}d^4x$, provided $\delta\Gamma^{\alpha}_{\mu\nu}$ vanishes on the boundary of the integration volume for $\forall\ \delta g_{\mu\nu}$. This condition must be verified for any 4-dimensional, infinitesimal coordinate transformation which leaves the coordinates on the boundary unchanged. It can be shown that, as a consequence of this assumption, $\delta g_{\mu\nu,\alpha} = 0$ for $\forall\ \delta g_{\mu\nu}$. Consequently,

$$\delta R = \delta g^{\mu\nu}R_{\mu\nu}\ + \text{surface terms}, \tag{17.41}$$

and the variation of the Einstein-Hilbert action (17.32) finally becomes

$$\delta I^{E-H} = \frac{c^3}{16\pi G}\int d^4x\ \sqrt{-g}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right]\delta g^{\mu\nu}\ . \tag{17.42}$$

By imposing

$$\delta I^{E-H} = 0, \qquad \forall\ \delta g^{\mu\nu}\ ,$$

we finally find Einstein's equations in vacuum

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0\ .$$

17.2 Einstein's equations with source

If the source of the gravitational field is some matter field, or some other field (for instance an electromagnetic field), the corresponding equations can be found by varying the total action with respect to the metric tensor $\mathbf{g}$

$$\delta I^{E-H} + \delta I^{fields} = 0 \qquad \forall\ \delta g^{\mu\nu}\ ,$$

where I^{E-H} and I^{fields} are given by eqs. (17.20) and (17.10), respectively. The variation δI^{fields} can easily be found using eq. (17.31)

$$\delta I^{fields} = \int d^4x\ \delta\left[\sqrt{-g}\ \ \mathcal{L}^{fields}\left(\mathbf{\Phi}^{(1)},\dots,\mathbf{\Phi}^{(A)},\dots,\nabla_\mu\mathbf{\Phi}^{(1)},\dots,\nabla_\mu\mathbf{\Phi}^{(A)},\dots,\mathbf{g}\right)\right] \tag{17.43}$$

$$= \int d^4x\ \sqrt{-g}\left[\frac{\partial\mathcal{L}^{fields}}{\partial g^{\mu\nu}} - \frac{1}{2}\mathcal{L}^{fields}g_{\mu\nu}\right]\delta g^{\mu\nu}\ . \tag{17.44}$$

In addition we know that

$$\delta I^{E-H} = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} \left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right] \delta g^{\mu\nu} \ , \tag{17.45}$$

therefore if we define

$$T_{\mu\nu} \equiv -2c \left[\frac{\partial \mathcal{L}^{fields}}{\partial g^{\mu\nu}} - \frac{1}{2} \mathcal{L}^{fields} g_{\mu\nu} \ , \right] \tag{17.46}$$

the variation of the total action can be written as

$$\delta I = \frac{c^3}{16\pi G} \int d^4x \sqrt{-g} \ \left[R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{8\pi G}{c^4} T_{\mu\nu} \right] \delta g^{\mu\nu} = 0 \ , \tag{17.47}$$

from which Einstein's equations

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

immediatly follow.