Particle Physics - Chapter 2
Hadron structure

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AA 21-22

last mod.
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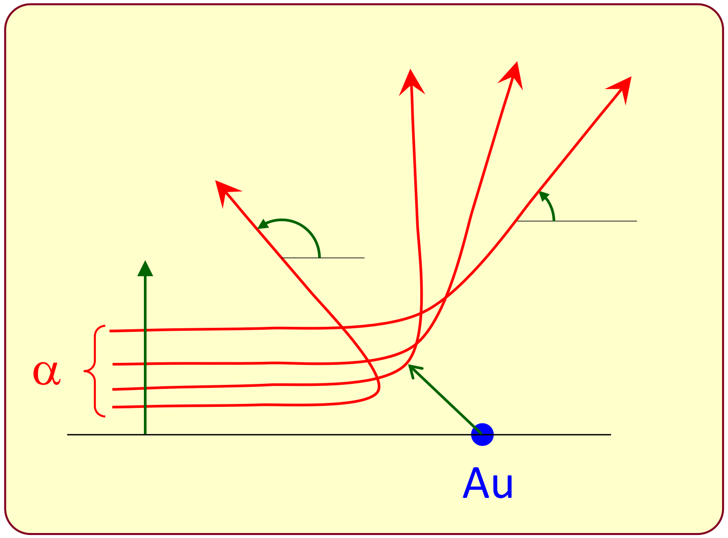
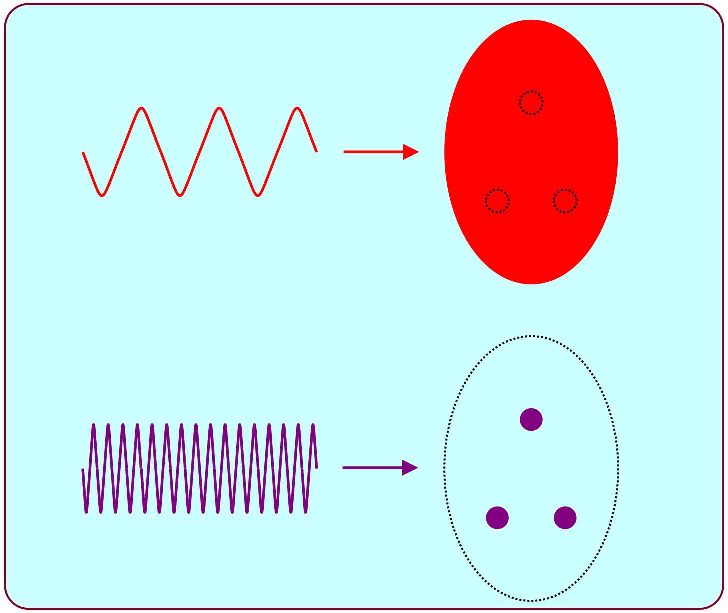
2 – Hadron structure





brief historical summary

"Hegel remarks somewhere that all great, world-historical facts and personages occur (…) twice. He has forgotten to add: the first time as tragedy, the second as farce." [Karl Marx, The 18th Brumaire of Louis Bonaparte]

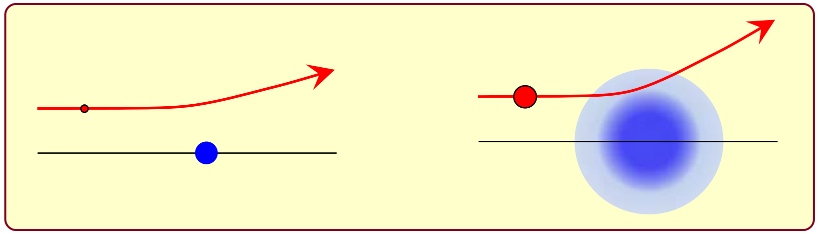
Despite this famous sentence, in this chapter a story is told, neither tragic nor farcical, which happened at least three times in the 20th century: *in a scattering experiment, a projectile probes the deep structure of the target; the scale of the observation depends on the energy of the probe:*

1. 1911 (Rutherford) α particles $\rightarrow$ gold (nucleus) [$\rightarrow$ *FNSN*];
2. 1950-60 (Hofstadter) e^- $\rightarrow$ H/D/He (nuclear structure);
3. 1965-80 (SLAC/CERN) e/ν $\rightarrow$ hadronic matter (quarks/partons)
4. 20xx [*possibly, maybe you*] a new *substructure* emerging ???

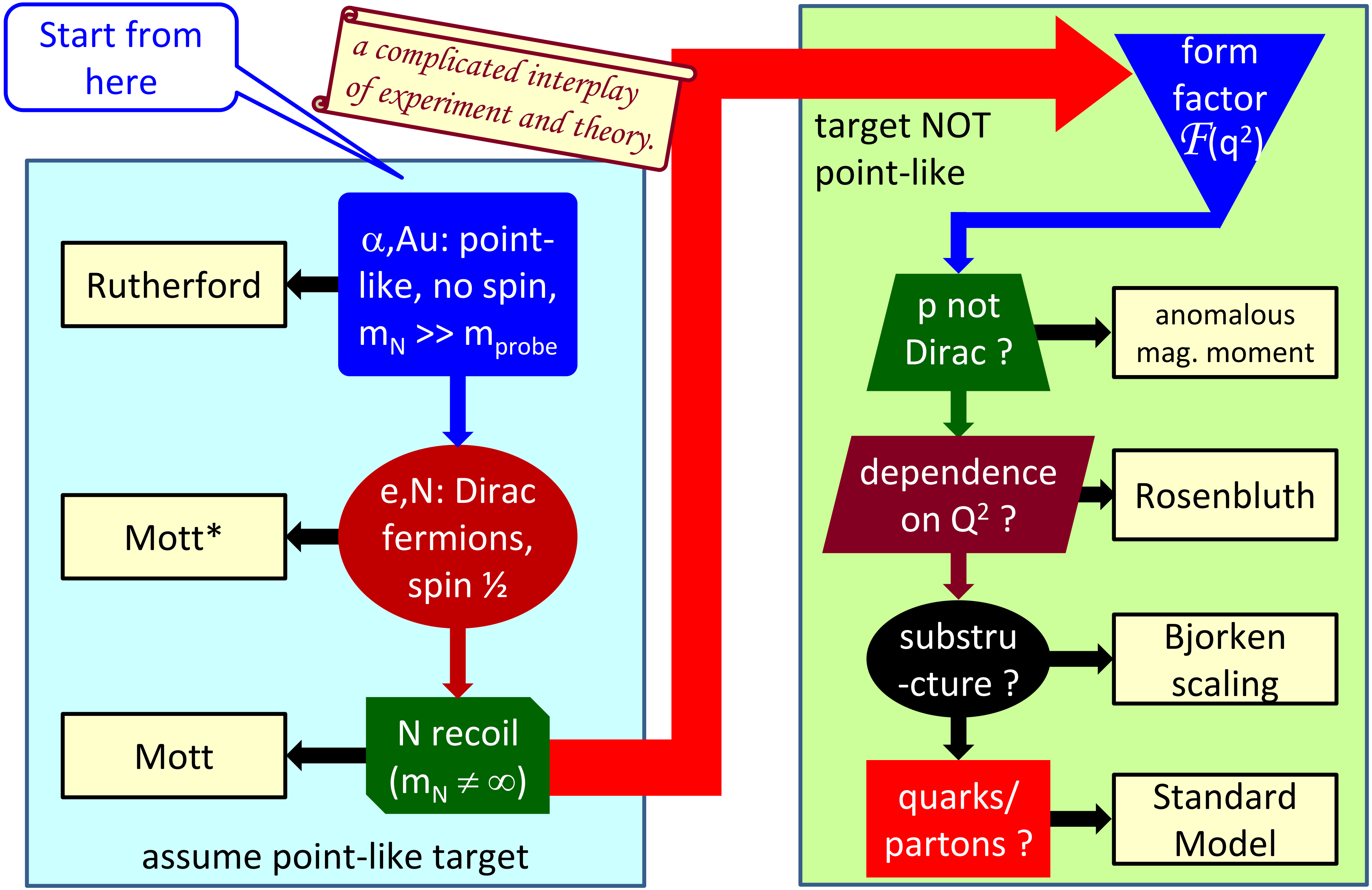
The deep meaning of the mechanism resides in Quantum Mechanics, which relates the space scale of a phenomenon with the (transverse) momentum of the scattered particles.

The role of technology is also important: the observation is possible because of powerful accelerators and detectors.

We will follow the history and therefore will study phenomena of ever smaller size [*look the contents page*].



the treasure map for scattering



the scattering experiment

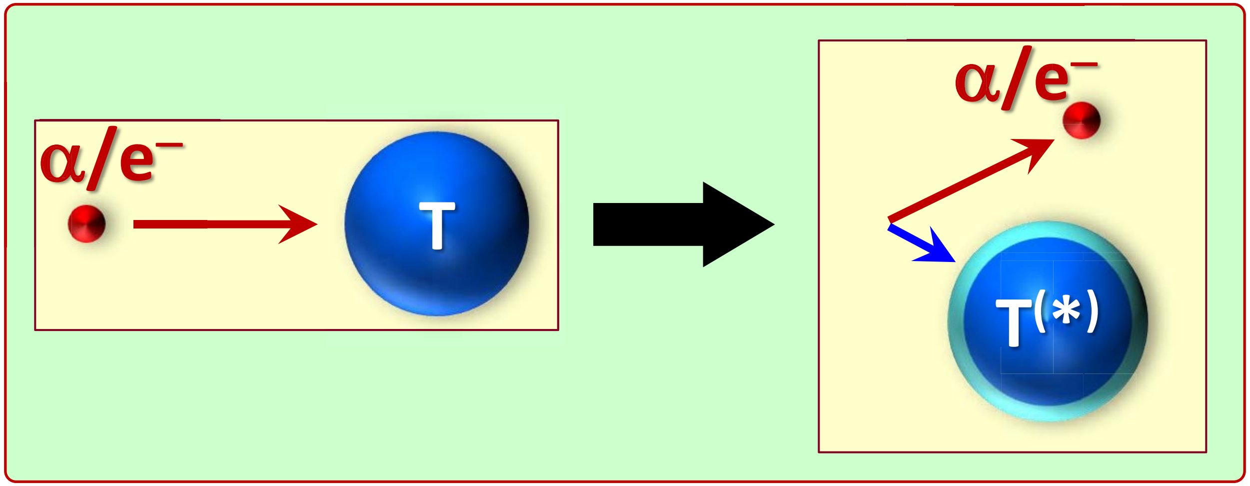
Q : is the target a **pointlike simple object** ? if not, how to probe its shape ?

A : *(à la Rutherford, but (a) he used α particles, (b) he did NOT see the nucleus size)*

- take a probe: e.g. an electron (e^-),
- study the scattering e^-T, [T=Nucl-eus/on]
- measure the cross section $\sigma(e^-T)$,
- … and the angular distribution of the e^-;
- … and detect the excited states or the final state hadronic system ("inelastic interactions").

Path:

1. study the kinematics (*);
2. compute $\sigma(e^-T)$ for pointlike nuclei in classical electrodynamics (Rutherford formula);
3. ditto in QM for spin ½ electrons and pointlike nuclei (Mott formula);
4. detect **deviations** from these models → derive informations on nuclear structure;
5. **new theory @ smaller distance (i.e. higher Q^2) → experiment → deviations → newer theory → … → … → (possibly ad infinitum)**



*(*) We call "kinematics" the equations which follow from space / angular momentum conservation and mass. The game is to study the "**dynamics**" after imposing the "kinematical" constraints.*

Fermi gas model

- Nuclei are bound states of protons (p) and neutrons (n).
- A simple model: the Fermi gas:

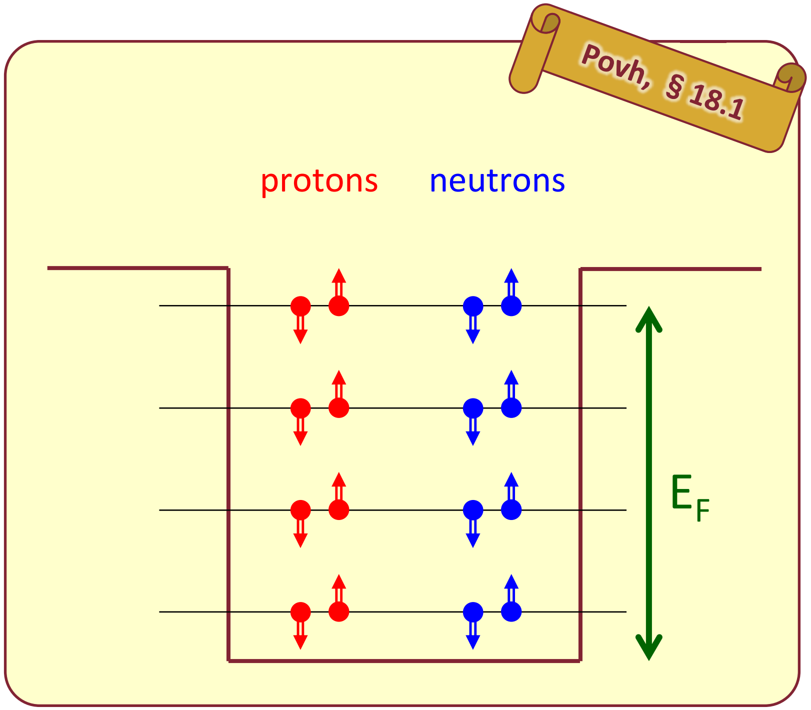
- p, n identical, but charge :
  - little spheres $r = r_0$, mass = m;
  - spin ½ fermions, pure Dirac-like;
  - bound inside the nucleus, otherwise free to move;
- define:
  - $n_{neutr.}$(= N), $n_{prot.}$ (= Z), A = N + Z,
  - p_{Fermi} (= p_F), E_{Fermi} (= E_F);

  $\rightarrow V_{Nucl}$ [$\propto$ A] = $4\pi r_0^3 A/3$;
- no e.m. interactions, only nuclear $\rightarrow$ N = Z = A/2, $p_F^p = p_F^n$, $E_F^p = E_F^n$ [better approx (not here):different interactions $\rightarrow p_F^p \neq p_F^n$];
- uncertainty principle $\rightarrow$ each p/n fills $V_{phase\ space} = [2\pi\hbar]^3$.

Therefore:

- well-shaped potential (⊔), identical for p/n, i.e. only interactions p↔p n↔n;
- Fermi statistics $\rightarrow$ two p/n per energy level (spin ⇑⇓);

[...*next page*...]



Fermi gas model: results

From those approximations, an elementary computation :

$$n^{n,\Uparrow} = n^{n,\Downarrow} = n^{p,\Uparrow} = n^{p,\Downarrow} = \frac{N}{2} = \frac{Z}{2} = \frac{A}{4} =$$

$$= \frac{\left[V_{space} V_{mom}\right]_{TOT}}{\left[V_{space} V_{mom}\right]_{each\ part}} = \frac{\frac{4}{3}\pi r_0^3 A \times \frac{4}{3}\pi p_F^3}{\left[2\pi\hbar\right]^3} =$$

$$= \frac{2 A r_0^3 p_F^3}{9\pi\hbar^3};$$

$$N = Z = \frac{\cancel{A}}{2} = \frac{4 \cancel{A} r_0^3 p_F^3}{9\pi\hbar^3}; \qquad p_F = \frac{\hbar}{r_0}\sqrt[3]{9\pi/8};$$

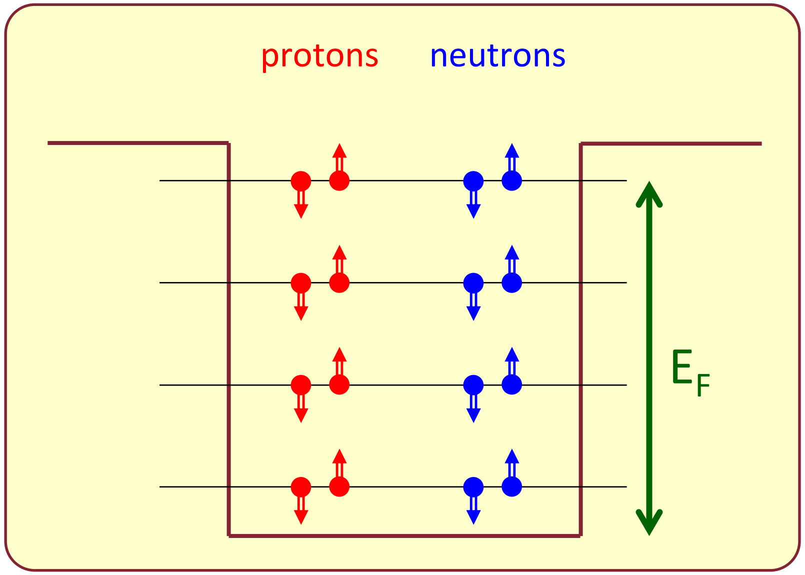
$$r_0 \approx 1.2 \text{ fm} \rightarrow \begin{cases} p_F \approx 250 \text{ MeV}; \\ E_F^{kin} = p_F^2/2m \approx 33 \text{ MeV}. \end{cases}$$

p, E << m, so non-relativistic approx

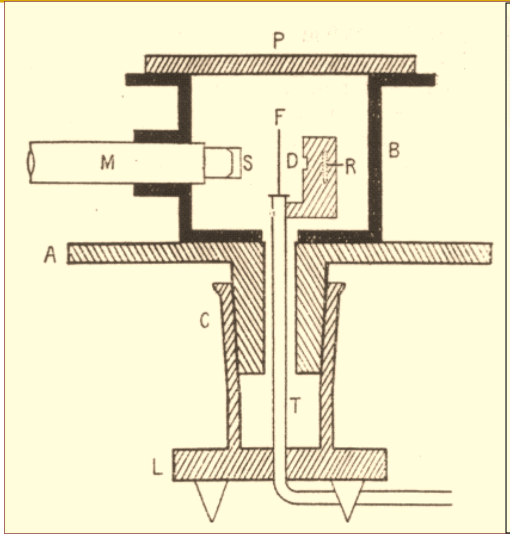
fit from form factors (see later)

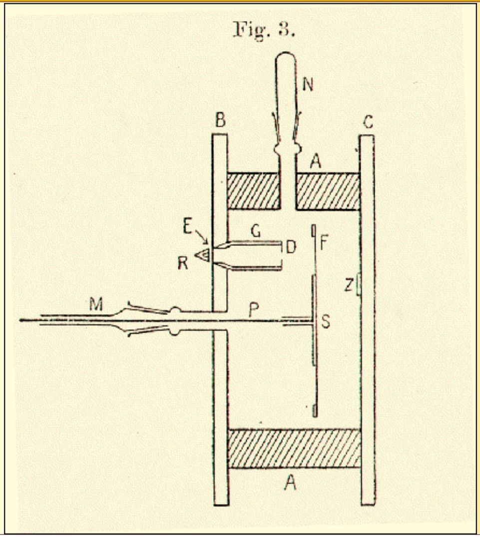
Conclusions :

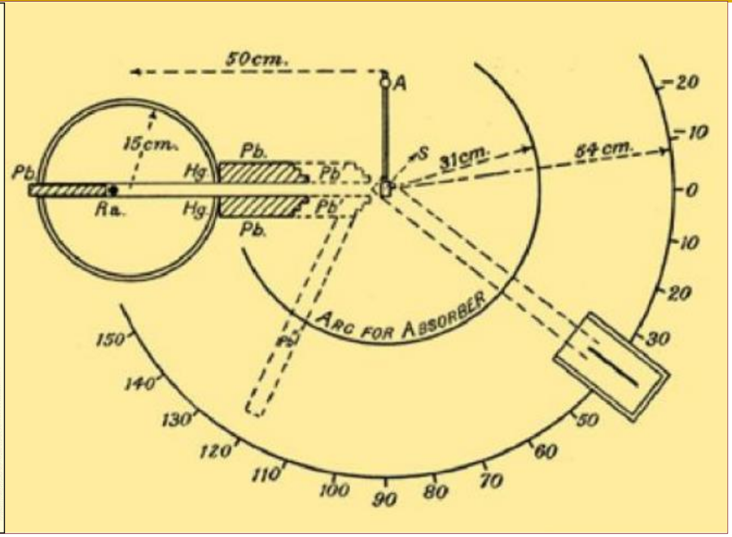
- $V_{space} \approx {}^4/_3 \pi r_0^3 A \rightarrow r_{nucl.} \propto A^{1/3}$;
- p_F, E_F not dependent on A (!!!);
- large p_F, small kin. energy;
- when p/n hit by probe ($e^\pm/\nu$), if E_{probe} >> 30 MeV → ignore Fermi motion.
- [*more elaborated model, e.g. add e.m. and spin interactions, etc. – see literature*]



Rutherford scattering







The birth of nuclear physics
(Manchester, 1908-13):

$\alpha(Z_\alpha=2,\ A_\alpha=4) \rightarrow Au(Z_{Au}=79,\ A_{Au}=197)$

- actually performed by H.Geiger and E.Marsden [*E.M. was 20 y.o. !*];
- alternative model by J.J.Thompson, with a diffused mass/charge ("soft matter");
- the first "fixed target" scattering

- *already discussed in FNSN (pag 25);*
- *do NOT repeat the math, simply recall the results;*
- *discussion of the physics;*
- *preparation for further steps.*

modern simulation (look):
https://phet.colorado.edu/en/


Lord Ernest Rutherford

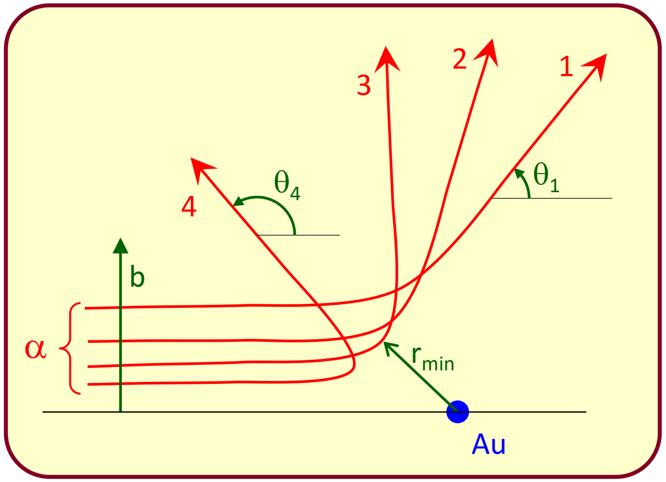
Rutherford scattering: in a nutshell

[*an incredible mix of genius, skill and luck*]

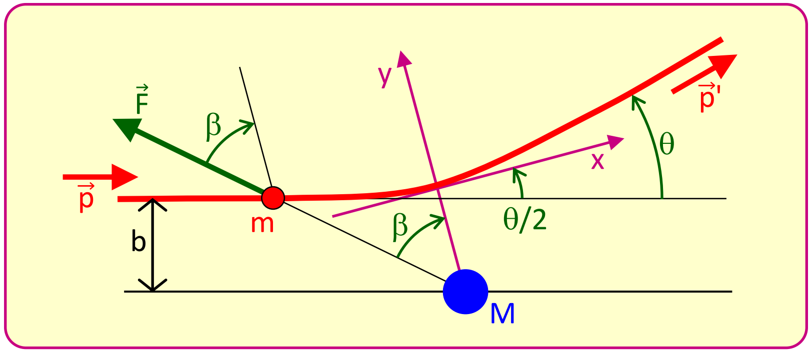
- α-particles (i.e. ionized He) $\rightarrow$ Au foil;
- $E_\alpha^{kin} \approx$ few MeV;
- sometimes, the α was scattered by $\theta > 90°$; *VERY* rare in reality, but impossible if matter were soft and homogeneous;
- only explanation: "matter" actually concentrated in small heavy bodies ("nuclei");

$\rightarrow$ the "matter" is essentially empty;

- how model the scattering ? Rutherford tried with a two-body scattering;
- notice: **Coulomb (electrostatic)**, **non-relativistic**, **no QM** (obviously);
- **success !!!** [*within their limited observation capabilities*]
- a key point: the nucleus is small enough, that the α "sees" always its full charge;
- [*remember the Gauss' theorem: if impact parameter $b > r_{Nucleus}$, only see an effective point-like charge*]
- but the matter is neutral ! yes, but the electrons are so light, that they cannot stop/deflect the α ($m_e/m_\alpha \approx 1/8{,}000$).



Rutherford scattering: the math



α (m, z) → nucleus (M, Z):

- $\vec{v}_{\alpha,init} = \vec{v}$, $\vec{v}_{\alpha,final} = \vec{v}\,'$, $\vec{v}_{nucleus} = 0$;
- $\vec{p} = m\vec{v}$, $\vec{p}\,' = m\vec{v}\,'$, m << M;
- Coulomb force only ($\vec{F}$);
- v << c → non-relativistic;
- elastic → $|\vec{p}\,'| = |\vec{p}|$;
- conserve E, ang. mom $\vec{L}$;
- $\Delta p_x = 0$ because of symmetry, only Δp_y matters;
- integral over β, the angle wrt $\hat{y}$;
- if attractive force (e.g. +–), M → the other focus of the hyperbola.

FNSN, 25

$$\Delta p = |\vec{p} - \vec{p}\,'| = 2p\sin(\theta/2);$$

$$|\vec{L}| = pb = |\vec{r}\times m\vec{v}| = \left|\vec{r}\times m\left(\frac{dr}{dt}\hat{r} + r\frac{d\beta}{dt}\hat{\beta}\right)\right| = mr^2\frac{d\beta}{dt};$$

$$\Delta p_y = 2p\sin(\theta/2) = \int_{-\infty}^{+\infty} dt F_y = \int_{-\infty}^{+\infty} dt \frac{zZe^2}{4\pi\varepsilon_0}\frac{\cos\beta}{r(t)^2} =$$

$$= \int_{-(\pi-\theta)/2}^{(\pi-\theta)/2} \frac{zZe^2}{4\pi\varepsilon_0}\frac{\cos\beta}{\cancel{r}^2}\frac{m\cancel{r}^2}{pb}d\beta = \frac{zZe^2}{2\pi\varepsilon_0}\frac{m}{pb}\cos(\theta/2);$$

$$\tan(\theta/2) = \frac{zZe^2}{4\pi\varepsilon_0}\frac{m}{p^2 b} \rightarrow db = -\frac{zZe^2}{4\pi\varepsilon_0}\frac{m}{p^2}\frac{d\theta}{2\sin^2(\theta/2)}.$$

$$d\sigma = 2\pi b\,db = 2\pi\left(\frac{zZe^2 m}{4\pi\varepsilon_0 p^2}\right)^2 \frac{d\theta}{2\tan(\theta/2)\sin^2(\theta/2)};$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{zZe^2 m}{4\pi\varepsilon_0}\right)^2 \frac{1}{4p^4\sin^4(\theta/2)} = \left(\frac{zZe^2 m}{2\pi\varepsilon_0}\right)^2 \frac{1}{|\vec{p}-\vec{p}\,'|^4}.$$

$$d[1/\tan(\theta/2)] = d[\cos(\theta/2)/\sin(\theta/2)]$$
$$= -d(\theta/2)[1+\cos^2(\theta/2)\sin^2(\theta/2)].$$

$$d\Omega = 2\pi\sin\theta d\theta = 4\pi\sin(\theta/2)\cos(\theta/2)d\theta$$

Rutherford scattering: more math

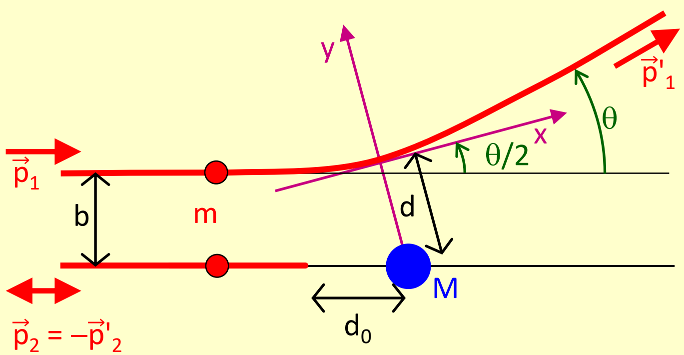
Useful formulas

$$d_0 = r_{min}(b=0) = \frac{zZe^2}{2\pi\varepsilon_0 mv^2};$$

$$\tan\left(\frac{\theta}{2}\right) = \frac{d_0}{2b};$$

$$d = r_{min}(b) = \frac{d_0 + \sqrt{d_0^2 + 4b^2}}{2} = \frac{d_0}{2}\left(1 + \frac{1}{\sin(\theta/2)}\right);$$

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{d_0^2}{16\sin^4(\theta/2)}} \xrightarrow{\theta\to 0} \frac{d_0^2}{\theta^4}.$$



- [if force attractive (e.g. +−), $\vec{F} \to -\vec{F}$, then $\theta \to -\theta$, but everything else equal, e.g. same $d\sigma/d\Omega$;]
- consider a particle $\vec{p}_2$ with b=0 → θ_2 = 180°;
  - ➢ define d_0 = "distance of closest approach", i.e. r_{min} (when $r=d_0$, the particle is at rest);
  - ➢ d_0 is computed from energy conservation;
- define $d_0 = (zZe^2)/(2\pi\varepsilon_0 mv^2)$ also for $b\neq 0$;
- write θ and $d\sigma/d\Omega$ as functions of d_0;
- define d as r_{min}, when $b\neq 0$;
- d is computed from E and $\vec{L}$ conservation [*hint in the box, v_0 is the velocity in d*]:

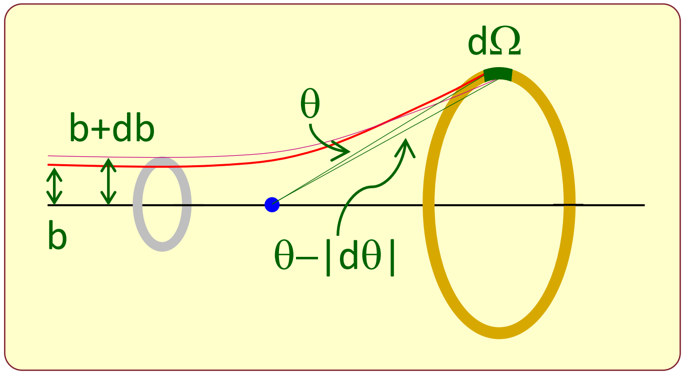
$$\vec{L}\text{ conserv} \to mbv = mdv_0 \to v_0/v = b/d$$

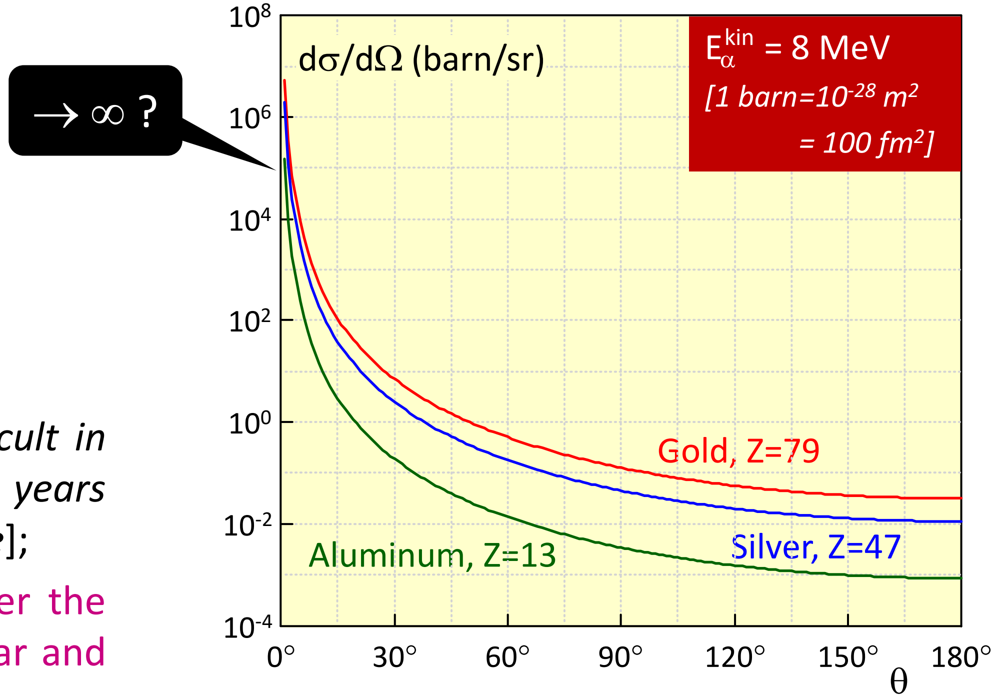
$$\text{E conserv} \to \tfrac{1}{2}mv^2 = \tfrac{1}{2}mv_0^2 + zZe^2/(4\pi\varepsilon_0 d) = \tfrac{1}{2}mv_0^2 + \tfrac{1}{2}mv^2 d_0/d$$

$$\to (v_0/v)^2 = (b/d)^2 = 1 - d_0/d \to$$

$$\to d^2 - dd_0 - b^2 = 0 \to d = \ldots.$$

Rutherford scattering: $d\sigma/d\Omega$



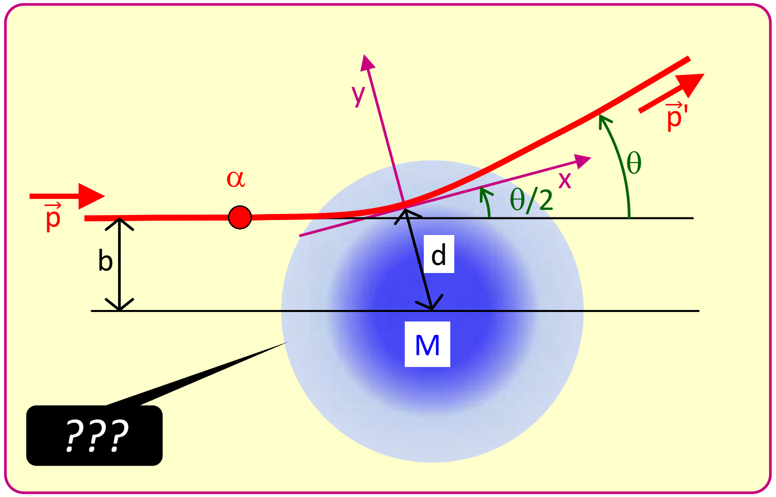


- [*the calculations above are *NOT* difficult in math: Newton could have done all 200 years earlier, had the correct model been made*];
- the real difficulty was to assess whether the matter is soft and continuous or granular and "empty";
- b large $\to$ θ small $\to$ $d\sigma/d\Omega$ $\to$ ∞ [*cutoff provided by other Au nuclei*].

A long and thorough investigation:

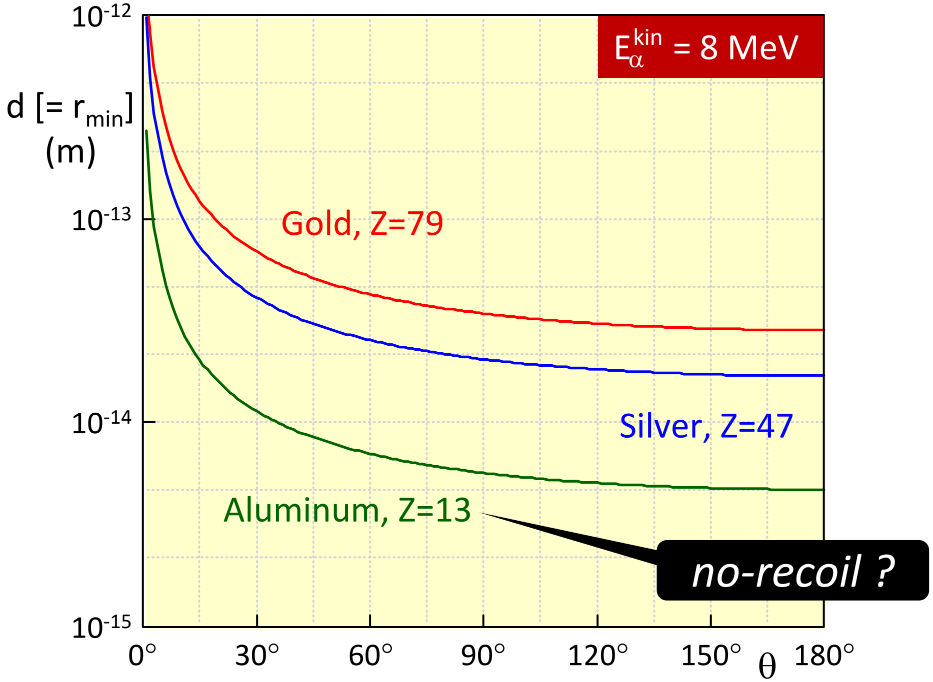
- 1909: found some events $\theta > 90°$: big shock;
- 1911: falsification of the Thomson model, correct assumptions, check of $d\sigma/d\Omega$ in the range 30°–50°;
- 1913: check of $d\sigma/d\Omega$ in the range 5°–150°;
- check that yield $\propto$ thickness of Au foil;
- other nuclei : check that yield $\propto Z^2$ [*roughly*];
- however Rutherford model clearly inconsistent in its "planetary" part: acceleration of charged electrons $\to$ radiation $\to$ collapse;
- after birth of QM, Rutherford computation redone in Born approx : $\to$ same $d\sigma/d\Omega$ [*big luck !*] + no more inconsistency [*next slides*].

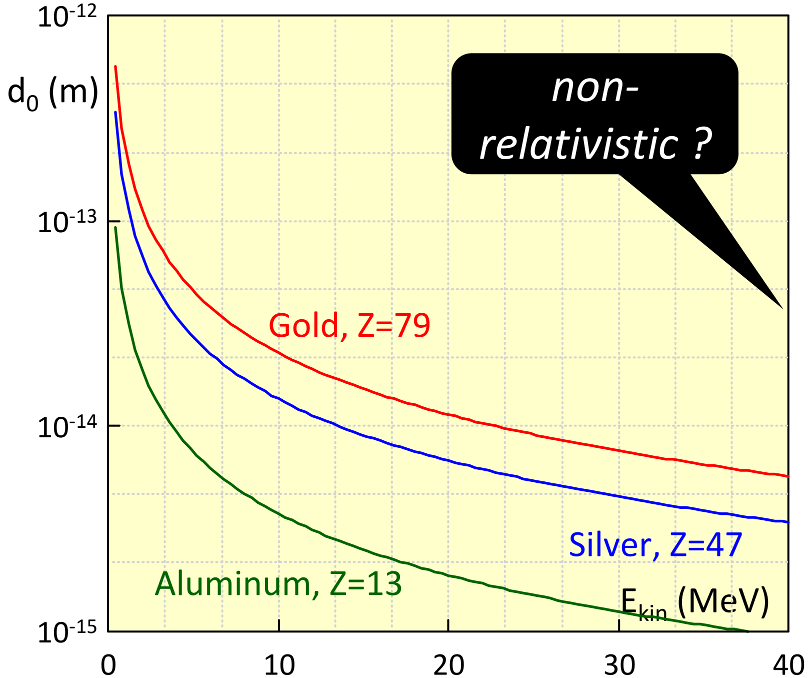
Rutherford scattering: $R_{nucleus}$



How large is the nucleus ?

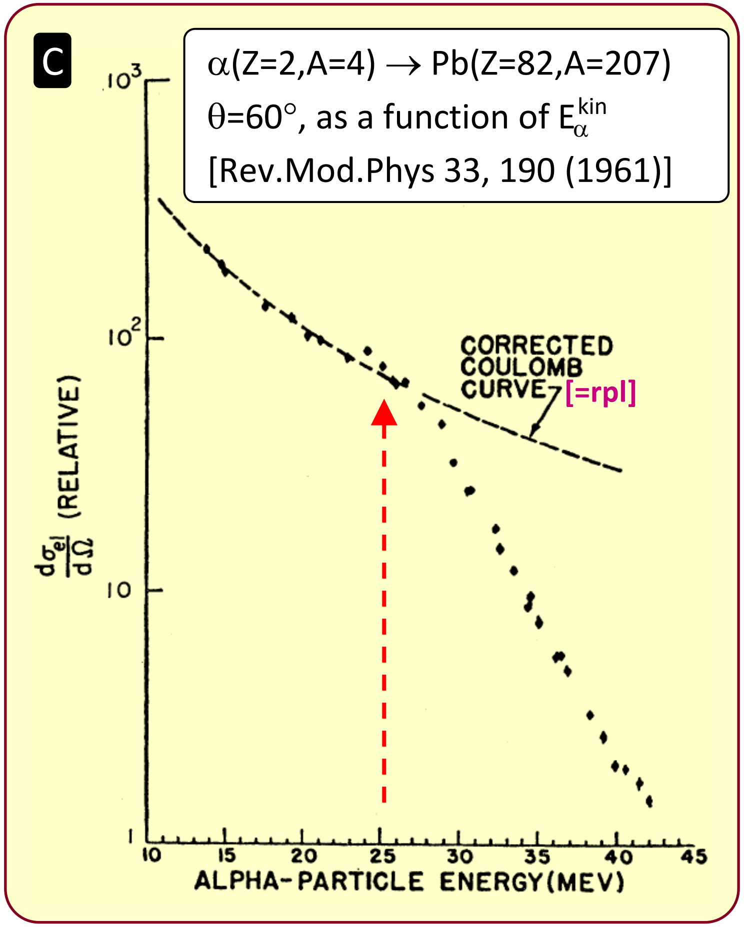
- [*remember the Gauss' theorem*]
- if the α trajectory is completely external to the nucleus, it does NOT probe its (*possible*) structure;
- the Rutherford experiment could only limit $R_{nucleus} < 10^{-14}$ m [*still an important result !*];
- to "see" 10^{-15} m $\rightarrow$ probes with $E_{kin} >$ 20÷30 MeV.

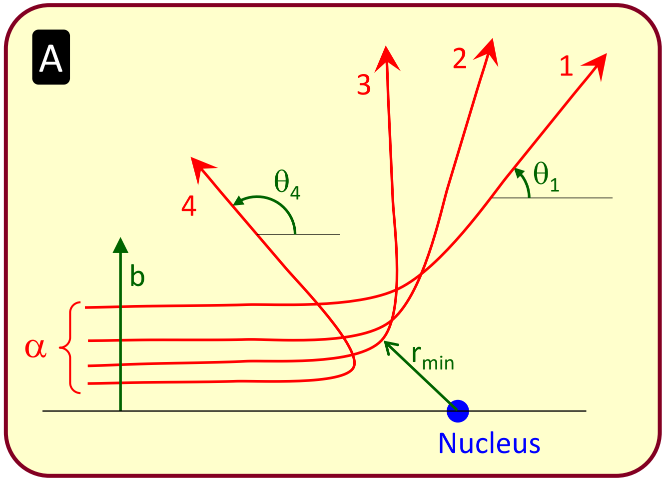


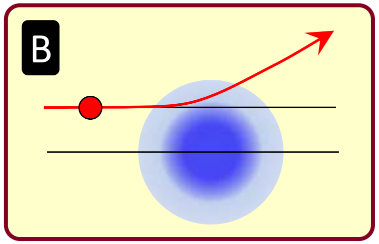


Rutherford scattering: measure $R_{nucleus}$

- plot [A]: b and r_{min} could *NOT* be measured directly for each event, but Rutherford point-like law (*rpl*) relates b $\leftrightarrow$ θ; in fact $b_{small} \leftrightarrow \theta_{large}$;
- plot [B]: the Gauss' theorem predicts a deviation from rpl, when (E_α^{kin} large) $\rightarrow$ ($r_{min} < R_{nucleus}$) $\rightarrow$ shielding $\rightarrow$ "smaller θ";
- plot [C] (**1961** !!!): a "Rutherford-like" scattering α-Pb; at θ=60°, deviation for E_α^{kin} > 25 MeV;
- at high θ, point-like target $\rightarrow$ larger σ, soft target $\rightarrow$ smaller σ (*deviations from rpl related to size of target*) [*please, remember*].







Q. find r_{min} for Pb, θ = 60°, E_α^{kin} = 25 MeV

A. r_{min} = [☞ formula] = 14 fm.

kinematics

This is a collection of kinematical computations. It is probably useful to have all in the same place. Notice that here we work in the LAB sys (= N at rest), not in the CM.

This chapter (and many others) deals with scattering. A "probe", usually assumed point-like (e.g. $e^\pm$) hits a hadronic complex system (a nucleus) [*see box*].

In the final state, the probe emerges unchanged, while the nucleus may or may not survive intact:

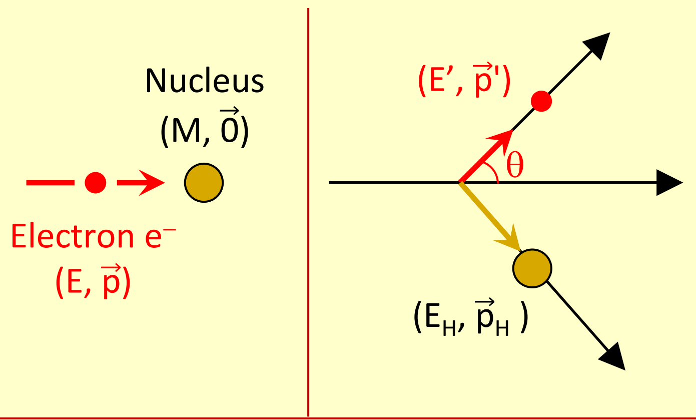
- elastic scattering, when the nucleus is unchanged, i.e. *identical initial and final state particles* (W=M);
- excitation, when the nucleus in the final state is excited, i.e. heavier (W = M* > M);
- a new hadronic system, with n particles (i=1…n):

$$E_H = \sum_{i=1}^{n} E_i; \qquad \vec{p}_H = \sum_{i=1}^{n} \vec{p}_i;$$

$$W = \sqrt{(E_H)^2 - (p_H)^2} = M_{had.\ sys.} > M.$$

The underlying idea is to study (*understand ?*) the structure of the hadrons by observing the scattering.

$$e^- N \rightarrow e^- H$$



$$\text{electron } e^- : \begin{cases} (E, & \vec{p}\,; & m) & \text{[init.]} \\ (E', & \vec{p}\,'; & m) & \text{[fin.]} \end{cases}$$

$$\text{had. sys.} : \begin{cases} (M, & \vec{0}; & M) & \text{[init.]} \\ (E_H, & \vec{p}_H; & W) & \text{[fin.]} \end{cases}$$

$$\text{4-mom cons.} \rightarrow \begin{cases} E + M = E' + E_H; \\ \vec{p} + \vec{0} = \vec{p}\,' + \vec{p}_H. \end{cases}$$

kinematics: elastic scattering

- To begin with, assume elastic scattering, i.e. "H" = N;
- Define, in the target nucleus ref.sys. :

$$\text{electron } e^\pm : \begin{cases} (E, & \vec{p}\,; & m) & \text{[init.]} \\ (E', & \vec{p}\,'; & m) & \text{[fin.]} \end{cases}$$

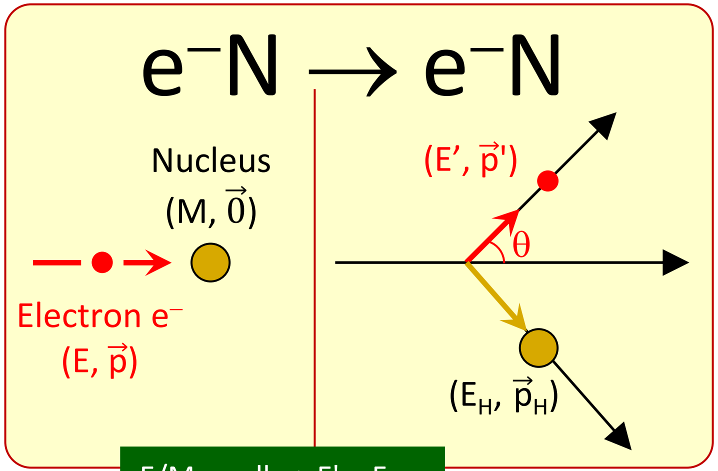
$$\text{nucleus } : \begin{cases} (M, & \vec{0}; & M) & \text{[init.]} \\ (E_H, & \vec{p}_H; & M) & \text{[fin.]} \end{cases}$$

- 4-mom cons. $\rightarrow \begin{cases} \vec{p} + \vec{0} = \vec{p}\,' + \vec{p}_H; \\ E + M = E' + E_H. \end{cases}$

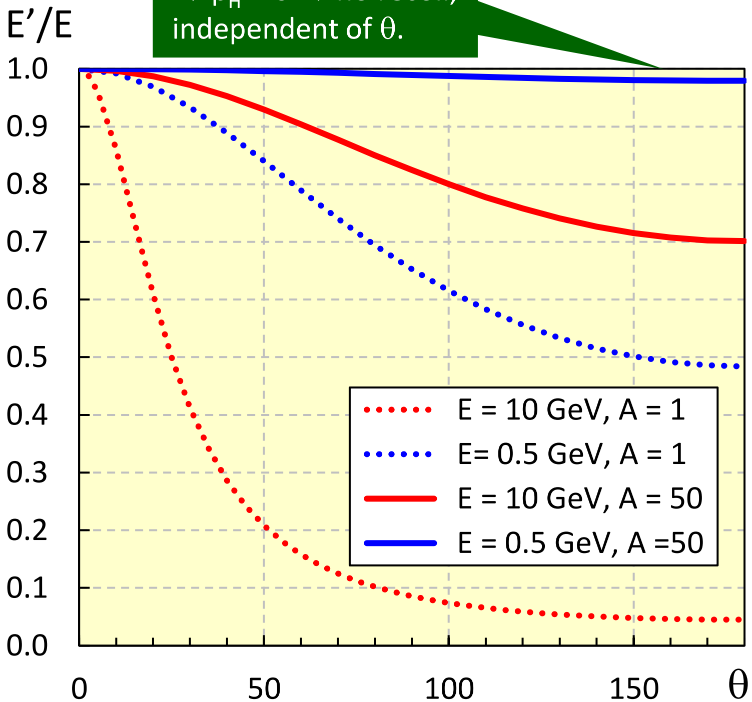
- The relation between the observed quantities (E, E', θ) is [*next slide*] :

$$E' = \frac{E}{1+\frac{E}{M}(1-\cos\theta)} = \frac{E}{1+\frac{2E}{M}\sin^2(\theta/2)} \approx |\vec{p}\,'|;$$

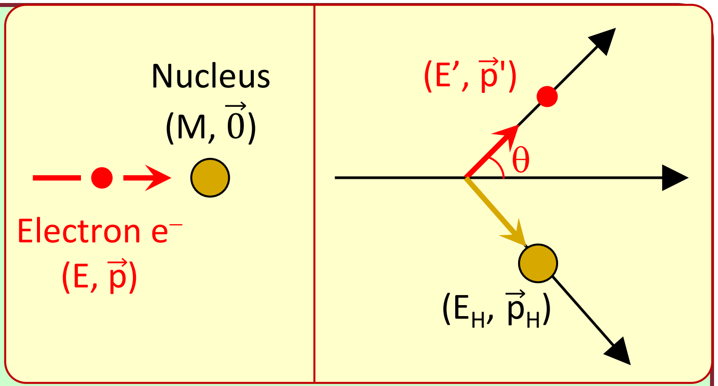
- Therefore, for known initial energy E and fixed M, the final state is defined by one independent variable (E' or θ).



E/M small → E' ≈ E → $p_H \approx 0$ → no recoil, independent of θ.



kinematics: elastic scattering - E' vs θ



$$\begin{cases} e^-_{init} & (E, \vec{p}; m); \\ N_{init} & (M, \vec{0}; M); \end{cases} \qquad \begin{cases} e^-_{fin} & (E', \vec{p}'; m); \\ H_{fin} & (E_H, \vec{p}_H; M); \end{cases}$$

4-momentum conservation
$$\begin{cases} E + M = E' + E_H \rightarrow E_H = E + M - E'; \\ \vec{p} + \vec{0} = \vec{p}' + \vec{p}_H \rightarrow \vec{p}_H = \vec{p} - \vec{p}'; \end{cases}$$

Square and subtract
$$\left\{ (E_H)^2 - (\vec{p}_H)^2 = M^2 = \left(E^2 + M^2 + E'^2 + 2EM - 2EE' - 2ME'\right) - \left(p^2 + p'^2 - 2pp'\cos\theta\right); \right.$$

Ultra-relativistic approx.
$(m_e << E, E') \rightarrow (p \approx E, p' \approx E')$
$$\begin{cases} \cancel{M}^2 = \cancel{E}^2 + \cancel{M}^2 + \cancel{E}'^2 + 2EM - 2EE' - 2ME' - \cancel{E}^2 - \cancel{E}'^2 + 2EE'\cos\theta; \\ 0 \quad = EM - EE' - ME' + EE'\cos\theta = EM - E'\left[E(1 - \cos\theta) + M\right]; \end{cases}$$

$$E' = \frac{EM}{M + E(1 - \cos\theta)} = \frac{E}{1 + \frac{2E}{M}\sin^2\left(\frac{\theta}{2}\right)}$$

q.e.d.

NB – The reaction is planar (why?). The final state is defined by 6 variables. There are 3 $(E, \vec{p})$ conservations and 2 $(m^2=E^2-p^2)$ rules. Therefore: 6-5=1 independent variable.

kinematics: Q^2 in elastic scattering

- in the following, $(E, \vec{p}, E', \vec{p}', m, M, \theta)$;

  $[m = m_e \text{ small} \rightarrow E \approx |\vec{p}|, \;\; E' \approx |\vec{p}'|]$
- new (not independent) variable:

  $\boxed{\vec{q} \equiv \vec{p} - \vec{p}'}$ "momentum transfer";

  $\left[E/M \text{ small} \rightarrow p' = p \rightarrow \;\; |\vec{q}| = 2|\vec{p}|\sin(\theta/2)\right]$
- relativistic equivalent (p and p' are 4-mom):

  $\boxed{q \;\; \equiv p - p'} \qquad \left[= (E - E', \vec{p} - \vec{p}')\right];$

  $-q^2 = -\left(2m_e^2 - 2EE' + 2|\vec{p}||\vec{p}'|\cos\theta\right) \approx$

  $\approx 4EE'\sin^2(\theta/2) = Q^2 \qquad [\text{i.e. } Q^2 > 0];$
- $\cancel{E}' = \dfrac{EM}{M + 2E\sin^2(\theta/2)} = \dfrac{EM}{M + Q^2/(2E')} =$

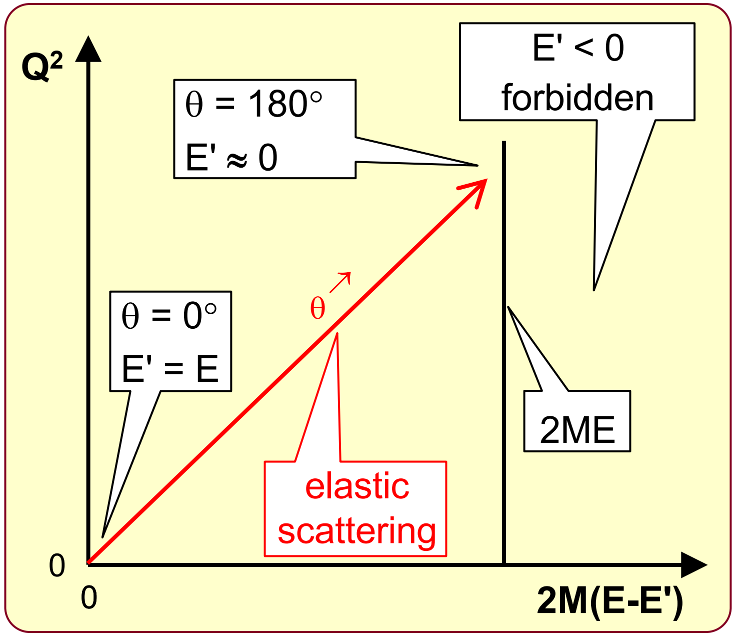
  $= \dfrac{2\cancel{E}'EM}{2E'M + Q^2} \rightarrow 2EM = 2E'M + Q^2$

  $\rightarrow \boxed{Q^2 = 2M(E - E')} \qquad \boxed{E' = E - Q^2/(2M)}$
- [*for elastic scattering one independent variable* $\rightarrow E' = E'(\theta) = E'(Q^2)$, $Q^2 = Q^2(E')$];

Study the kinematical limits:

- $\theta = 0°$: $E' = E$; $Q^2 = 0$;
- $\theta = 180°$: $E - E' = E\dfrac{M+2E}{M+2E} - \dfrac{EM}{M+2E} = \dfrac{2E^2}{M+2E}$

  $(E >> M)$: $E - E' \approx E \rightarrow E' \approx 0$;
- in conclusion <u>E > E' > "0"</u>.
- Plot Q^2 vs $2M(E-E')$: <u>only a segment allowed</u> [*useless for elastic scatt., but ...*]:



kinematics: why $|\vec{q}|$, Q^2

an advance of dynamics

The variable $\vec{\mathbf{q}}$ is *very* important:

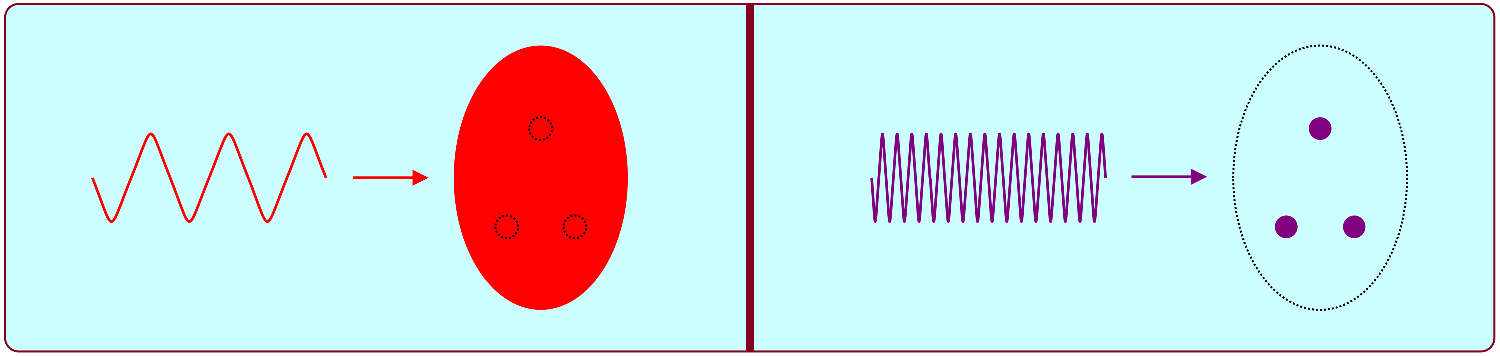
- [if relativistic, use Q^2 or its root $\sqrt{Q^2}$];
- it is related to the deBroglie wavelength of the probe: $\bar{\lambda} = \hbar/|\vec{q}|$;
- it represents the "scale" of the scattering;
- i.e. structures smaller than $\bar{\lambda} \sim 1/|\vec{q}|$ are not "visible" to the probe;
- [the uncertainty principle $\Delta p \Delta x \geq \hbar/2$ leads to the same conclusion – *actually it is exactly the same argument*;

Comments:

- large $|\vec{q}| \rightarrow$ large E, but not necessarily the opposite: high-energy & large distance processes do exist;
- the quest for smaller scales leads inevitably to larger Q^2 and therefore to larger E [$\rightarrow$ money and resources...]

____

[as usual] sometimes in the literature the notation is confusing: $Q^2 = -t$, see later.



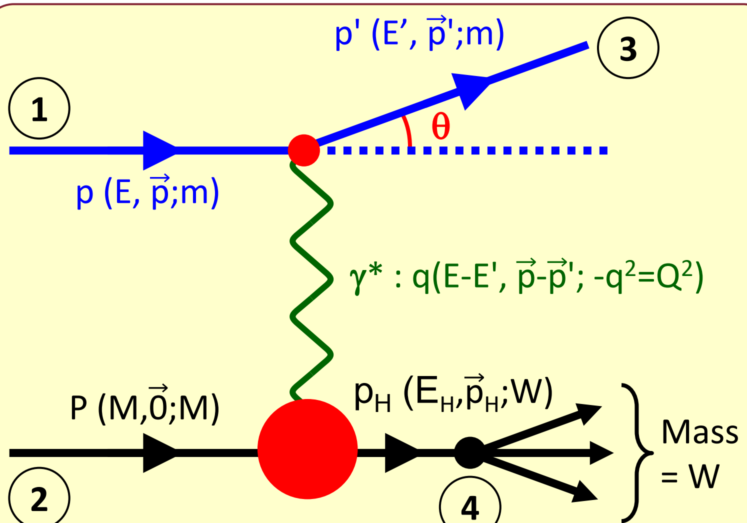
- popular understanding:
  higher Q^2 $\rightarrow$ smaller distance $\rightarrow$
  $\rightarrow$ "better microscope".
- conclusion:
  Q^2 is an important variable, possibly the most important in modern particle physics.

kinematics: the inelastic case

[in general, $\ell N \to \ell' H$ (ℓ,ℓ' generic leptons); the kinematics is the same, if E_ℓ , $E_{\ell'}$ >> m_ℓ , $m_{\ell'}$]

Kinematical variables (**ℓ N → ℓ' H**) :

- [$\ell'=\ell$, H=N → elastic];
- 4-mom. in LAB sys (≡ had CM);
- $p_1 = p$, $p_2 = P$, $p_3 = p'$, $p_4 = p_H$;
- q = p – p' [as in previous slides];



Lorentz – invariant variables:

- $\nu = q \cdot P/M = E - E'$ [= energy lost by e⁻];
- $Q^2 = - q^2 = 2(EE' - pp'\cos\theta) - 2m^2 \approx 4\,EE'\sin^2(\theta/2)$ [= – module of the 4-momentum transfer];
- $x = Q^2 / (2M\nu)$ [later : x-Bjorken x_B, the fraction of the hadron 4-momentum carried by the interacting parton];
- $y = (q \cdot P) / (p \cdot P) = \nu / E$ [= the fraction of the energy lost by the lepton in the target frame];
- $W^2 = (p_H)^2 = (P + q)^2 = M^2 - Q^2 + 2M\nu$ [=(mass)² of the hadron system in the final state] : W = M if elastic;
- $s = (p+P)^2 = (p'+p_H)^2 \approx M(M+2E)$ [the (energy)² in the CM].

[*computations in next slide*]

kinematics: Q^2, ν, x, y, W^2

$$\begin{cases} e^-_{init} & p\ (E,\vec{p};m_\ell); \\ N_{init} & P\ (M,\vec{0};M); \end{cases} \qquad \begin{cases} e^-_{fin} & p'\ (E',\vec{p}';m_\ell); \\ H_{fin} & p_H\ (E_H,\vec{p}_H;W); \end{cases}$$

p, p', P, P_H, q, Q^2, M, ν, x, y, W^2 Lorentz invariant;

E,E', ... Lab sys (= P at rest).

$$q = p - p' = (E - E', \vec{p} - \vec{p}');$$

m << E,M (safe approx.)

$$q^2 = m^2 + m^2 - 2EE' + 2pp'\cos\theta \approx -2EE'(1-\cos\theta) = -4EE'\sin^2\left(\frac{\theta}{2}\right) \equiv -Q^2;$$

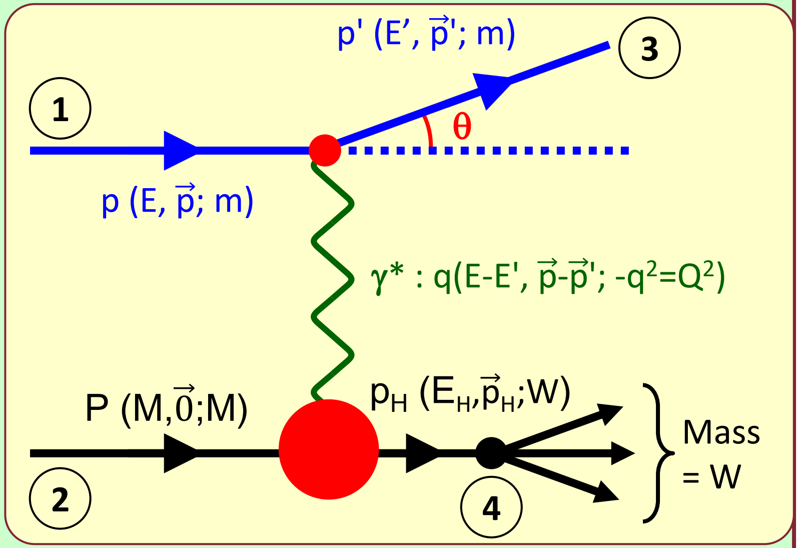
$$\nu \equiv \frac{q\cdot P}{M} = \frac{(E-E')M}{M} = (E-E');$$

warning: x_B is very interesting, *see later*

$$x \equiv \frac{Q^2}{2M\nu}; \qquad y \equiv \frac{q\cdot P}{p\cdot P} = \frac{(E-E')M}{EM} = \frac{E-E'}{E} = \frac{\nu}{E};$$

$$W^2 = p_H^2 = (P+q)^2 = M^2 - Q^2 + 2M\nu;$$

$$s = (p+P)^2 = (p'+p_H)^2 \approx \cancel{m}^2 + M^2 + 2p\cdot P = M^2 + 2ME.$$



kinematics: the inelastic case - remarks

Remarks :

- a lot of kinematical relations, e.g.

  $W^2 = M^2 + 2MEy(1-x);$

  $Q^2 = 2MExy;$

  $s = M^2 + m^2 + Q^2/(xy);$

- in the <u>elastic case</u> eN $\rightarrow$ eN [ep $\rightarrow$ ep], ν and Q^2 are NOT independent :

  $W^2 = M^2 = (P + q)^2 = M^2 - Q^2 + 2\,M\nu$

  $\rightarrow\ Q^2 = 2M\nu \rightarrow Q^2 / (2M\nu) = x = 1;$

- therefore (obviously) in the elastic case, there is only <u>one</u> independent parameter (E' or θ, choice according to the meas.);

- instead, in the <u>inelastic scattering</u> :

  $Q^2 = M^2 + 2\,M\nu - W^2 =$
  $= 2M\nu - (W^2 - M^2) \leq 2M\nu \rightarrow x \leq 1;$

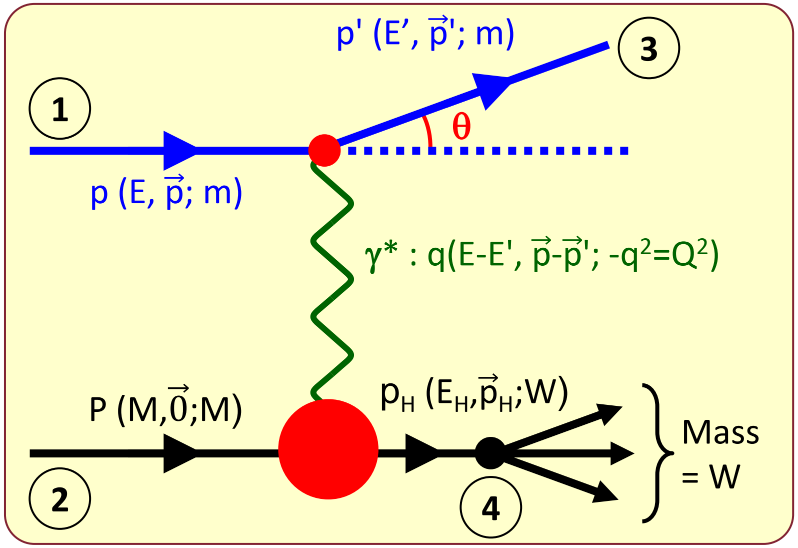
  if W not fixed, Q^2 and ν are independent;

- therefore, in the inelastic case, there are <u>two</u> independent variables;

- in the analysis, choose two among all variables, according to convenience, e.g.:

  (E', θ), (Q^2, ν), (x, y).

$$\boxed{\begin{aligned} Q^2 &= (\vec{p}-\vec{p}\,')^2 - (E-E')^2 = (\vec{p}_H)^2 - (E_H - M)^2 = \\ &= (\vec{p}_H)^2 - E_H^2 - M^2 + 2E_H M = 2E_H M - 2M^2 = \\ &= 2M(E_H - M) \xrightarrow{\text{elastic}} 2MT \\ E_H &= \frac{Q^2}{2M} + M; \quad \frac{E_H}{M} = 1 + \frac{Q^2}{2M^2} \xrightarrow{Q^2 \ll M^2} 1 \\ & \qquad\qquad\qquad\qquad\qquad \text{(elastic, no recoil)} \end{aligned}}$$

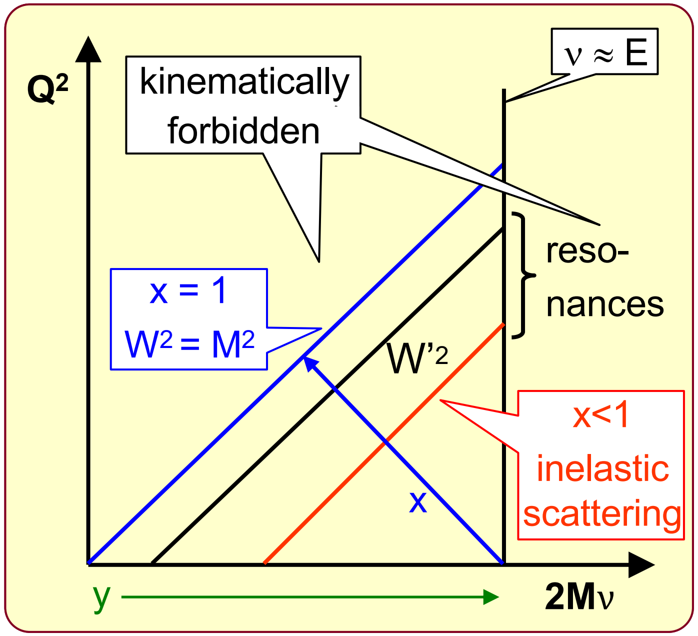


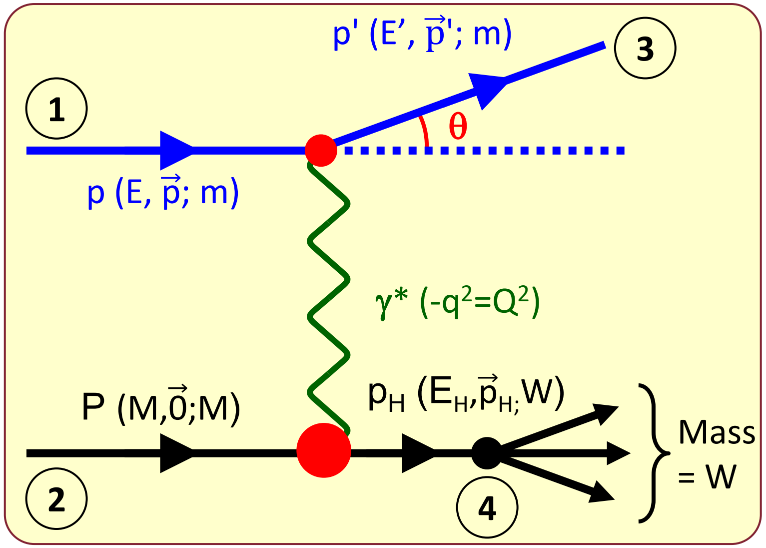
kinematics: deep inelastic scattering

Redefine the kinematics of the scattering process in the plane (Q^2 vs ν) [more precisely (Q^2 vs $2M\nu$)]:

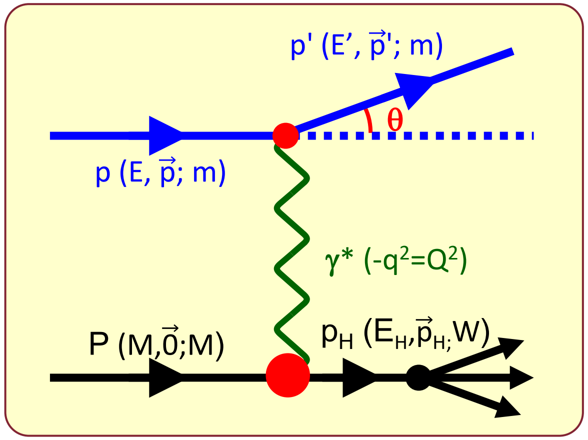
- both are Lorentz-invariant [*but usually used in the lab. frame, where the initial state hadron is at rest*] ;
- $Q^2 = 4\,EE' \sin^2(\theta/2) \geq 0 \rightarrow$ only the 1st quadrant;
- $\nu\ = E - E' \rightarrow 0 \leq \nu \leq E \rightarrow$ only a band is allowed;
- $x\ = Q^2\ /\ (2M\nu) \leq 1 \rightarrow 0 \leq x \leq 1 \rightarrow$ only "lower triangle";
- $y\ = (q \cdot P)\ /\ (p \cdot P) = \nu\ /\ E \rightarrow 0 \leq y \leq 1$;
- $W^2\ = M^2 + 2M\nu - Q^2\ \rightarrow$ the bisector x=1 ("/") defines the elastic scattering, where $W^2 = M^2$;
- on the bisector, only θ varies : $\theta = 0 \rightarrow Q^2 = \nu = 0$;
- the loci W'^2 = constant are lines parallel to the bisector $\rightarrow$ some of them define the excited states (one shown in fig.);
- at higher distance from the bisector we have the <u>deep inelastic scattering</u> (*DIS*) and (possibly) new physics.

[*see next slide*]



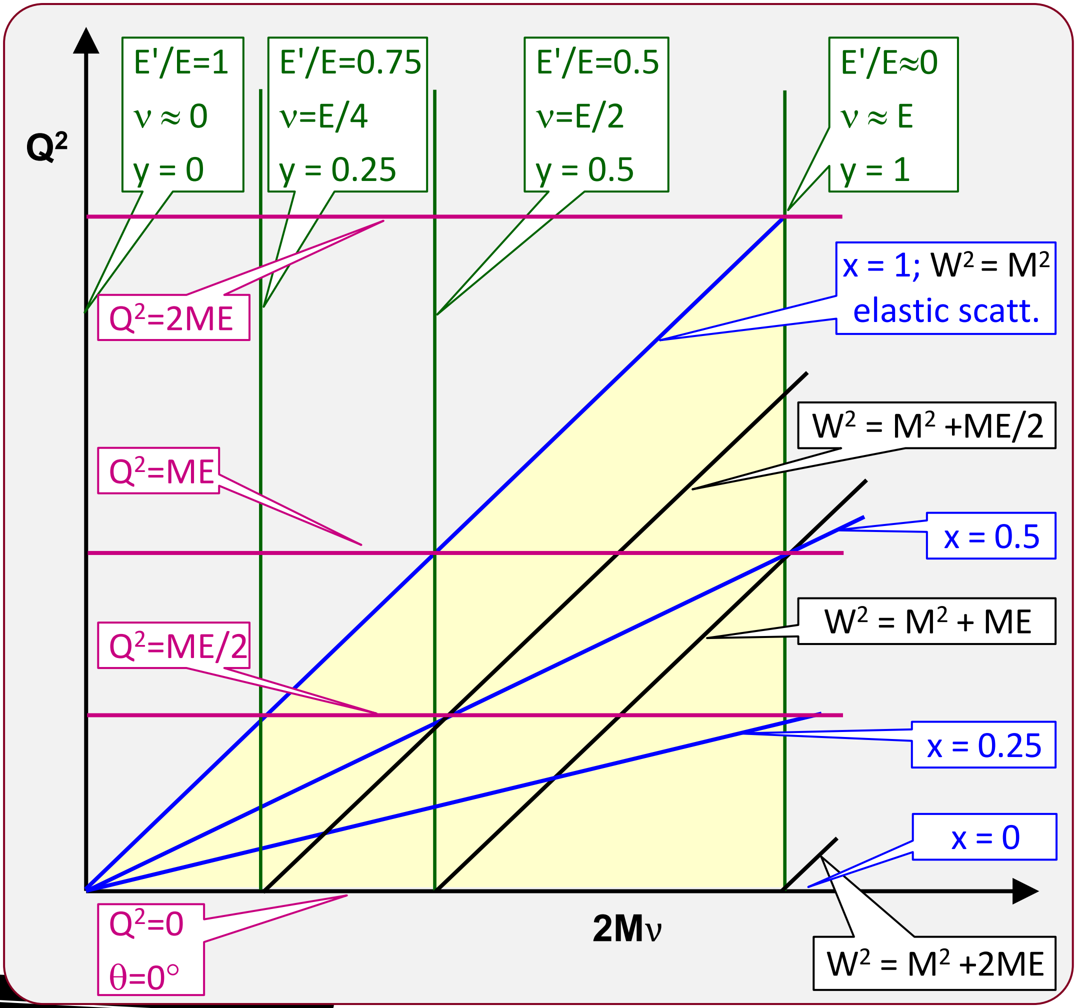


kinematics: a summary



0	<	x	<	1
0	<	y	<	1
0	<	ν	<	E
M^2	<	W^2	<	M^2+2ME
0	<	Q^2	<	2ME
0	<	E'	<	E
0°	<	θ	<	180°

limits (some only if E >> M).



You are kindly (but strongly) requested to look carefully to this slide and get used to these variables.

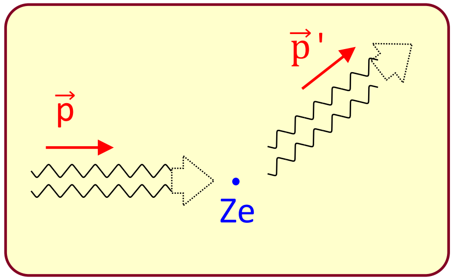
THANKS.

"higher Q^2"

elastic scattering e-N : Rutherford + q.m.

In the '20s QM entered in the game;

- Rutherford formula works also in QM;
- non-relativistic q.m. + Born approx.;
- Coulomb potential;
- initial (i) and final (f) particle as plane waves [*see introduction + box*];
- negligible recoil;
- $\vec{q} = |\vec{p} - \vec{p}'|$ (as usual);
- $\hbar = c = 1$;
- $V(r=\infty)$ does NOT contribute, because of other nuclei → in the last integration, do not use the value at $r=\infty$ [YN1, 135 has a cutoff "μ"].



$$V(r) = -\frac{zZ\alpha}{r}; \quad \vec{q} = \Delta\vec{p} = \vec{p} - \vec{p}\,'; \quad q = |\vec{q}| = 2p\sin(\theta/2);$$

$$\psi_i = e^{i\vec{p}\cdot\vec{r}}/\sqrt{\Phi}; \quad \psi_f = e^{i\vec{p}\,'\cdot\vec{r}}/\sqrt{\Phi}; \quad \frac{dn}{dE'} = \frac{4\pi p'^2\,\Phi}{(2\pi)^3 v'};$$

$$\mathcal{M}_{fi} = \langle \psi_f | V(\vec{r}) | \psi_i \rangle = \frac{1}{\Phi}\int e^{-i\vec{p}\,'\cdot\vec{r}}\,V(\vec{r})\,e^{i\vec{p}\cdot\vec{r}}\,d^3\vec{r} =$$

$$= -\frac{1}{\Phi}\iiint \frac{zZ\alpha}{r} e^{i\vec{q}\cdot\vec{r}} r^2 dr \sin\theta\, d\theta\, d\varphi = -\frac{4\pi}{\Phi}\frac{zZ\alpha}{q^2};$$

$$\boxed{\frac{d\sigma}{d\Omega}} = \frac{1}{4\pi}\left[2\pi|\mathcal{M}_{fi}|^2 \frac{dn}{dE'}\frac{\Phi}{v'}\right] \xrightarrow{v'\to c=1,\; p'=E'}$$

$$= \frac{1}{2}\left|\frac{4\pi}{\Phi}\frac{zZ\alpha}{q^2}\right|^2 \frac{\Phi E'^2}{2\pi^2}\Phi = \boxed{\frac{4z^2Z^2\alpha^2E'^2}{q^4}}$$

FNSN, 58

$$\int_0^{2\pi} d\varphi \int_0^\infty r dr \int_{-1}^{1} d\cos\theta\, e^{iqr\cos\theta} = 2\pi\int_0^\infty dr \int_{-r}^{r} e^{iqt} dt \quad [t = r\cos\theta]$$

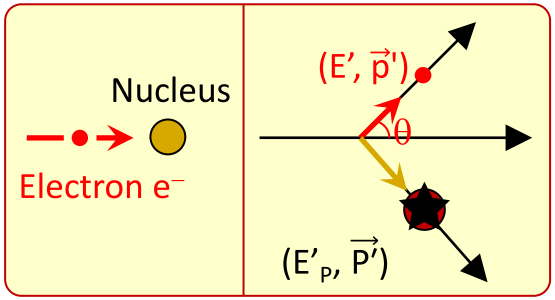
$$= \frac{2\pi}{iq}\int_0^\infty dr\left(e^{iqr} - e^{-iqr}\right) = \frac{2\pi}{iq}\frac{1}{iq}\left[e^{iqr} + e^{-iqr}\right]^{r=0} = -\frac{4\pi}{q^2}.$$

elastic scattering e-N : $\sigma_{Mott^{(*)}}$

- However, the scattering α-Nucleus takes place between two nuclei (e.g. He^{++}-Au);
- not suitable for measuring a (possible) nucleus structure → replace the α with a better (?) point-like probe: electron (e^-);
- the dynamics of the eN scattering can be described by the Rutherford formula with an adjustment [*later*], due to Mott :

$$\left[\frac{d\sigma}{d\Omega}\right]^*_{\text{Mott}} = \left[\frac{d\sigma}{d\Omega}\right]_{\text{Rutherford}} \times \left(1 - \beta^2 \sin^2\frac{\theta}{2}\right) \rightarrow$$

$$\xrightarrow{\beta=|\vec{p}|/E\rightarrow 1} \left[\frac{d\sigma}{d\Omega}\right]_{\text{Rutherford}} \cos^2\frac{\theta}{2} = \frac{4Z^2\alpha^2E'^2}{|\vec{q}|^4}\cos^2\frac{\theta}{2}.$$



- similar to the Rutherford formula, the Mott* cross-section neglects
  a) the nucleus dimension, if any;
  b) its recoil*;
- unlike Rutherford, Mott takes into account the e^- spin (=½).

NB The "*" in the name "Mott*" means that the "no-recoil" approximation is used → leave it out when the recoil is considered ("Mott*" → "Mott"].

Sir Nevill Francis Mott

elastic scattering e-N : helicity

The $\cos^2(\theta/2)$ factor in $[d\sigma/d\Omega]_{Mott}$ comes from Dirac equation; it is understood by considering the extreme case of θ~180°.

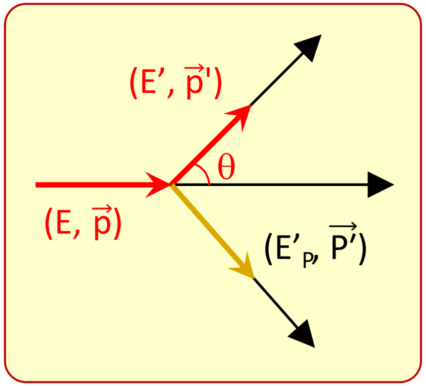
For relativistic particles ($\beta \to 1$), the helicity h (the projection of spin along momentum) is conserved :

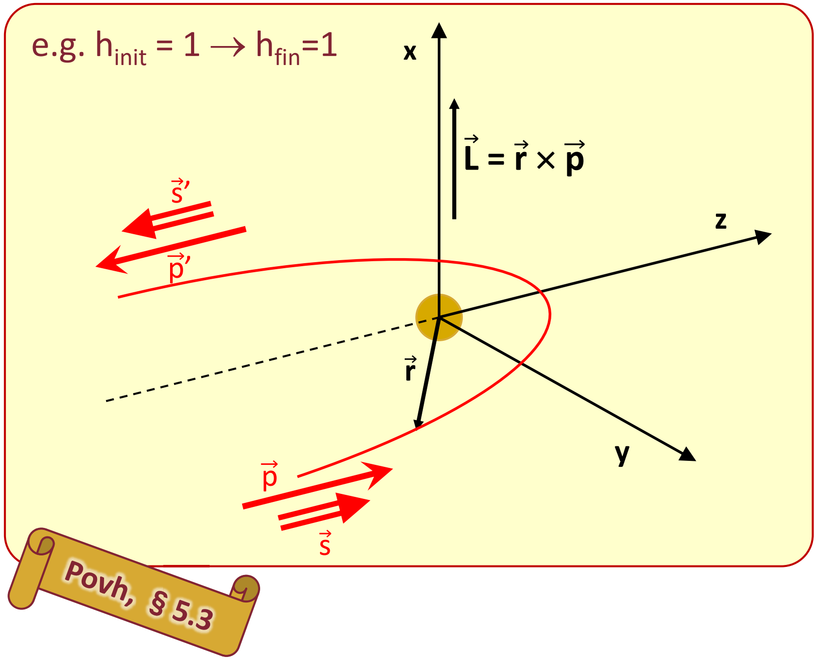
$$h = \frac{\vec{s} \cdot \vec{p}}{|\vec{s}| \cdot |\vec{p}|}.$$

The conservation requires the "spin flip" of the electron between initial and final state, because the momentum also flips at θ=180°.

In this condition, the angular momentum is NOT conserved, if the nucleus does NOT absorb the spin variation (e.g. because it is spinless). Therefore the scattering for $\theta \approx 180°$ is forbidden.

The factor $\cos^2(\theta/2)$ in the Mott formula is connected to the spin and describes the magnetic part of the interaction.





elastic scattering e-N : experiment

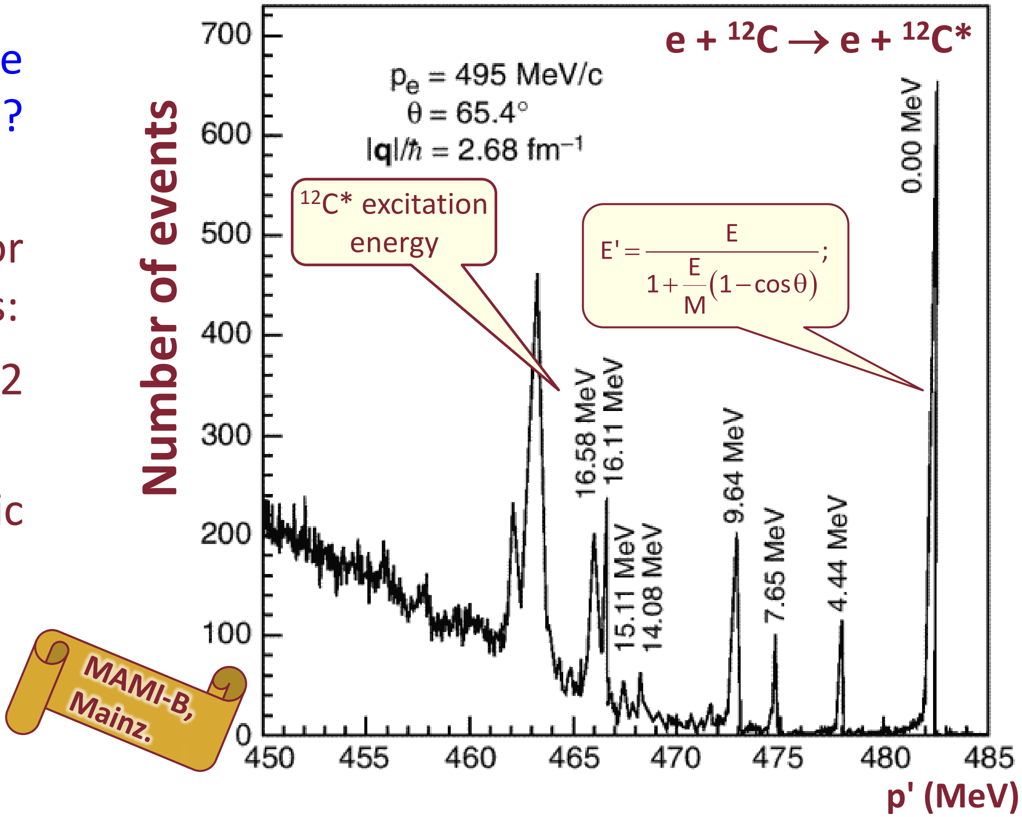
Is the experiment consistent with the kinematics of the elastic scattering ? Get e + ^{12}C data.

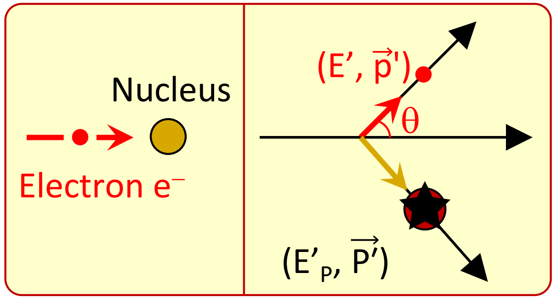
The plot of the number of events, for fixed E_{init} at fixed θ, shows many peaks:

- the expected elastic ($E' \approx p' = 482$ MeV),
- a rich structure, due to inelastic scattering:

$$e + {}^{12}C \rightarrow e + {}^{12}C^*$$

[$^{12}C^*$ = excited carbon, mass M*].





- the expected elastic [$e + {}^{12}C \rightarrow e + {}^{12}C$] is there;
- but "*more things in heaven, than in your philosophy*";
- **back to elastic scattering !**
- kinematics ok, dynamics ?

→ measure $d\sigma/d\Omega$ vs θ !!!

form factors: definition

- The experimental $d\sigma/d\Omega$ agrees with the Mott one only for small θ, i.e. small $|\vec{q}|$;
- otherwise, the cross section is "funny";
- possibly the reason is the structure of the nucleus, which results in a smaller effective charge, as seen by the projectile (Gauss' theorem);

  define $\boxed{\rho(\vec{x}) = Ze f(\vec{x})}$, $\int f(\vec{x}) d^3x = 1$;

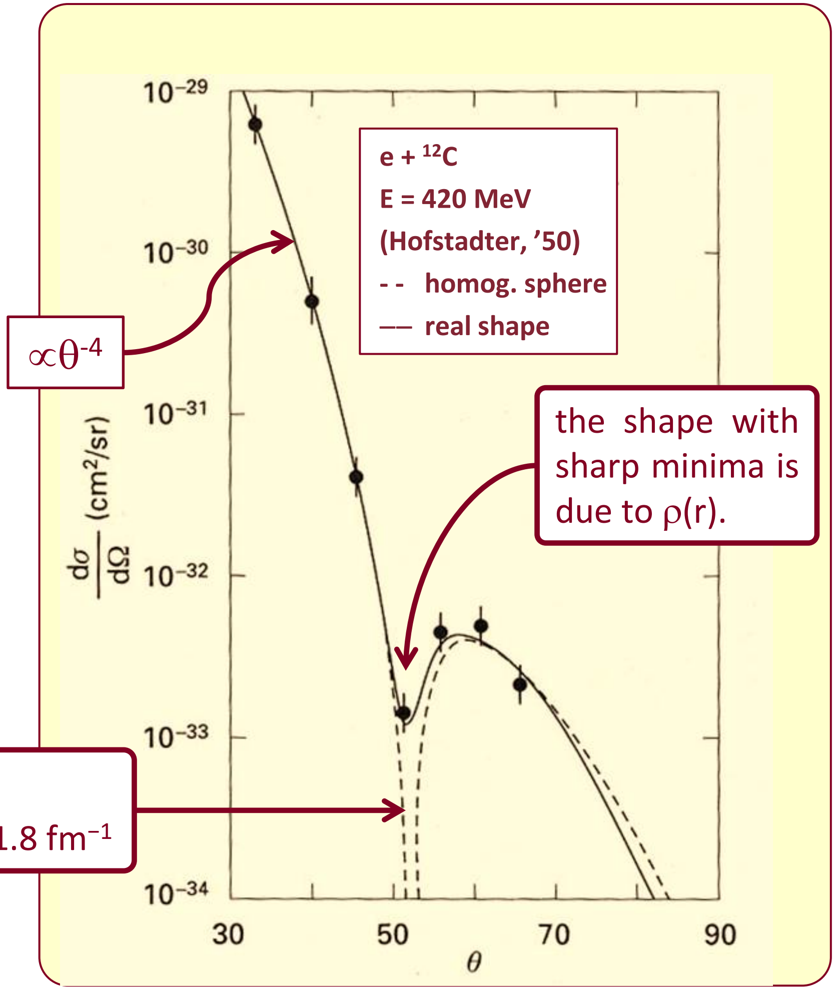
$\rightarrow$ define the **form factor** $\mathcal{F}(\vec{q})$, as the **Fourier transform of the charge distribution function**:

$$\boxed{\mathcal{F}(\vec{q}) = \int e^{\left(i\frac{\vec{q}\cdot\vec{x}}{\hbar}\right)} f(\vec{x}) d^3x}; \qquad \boxed{\vec{q} = \vec{p} - \vec{p}\,'};$$

- pointlike: $f(\vec{x}) = \delta(\vec{x}) \rightarrow \mathcal{F}(\vec{q}) = 1$.
- if $\rho(\vec{x})$ depends only on $|\vec{x}|$ [*next slides*]:

$$\left[\frac{d\sigma}{d\Omega}\right]_{exp} = \left[\frac{d\sigma}{d\Omega}\right]^*_{Mott} \times \left|\mathcal{F}(q^2)\right|^2.$$

form factors are measurable, at least in principle.

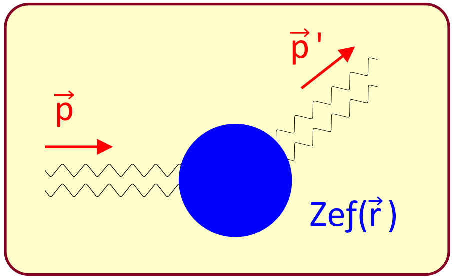


[*in the following, we will discuss only the case with spherical symmetry* $\rho(r)$, *when* $\mathcal{F}(\vec{q})$ depends on $q=|\vec{q}|$].

form factors: qm definition

q.m. calculation [Thomson, 166]

- non-relativistic q.m. + Born approx.;
- Coulomb potential;
- negligible recoil;
- initial (i) and final (f) particle as plane waves with ƛ << nucleus size [*see little box*];
- charge distribution $f(\vec{r})$, normalized to 1;
- $\vec{q} = \vec{p} - \vec{p}\,'$ and $\mathcal{F}(q^2)$ as defined before.



$$V(\vec{r}) = -\int d^3\vec{r}\,' \frac{Z\alpha f(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|};$$

$$\psi_i = e^{i(\vec{p}\cdot\vec{x}-Et)}/\sqrt{\Phi}; \qquad \psi_f = e^{i(\vec{p}\,'\cdot\vec{x}-Et)}/\sqrt{\Phi};$$

$$\mathcal{M}_{fi} = \langle\psi_f|V(\vec{r})|\psi_i\rangle = \frac{1}{\Phi}\int e^{-i\vec{p}\,'\cdot\vec{r}}V(\vec{r})e^{i\vec{p}\cdot\vec{r}}d^3\vec{r} =$$

$$= -\frac{1}{\Phi}\int\int e^{i\vec{q}\cdot\vec{r}}\frac{Z\alpha f(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}d^3\vec{r}\,'d^3\vec{r} =$$

$$= -\frac{1}{\Phi}\int\int e^{i\vec{q}\cdot(\vec{r}-\vec{r}\,')}e^{i\vec{q}\cdot\vec{r}\,'}\frac{Z\alpha f(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|}d^3\vec{r}\,'d^3\vec{r} =$$

$$= \left[-\frac{1}{\Phi}\int e^{i\vec{q}\cdot\vec{R}}\frac{Z\alpha}{4\pi|\vec{R}|}d^3|\vec{R}|\right]\times\left[\int f(\vec{r}\,')e^{i\vec{q}\cdot\vec{r}\,'}d^3\vec{r}\,'\right] =$$

$$= \mathcal{M}_{fi}^{point}\times\mathcal{F}(q^2)$$

$$\boxed{\vec{R} = \vec{r}-\vec{r}\,'}$$

$$\rightarrow \boxed{\left[\frac{d\sigma}{d\Omega}\right]_{\text{non-point}} = \left[\frac{d\sigma}{d\Omega}\right]_{\text{point}} \times \left|\mathcal{F}(q^2)\right|^2}.$$

Φ = volume

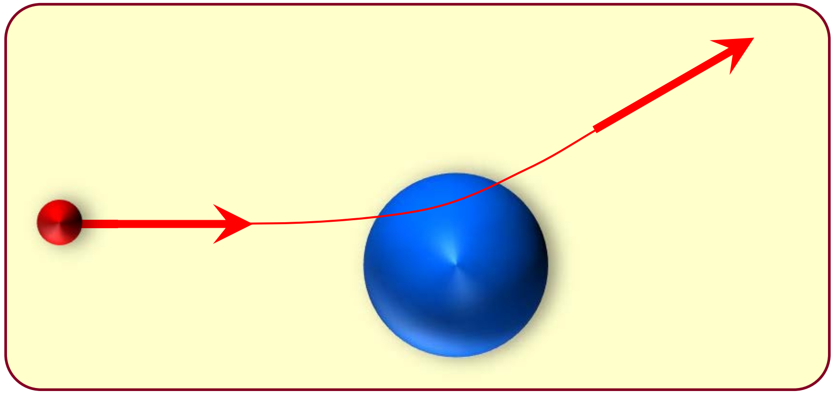
form factors: radial symmetry

In principle, the function ρ(r) may be computed by measuring $\mathcal{F}(q^2)$ and then, e.g. numerically:

$$\rho(r) = \frac{Ze}{(2\pi)^3} \int_{\boxed{\text{all q}}} \mathcal{F}(q^2) e^{-iqr} d^3q.$$

However, the range of q accessible to experiments is limited; therefore, the behavior of $\mathcal{F}(q^2)$ for q^2 large (i.e. r small, *the interesting region*) has to be extrapolated with reasonable assumptions.

In the next slides, examples of ρ(r) and $\mathcal{F}(q^2)$ are computed (e.g. the case of a homogeneous sphere of radius R).



Compute the symmetrical case[(1)]; neglect the nuclear recoil :

$$\mathcal{F}(q^2) = \frac{1}{S}\int e^{i\vec{q}\cdot\vec{x}} f(\vec{x}) d^3x = \qquad [f(\vec{x}) = f(r) \rightarrow]$$

$$= \frac{2\pi}{S}\int_0^\infty f(r)\, r^2 dr \int_{-1}^{1} e^{iqr\cos\theta} d\cos\theta =$$

$$= \frac{2\pi}{S}\int_0^\infty f(r)\, r^2 \frac{2}{2}\frac{1}{iqr}\left[e^{iqr} - e^{-iqr}\right] dr =$$

$$= \frac{4\pi}{S}\int_0^\infty f(r)\, r^2 \frac{\sin(qr)}{qr} dr;$$

$$S = 4\pi\int_0^\infty f(r)\, r^2 dr \qquad [=1 \text{ if normalized}];$$

---

(1) dσ/dΩ, both Rutherford and Mott, is scale-independent. However, if ρ(r) depends on a scale (e.g. by a sphere radius), form factors break the scale invariance of the dynamics.

form factors: examples

$f(\vec{r}) = f(|\vec{r}|)$

$$f(r) = \frac{1}{(2\pi)^3}\int \mathcal{F}(q^2)e^{-iqr}\,d^3q \qquad\qquad \mathcal{F}(q^2) = 4\pi\int_0^\infty f(r)\,r^2\,\frac{\sin(qr)}{qr}\,dr$$

Charge distribution	f(r)	form factor	$\mathcal{F}(q^2)$	example
point-like	$\delta(r)/(4\pi)$	constant	1	$e^\pm$
exponential	$(a^3/8\pi)$ $\exp(-ar)$	dipolar	$1/(1+q^2/a^2)^2$	p [1]
gaussian	$[a^2/(2\pi)^{3/2}]$ $\exp(-a^2r^2/2)$	gaussian	$\exp[-q^2/(2a^2)]$	^{6}Li
homog. sphere	$3/(4\pi R^3)$ $r \leq R$; 0 $r > R$	oscill.	$3\alpha^{-3}(\sin\alpha - \alpha\cos\alpha)$ $\alpha = \lvert q \rvert R$	– (see)
sphere with soft surface	$\rho_0 / [1 + e^{(r-c)/a}]$	oscill.		^{40}Ca

Fermi (Woods-Saxon) function

[1] the proton shape depends on Q^2: from a pointlike body to a quark/gluon composite.

form factors: homogeneous sphere

Homogeneous sphere with unit charge :

$$\rho(r) = f(r) = \begin{cases} \rho_0 = \dfrac{3}{4\pi R^3} & r \le R \\ 0 & r > R \end{cases}$$

$$\mathcal{F}(q^2) = 4\pi \int_0^\infty f(r)\, r^2 \frac{\sin(qr)}{qr} dr =$$

$$= \frac{4\pi\rho_0}{q} \int_0^R r \sin(qr) dr = \qquad \boxed{w = qr; \bar{W} = qR}$$

$$= \frac{4\pi\rho_0}{q^3} \int_0^{\bar{W}} w \sin w\, dw = \frac{4\pi\rho_0}{q^3} \left[\sin w - w \cos w\right]_0^{\bar{W}} =$$

$$= \frac{4\pi\rho_0}{q^3} \left[\sin(qR) - qR\cos(qR)\right] =$$

$$= \frac{3}{q^3 R^3} \left[\sin(qR) - qR\cos(qR)\right]$$

if qR [= t] $\to 0$

$\mathcal{F} \approx 3/t^3$ [$(t - t^3/6)$ –

– $t(1-t^2/2)$] = 1.

first minimum :

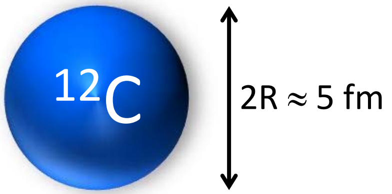
$qR = \tan(qR)$

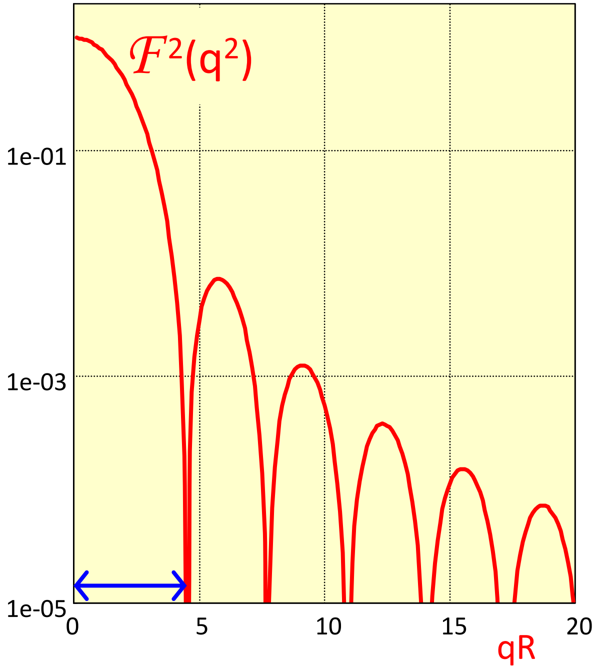
$\to qR \approx 4.5$

By comparing the first minimum with the experiment of ^{12}C ($q/\hbar \approx 1.8$ fm^{-1}), we get :

$$R \approx 4.5\, r_{min} = 4.5/1.8 \approx 2.5 \text{ fm}$$

i.e. ^{12}C is approximately a sphere with radius of 2.5 fm.





form factors: $<r^2>$

Study the behavior for $q \to 0$:

$$\mathcal{F}(q^2) = \iiint e^{iqr\cos\theta} f(r) r^2 dr d\cos\theta d\varphi =$$

$$= 2\pi \int_0^\infty f(r) r^2 dr \int_{-1}^{1} \begin{bmatrix} 1 + iqr\cos\theta - \\ -\frac{1}{2}(qr)^2 \cos^2\theta + \ldots \end{bmatrix} d\cos\theta =$$

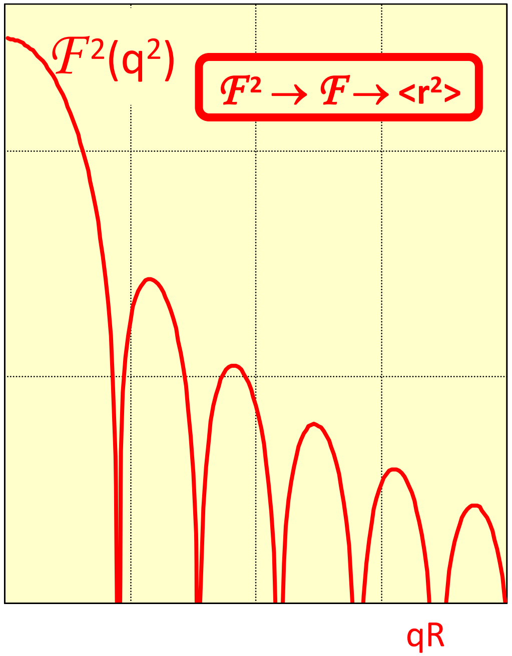
$$= 4\pi \int_0^\infty f(r) r^2 dr + 0 - \frac{4\pi}{6} q^2 \int_0^\infty f(r) r^4 dr + \ldots =$$

$$= 1 - \frac{1}{6} q^2 < r^2 > + \ldots$$

with $$< r^2 > \equiv \iiint r^2 f(\vec{x}) d^3x = 4\pi \int_0^\infty r^2 f(r) r^2 dr.$$

$$\to \quad \boxed{r_{RMS} = \sqrt{< r^2 >} = \sqrt{-6 \left. \frac{d\mathcal{F}(q^2)}{dq^2} \right|_{q^2=0}}}.$$

The parameter $<r^2>$ is a measure of the $(\text{size})^2$ of the [*charge of the*] particle.



form factors: solution

Simple problem : check that for the homogeneous sphere, both directly and from the definition :

$\langle r^2 \rangle = 3R^2/5$.

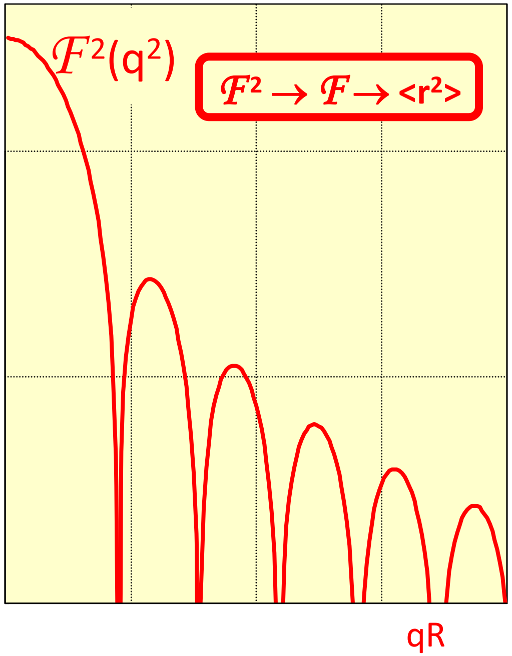
$$\left\langle r^n \right\rangle = \frac{1}{V}\iiint r^n d^3x = \frac{4\pi}{V}\int_0^R r^n r^2 dr =$$

$$= \frac{4\pi}{V}\frac{R^{n+3}}{n+3} = \frac{4\pi R^{n+3}}{n+3}\frac{3}{4\pi R^3} =$$

$$= \frac{3}{n+3}R^n$$

$$\xrightarrow{n=2} \left\langle r^2 \right\rangle = \frac{3}{5}R^2$$

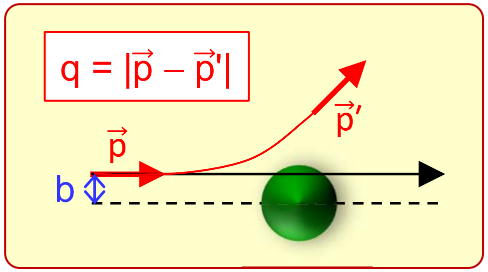
[qed, too easy to enjoy]



form factors: q→0 vs q→∞

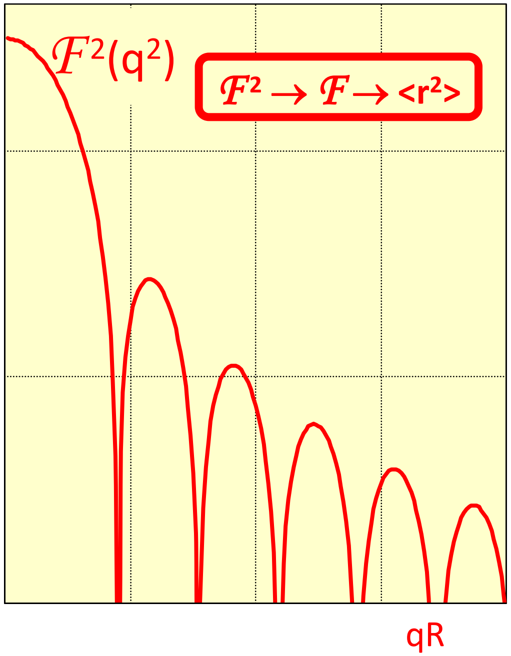
The limits q → 0, → ∞ have a deep meaning:

- q is (approximately) the conjugate variable of b, the impact parameter of the projectile wrt the target center:
  - → for q very small (i.e. b very large), the target behave as a point-like object;
  - → for q quite small (i.e. b quite large) it behaves as a coherent homogeneous charged sphere with radius $\sqrt{<r^2>}$;
  - → large q probes the nucleus at small b;
- "new physics" (a substructure emerging at very small distance) requires very large q, which in turn is only possible if a large projectile energy is available.



The same story has repeated many times, from Rutherford to the LHC, but at smaller b (i.e. larger q). This fact is the main justification for higher energy accelerators …

… and (unfortunately) larger experiments, larger groups, more expensive detectors, politics, troubles, … [*the usual "laudatio temporis acti", forgive me*]



form factors: shape of nuclei

Summary of systematic study of the form factors for nuclei [just results, no details]:

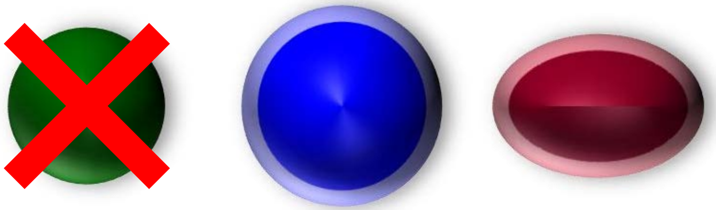
- heavy nuclei :
  - NOT "homogeneous spheres" with a sharp edge;
  - similar to **spheres with a soft edge**;
  - charge distribution is well reproduced by a standard Fermi function :

    $\rho_{charge}(r) = \rho_0 \,/\, [1 + e^{(r-c)/a}]$;
  - for large A (see figure) :

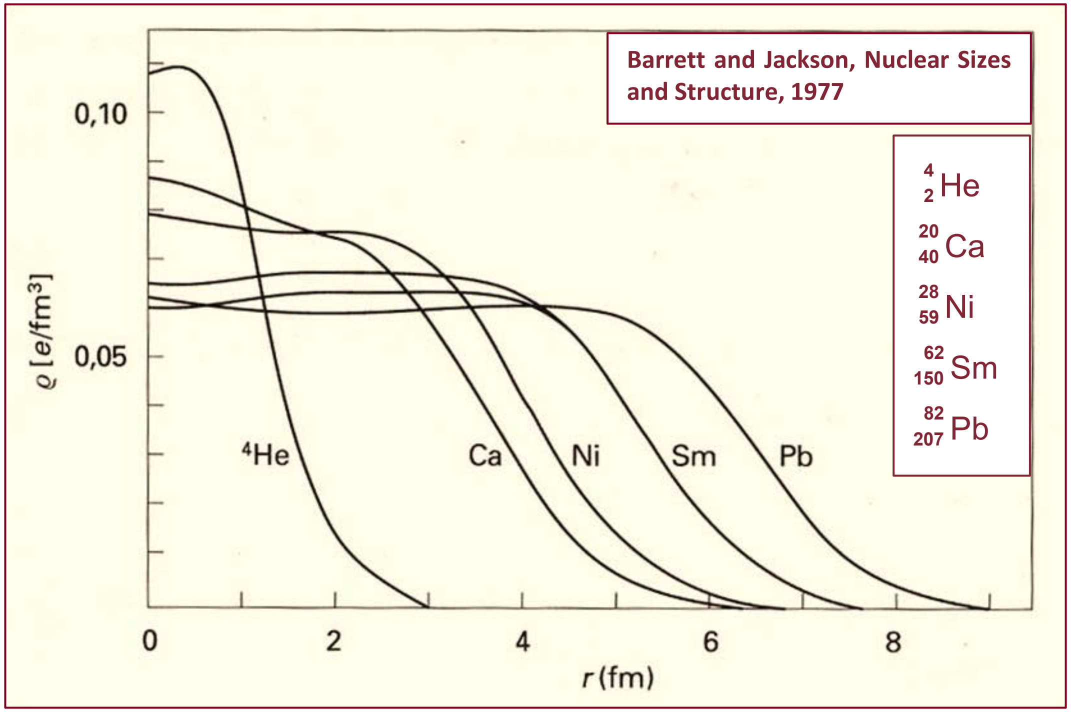
    $c \approx 1.07 \text{ fm} \times A^{1/3}$ ["radius"]

    $a \approx 0.54 \text{ fm}$ ["skin"];

$V_{nucleus} \propto A \rightarrow c \approx r_{nucleus} \propto A^{1/3}$



- light nuclei (^{4}He, ^{6,7}Li, ^{9}Be) more Gaussian-like;
- all these nuclei have spherical symmetry;
- lanthanides (rare earths) are more like ellipsoids [*think to an experiment to show it*].



form factors: nuclear density

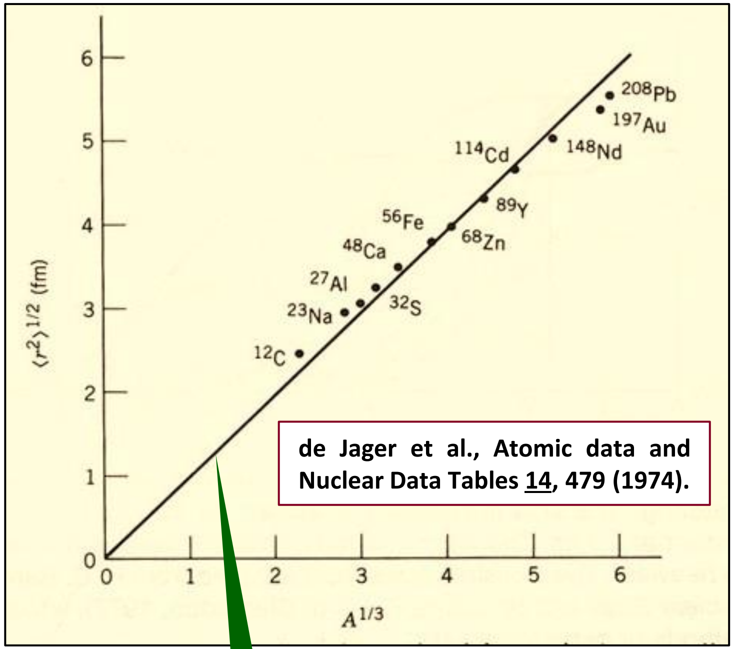
Compute the nuclear densities of p and n [$q_p\,\rho_Q = dq/dV$, $m_p\,\rho_p = dm_p/dV$] :

- assume homogeneous and equal distribution of p and n;
- then:
  - $\rho_Q = \rho_p$ = proton density;
  - ρ_n = neutron density = ρ_p;
  - ρ_T = nuclear density = $\rho_p + \rho_n$;
- compute :
  - $\rho_T = \rho_p + \rho_n = \rho_p + N\,\rho_p / Z = A\,\rho_Q/Z$;
  - $A = V\rho_T = {}^4/_3\,\pi R^3 \rho_T$;
  - ρ_T = 0.17 nucleons / fm³ (from ρ_0 of previous slide);
- $\frac{4\pi}{3}R^3 = \frac{4\pi}{3}r_0^3 A \quad \rightarrow$

$$\boxed{r_0 = \frac{R}{\sqrt[3]{A}} = \sqrt[3]{\frac{3\cancel{A}}{4\pi\rho_T}\frac{1}{\cancel{A}}} \approx 1.12\text{ fm}}.$$

- in fair agreement with "c" [*previous slide*] and with the slope of the fig.:

$$r_0|^{exp} = 1.23\text{ fm}.$$



de Jager et al., Atomic data and Nuclear Data Tables 14, 479 (1974).

for light nuclei, the model is NOT valid: do NOT plot them.

e-N scattering: higher energy

Probing smaller space scales requires larger energies, both in the initial and final state [*today experiments work at the TeV scale* → *~10^{-18} m = 10^{-3} fm*].

High-energy + q.m. corrections to the Rutherford formula [1st already discussed]:

- consider the electron spin [*Rutherford had only bosons !!!*];
- include the target recoil in the Mott cross section [Perkins-1971, 197];
- use 4-vectors p and p' to describe the scattering [instead of $\vec{p}$ and $\vec{p}'$]:

$$q^2 = (p-p')^2 = 2m^2 - 2\left(EE' - |\vec{p}||\vec{p}'|\cos\theta\right)$$
$$\approx -4EE'\sin^2(\theta/2);$$
$$Q^2 \equiv -q^2 \approx 4EE'\sin^2(\theta/2).$$

- for scattering eN, consider the magnetic moment of the nucleons, by introducing the parameter $\tau = Q^2/(4M^2)$ [*next slide*].

Description of the scattering

↓ no electron spin, no magn. moment, notice E'

$$\left[\frac{d\sigma}{d\Omega}\right]_{\text{Rutherford}} = \frac{4Z^2\alpha^2E'^2}{|\vec{q}|^4};$$

↓ + electron spin

$$\left[\frac{d\sigma}{d\Omega}\right]^*_{\text{Mott}} = \left[\frac{d\sigma}{d\Omega}\right]_{\text{Rutherford}} \times \boxed{1-\beta^2\sin^2\frac{\theta}{2}};$$

$$\boxed{\approx \cos^2(\theta/2)}$$

↓ + recoil

$$\left[\frac{d\sigma}{d\Omega}\right]_{\text{Mott}} = \left[\frac{d\sigma}{d\Omega}\right]^*_{\text{Mott}} \times \frac{E'}{E};$$

↓ + magn. moment

$$\boxed{\text{"}\tau\text{"}}$$

$$\left[\frac{d\sigma}{d\Omega}\right]_{\text{point, spin½}} = \left[\frac{d\sigma}{d\Omega}\right]_{\text{Mott}} \times \left(1 + 2\boxed{\frac{Q^2}{4M^2}}\tan^2\frac{\theta}{2}\right).$$

e-N scattering: magnetic moments

For particles of mass m, charge e:
- point-like,
- spin ½;

the Dirac equation assigns an intrinsic magnetic dipole moment

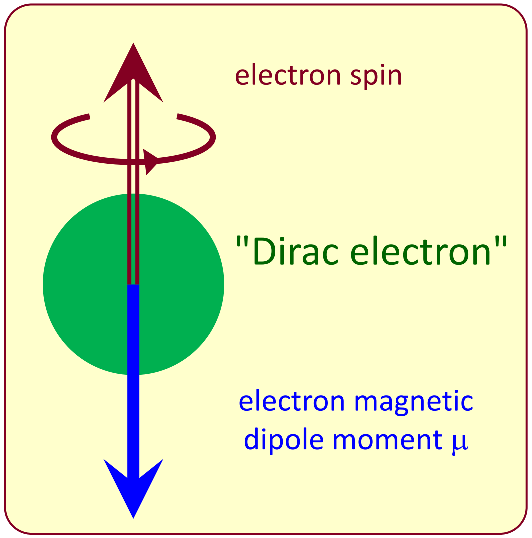
$\mu_C = g\, e\, \hbar\, /\, (4\, m)$;

g = "gyromagnetic ratio" = 2;

- an ideal "Dirac-electron" has a magnetic dipole moment

  $\mu_e = e\hbar/(2m_e) \approx 5.79 \times 10^{-5}$ eV/T;

- the first measurements roughly confirmed this value.
- for neutral particles (neutron ?) $\mu_N = 0$;
- this effect adds to the cross-section a term, corresponding to the "spin flip" probability, proportional to [Povh § 6.1]:
  - $\sin^2(\theta/2)$ [*cfr. the "Mott* factor"*];
  - $1/\cos^2(\theta/2)$ (to remove the non-flip dependence);
  - μ_N^2 ($\propto 1/M^2$);
  - Q^2 (mag field induced by the e)^2;
  - $$\left[\frac{d\sigma}{d\Omega}\right]_{\text{point, spin½}} = \left[\frac{d\sigma}{d\Omega}\right]_{\text{Mott}} \times \left(1 + 2\frac{Q^2}{4M^2}\tan^2\frac{\theta}{2}\right).$$
- Therefore the spin-flip is particularly relevant for large Q^2 and large θ.



e-N scattering: anomalous magnetic moments

In the nuclei and nucleons sector the experiments measured the following quantities :

- ☺ nuclear magnetism is a combination of the intrinsic magnetic moments of the nucleons and their relative orbital motions;
- ☺ all nuclei with Z=even and N=even have $\mu_{nuclei} = 0$;
- ➢ define for the nucleons (proton and neutron) the Dirac value

  $\mu_N = e\hbar/(4m_N) \approx 3.1525\times10^{-14}$ MeV/T;
- ➢ if p and n were ideal Dirac particles, they should have

  $\mu_p = 2\mu_N, \qquad \mu_n = 0,$

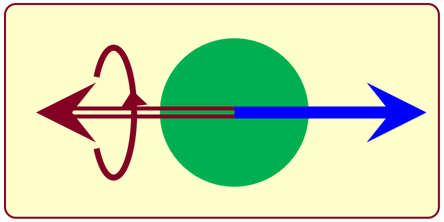
  i.e. in conventional notation

  $g_p/2 = \mu_p/\mu_N = 1, \qquad g_n/2 = 0;$

- ☹ instead, experiments found *anomalies*

  $g_p/2 = +(2.7928473508 \pm 0.0000000085),$

  $g_n/2 = -(1.91304273 \pm 0.00000045);$
- ☺ therefore, there are other effects which contribute to the magnetic moments, i.e. p and n are NOT ideal spin-½ point-like Dirac particles;
- ☺ [maybe] they are NOT point-like;
- ☺ in this case, their "g" is due to their (possibly complicated) internal structure, in analogy with the nuclear case.



e-N scattering: Rosenbluth cross-section

In the eN scattering, the main contribution is from single photon exchange [see fig.].

The **eeγ* vertex** is well under control, with three point-like, well-understood particles.

Instead, the **NN'γ* vertex** is the unknown, due to the internal structure of the proton.

Strategy : assume a simpler process (N = Dirac fermion), compare it with exp., then modify the theory, inserting parameters which model the nucleon structure.

Take also into account the spin and magnetic moment, both of the electron and the nucleon.

"Generalize" the cross section by defining the **Rosenbluth cross-section**, function of TWO form factors, both dependent on Q^2:

- $G_e(Q^2)$ for the electric part (no spin-flip);
- $G_M(Q^2)$ for the magnetic one (spin-flip).

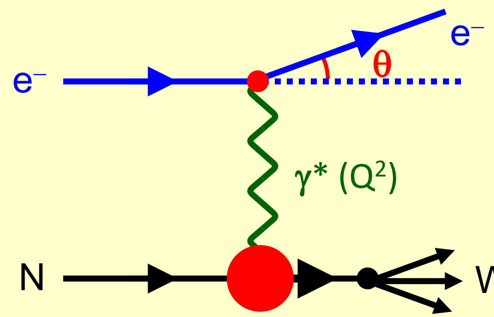
[formerly : $G_e(Q^2) = \mathcal{F}(Q^2)$, no G_M].

For a charged Dirac fermion f_D, proton, neutron :

- f_D : $G_E^f(\text{any } Q^2) = 1$, $G_M^f(\text{any } Q^2) = 1$;
- p : $G_E^p(Q^2 = 0) = 1$, $G_M^p(Q^2=0) \approx 2.79$;
- n : $G_E^n(Q^2 = 0) = 0$, $G_M^n(Q^2=0) \approx -1.91$.

$$\left[\frac{d\sigma}{d\Omega}\right]_{\text{Rosenbluth}} = \left[\frac{d\sigma}{d\Omega}\right]_{\text{Mott}} \times \left(\frac{G_E^2 + \tau G_M^2}{1+\tau} + 2\tau G_M^2 \tan^2\frac{\theta}{2}\right);$$

$$\tau = \frac{Q^2}{4M^2}; \qquad G_E = G_E\left(Q^2\right); \qquad G_M = G_M\left(Q^2\right).$$



e-N scattering: remarks on $\sigma_{Rosenbluth}$

A non-exhaustive personal classification(*) of "physics formulae":

1. "principles" [$\vec{F} = m\vec{a}$] – They require the a-priori knowledge of all entities involved; <u>not</u> direct empirical laws;
2. "natural laws" [the gravitational/Hooke law] – (semi-)empirical descriptions of the behavior of the Nature;
3. "positions" [$K = ½mv^2$] – They <u>define</u> a new entity, using other well-known entities;
4. "theorems" [the Gauss law] – Relations among well-known entities, math derived from other laws;
5. … other types (???) …

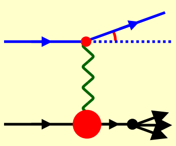
(*) This is too simplistic [*I know*], e.g. one of the Gauss and Coulomb law may be chosen as a law and the other as a derived theorem; the same for the positions; but in a given approach, the above classification is approximately correct.

The "Rosenbluth formula" is another type of math-logical relation:

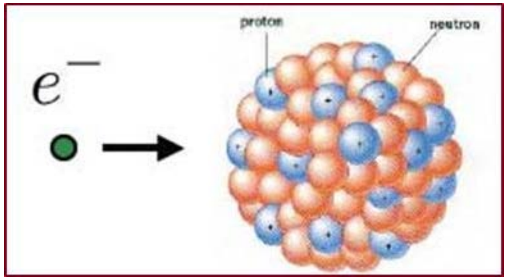
- it is a model, which includes some constraints (e.g. the θ dependence cannot be modified);
- but it is "open" (e.g. G_E and G_M depends on the unknown Nucleon structure);
- it contains in-se no full predictive power;
- but it is a powerful working tool to study the phenomena and incorporate new knowledge in a (quasi-)formal theory.

A "frontier" approach, quite common in modern research, which requires some care by the users/students.

$$\left[\frac{d\sigma}{d\Omega}\right]_{Rosenbluth} = \left[\frac{d\sigma}{d\Omega}\right]_{Mott} \times \left(\frac{G_E^2 + \tau G_M^2}{1+\tau} + 2\tau G_M^2 \tan^2\frac{\theta}{2}\right).$$



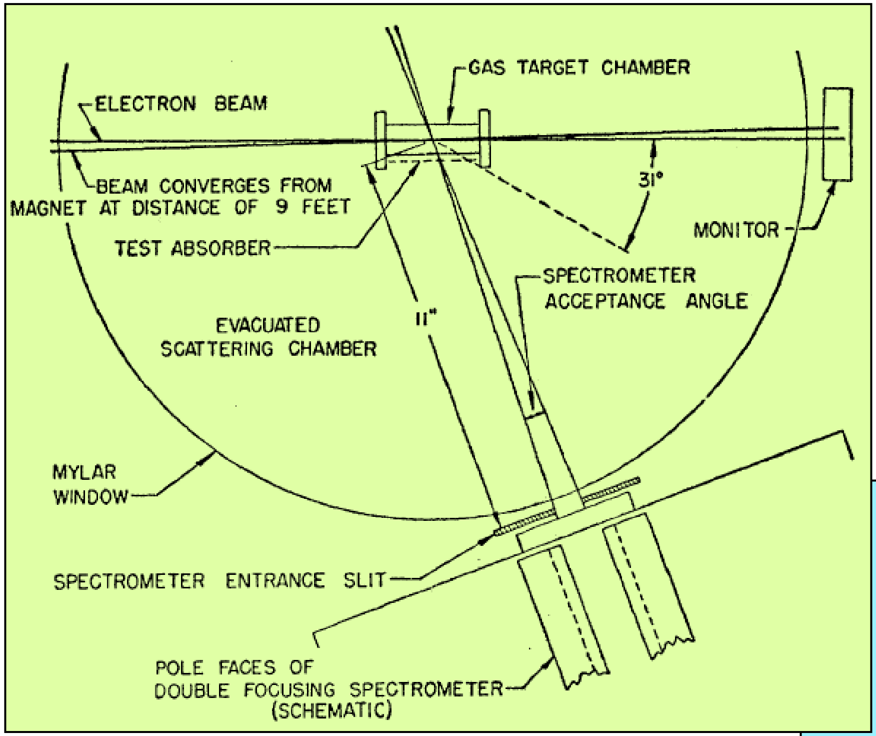
Proton structure: Mark 3 Linac



Maybe you think that this is old and obsolete;
in this case, go and look:
https://home.cern/news/news/physics/meet-amber

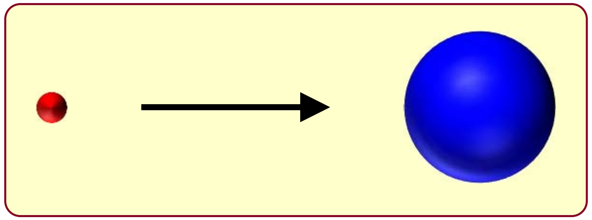
Mark 3 electron Linac – Stanford University – 1953

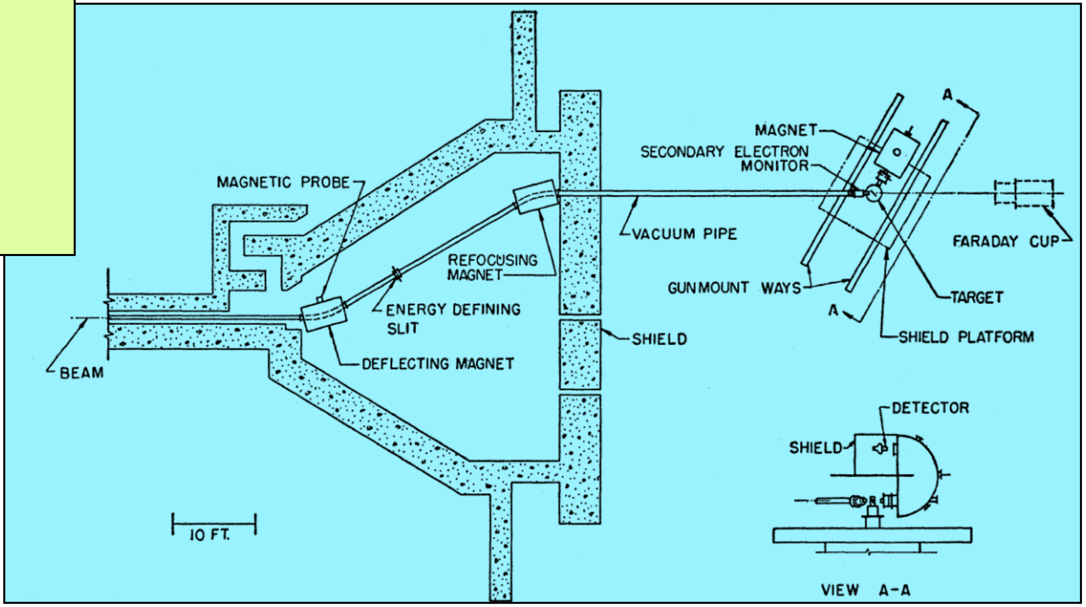
Proton structure: setup



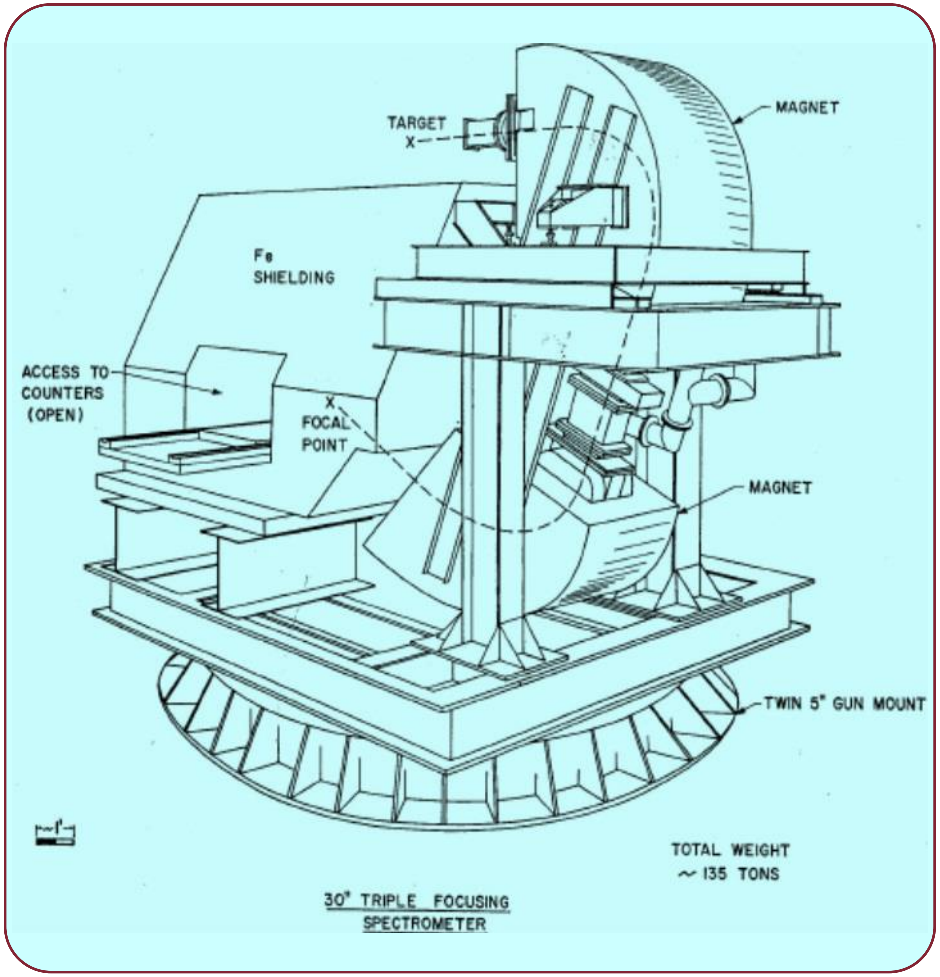
Stanford - 1956

Robert Hofstadter



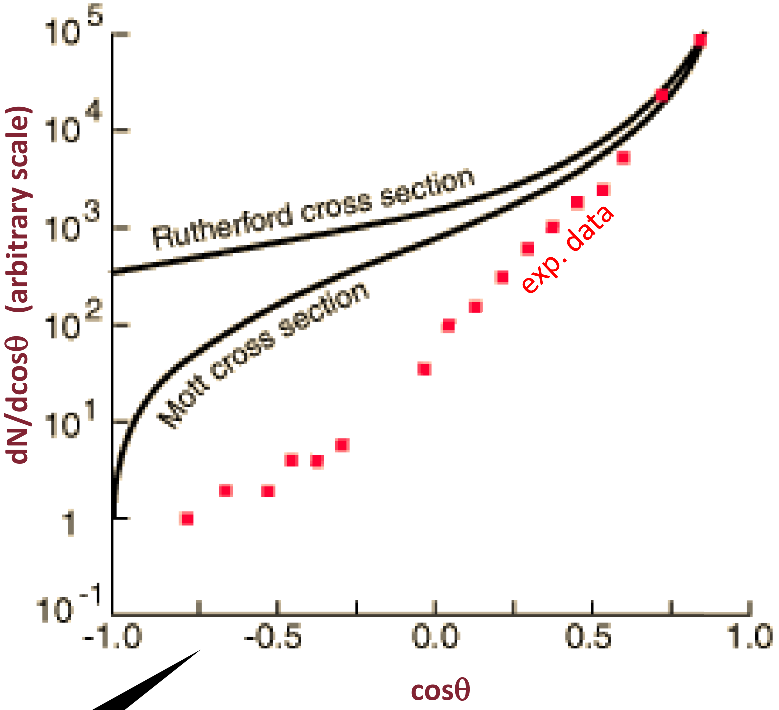


Proton structure: Mark 3 detector



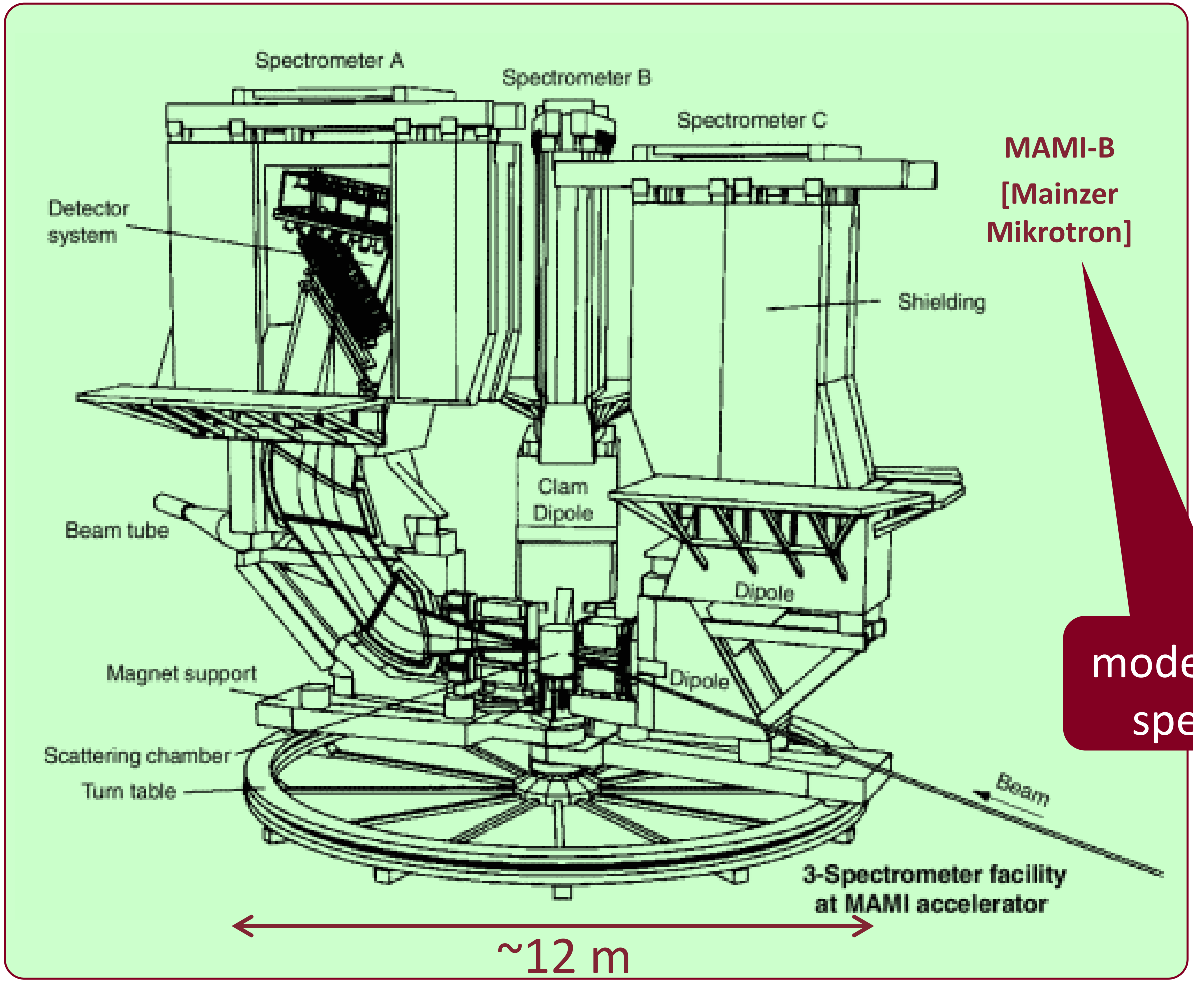
Hofstadter et al., Phys. Rev. 92, 978 (1953)

p(e⁻) = 125 MeV



A summary of Hofstadter experiments, see later

Proton structure: MAMI-B



MAMI-B [Mainzer Mikrotron]

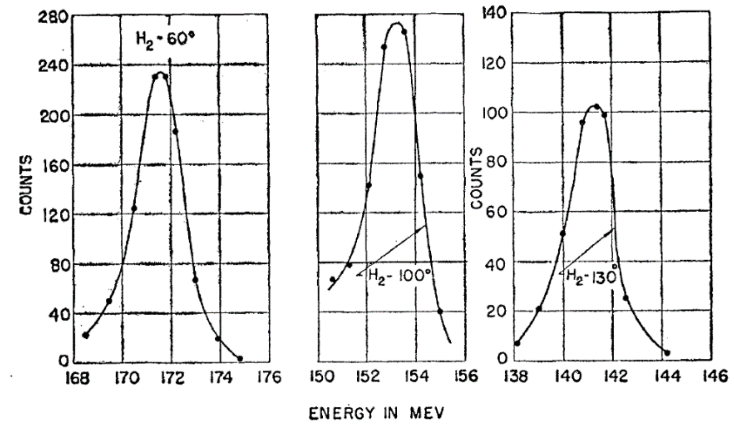
modern magnetic spectrometer

Proton structure: quality check

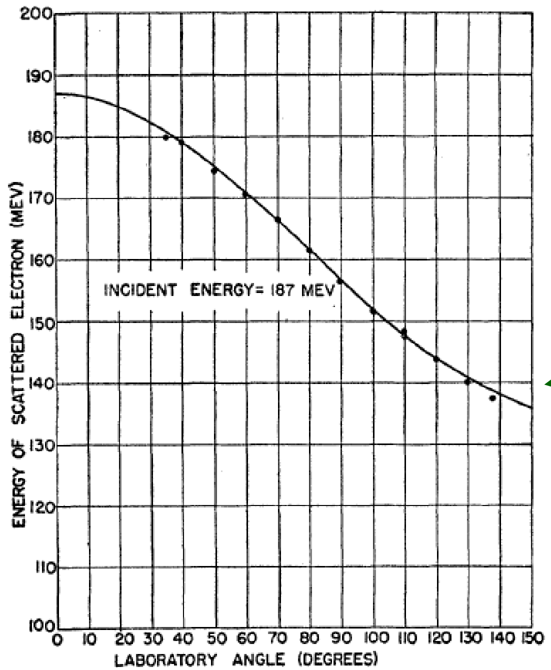
In 1956 the Hofstadter spectrometer measured the elastic <u>ep → ep</u>. It measured θ in the range 35°-138°, and therefore Q^2, using the relations :

$$E' = \frac{E}{1+E(1-\cos\theta)/M};$$

$$Q^2 = 2EE'(1-\cos\theta).$$



Plot E' for E = 185 MeV at fixed θ (60°, 100°, 130°) [*in a perfect experiment, expect* δ_{Dirac}].



Show the plot E' = E'(θ).

Result:

- <u>Kinematics</u> ok. Experiment under control.
- **Study the <u>dynamics</u>**.

Proton structure: results

Study $[d\sigma/d\Omega]_{Lab}$ ($\rightarrow$ legend):

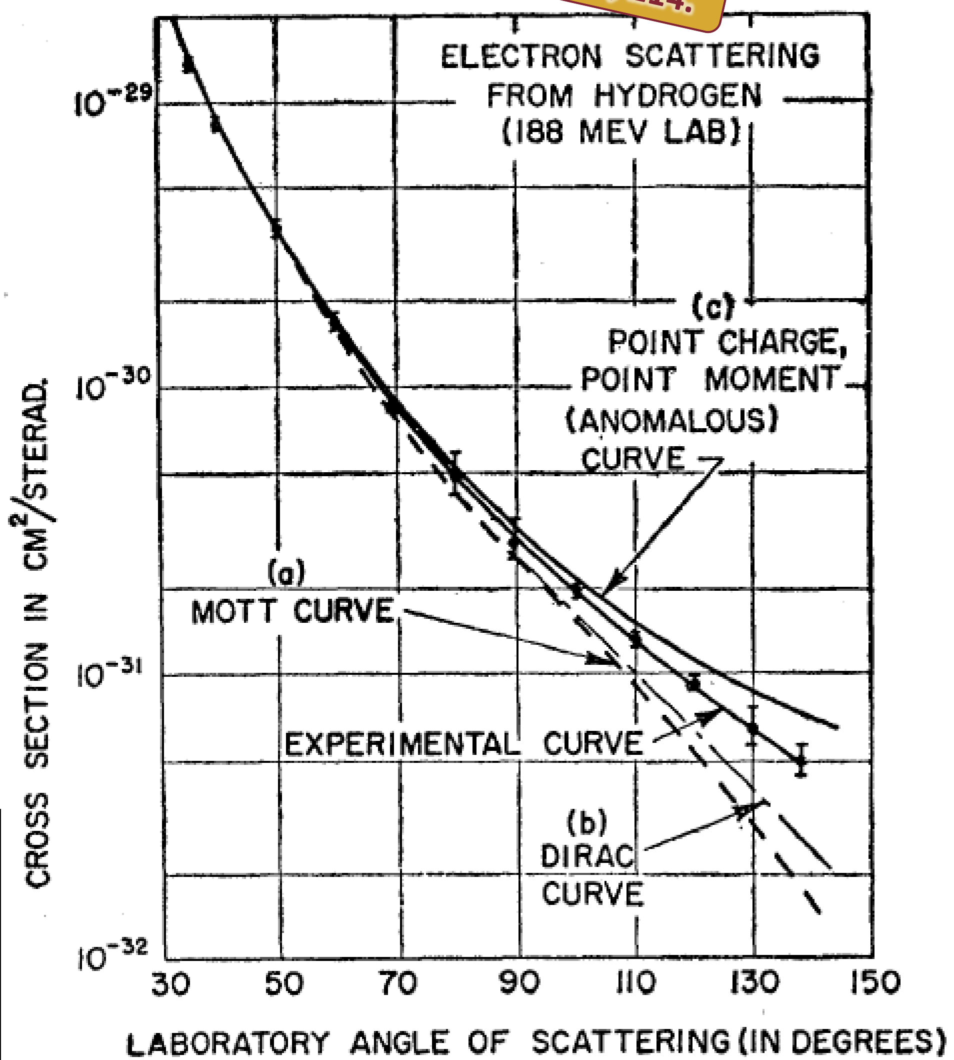
- small θ (= small Q^2 $\rightarrow$ $d\sigma/d\Omega$ independent from G_M): all formulas agree $\rightarrow$ $G_E(Q^2=0) \approx 1$;
- large θ (= large Q^2, small distance, $d\sigma/d\Omega$ dependent on G_M): it **disagrees** with ANY theoretical prediction $\rightarrow$ G_E, G_M ?;
- the disagreement with (a) and (b) was foreseen (proton $g_p \neq 2$);
- the one with (c) shows a dependence on Q^2 (on scale) $\rightarrow$ proton is NOT point-like;
- Hofstadter measured ($r_{rms} \equiv \sqrt{<r^2>}$, see) :

$$r^p_{rms} = (0.77 \pm 0.10) \times 10^{-15} \text{ m};$$
$$r^\alpha_{rms} = (1.61 \pm 0.03) \times 10^{-15} \text{ m}.$$

… and got the 1961 Nobel Prize in Physics.

LEGEND	(a) Mott	(b) Dirac	(c) A-Dirac	(d) Exp.
G_E	1	1	1 fix	$G_E(Q^2) \approx 1$
G_M	no	1	2.79 fix	$G_M(Q^2)$?
point-like p ?	yes	yes	"yes" ?	no
fit low Q^2 ?	yes	yes	yes	def.
fit high Q^2 ?	no	no	no	def.

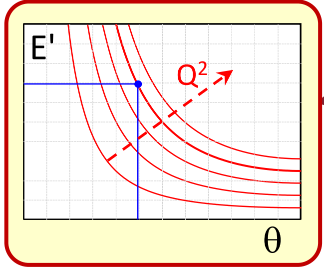
Rev.Mod.Phys.,28, 214.



Proton structure: $G_{E,M}^{p,n}$ vs Q^2

Write the Rosenbluth formula, <u>at fixed Q^2</u>, :

$$\left[\frac{d\sigma}{d\Omega}\right]_{\text{Rosenbluth}} \Bigg/ \left[\frac{d\sigma}{d\Omega}\right]_{\text{Mott}} = \left(\frac{G_E^2+\tau G_M^2}{1+\tau} + 2\tau G_M^2 \tan^2\frac{\theta}{2}\right).$$



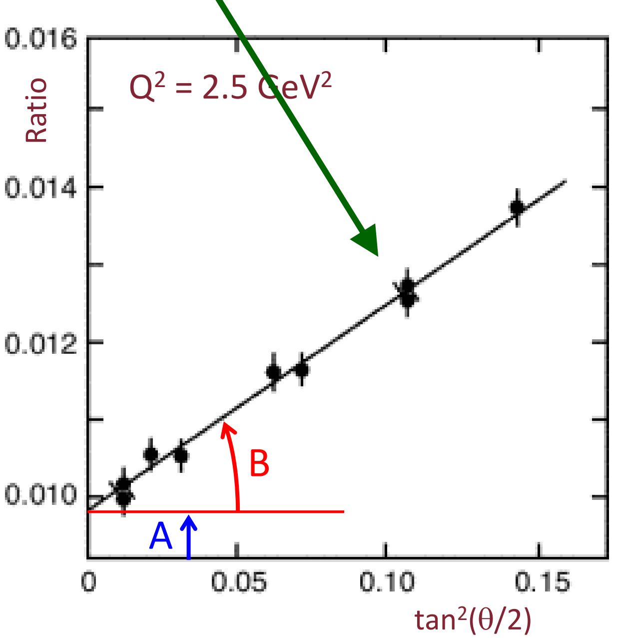
remember:
$Q^2 = 4EE'\sin^2(\theta/2)$

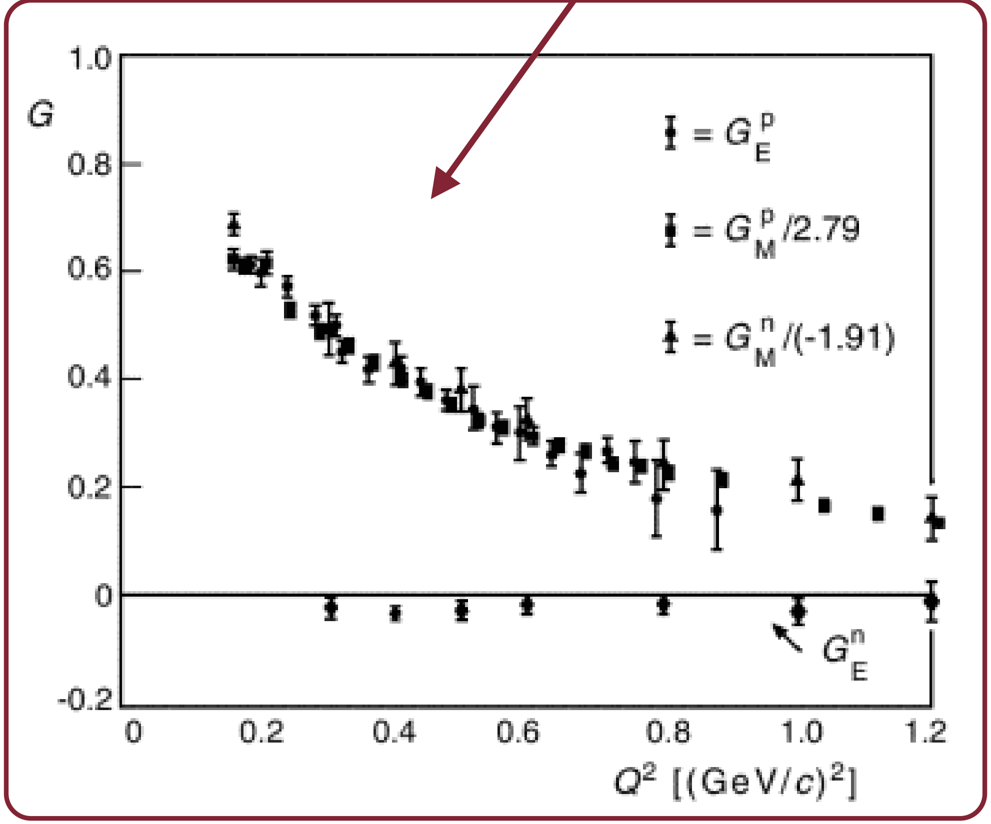
→ Ratio(E, θ, fixed Q^2) = A + B $\tan^2(\theta/2)$;

→ measure (A, B at fixed Q^2) vs $\tan^2(\theta/2)$;

→ get G_E^p, G_M^p, (G_E^n, G_M^n) at fixed Q^2 (example shown)

By repeating it at many Q^2, the full dependence can be measured (SLAC, '60s).





Proton structure: $G^{p,n}_{E,M}$ - remarks

- The fig. shows that the electric and magnetic form factors tend to a "universal" function of Q^2, with a **dipolar** shape :

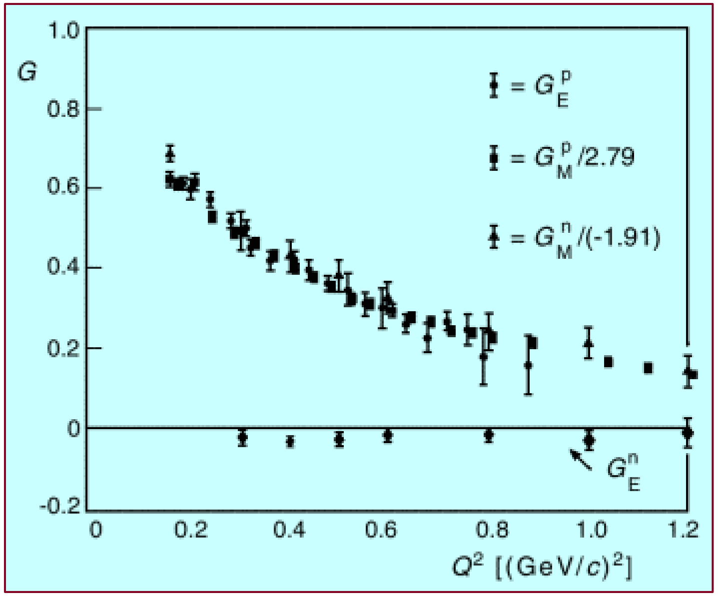
$$G_E^p(Q^2) \approx \frac{G_M^p(Q^2)}{2.79} \approx \frac{G_M^n(Q^2)}{-1.91} \approx G(Q^2) = $$
$$= \frac{1}{\left(1+Q^2/A^2\right)^2}; \qquad A^2 \approx 0.71\ \text{GeV}^2$$

- From the curve, it is possible to derive the function $\rho(r)$, at least where the 3- and 4-momentum coincide, i.e. at small Q^2. It turns out :
  $\rho(r) \approx \rho_0\, e^{-ar}$, $a \approx 4.27\ \text{fm}^{-1}$.

- The nucleons do NOT look like ~~point-like particles~~, nor ~~homogeneous spheres~~, but like diffused non-homogeneous systems.

- From the values at $Q^2=0$:

$$\langle r^2 \rangle_{dipole} = -6\hbar^2 \left.\frac{dG(q^2)}{dq^2}\right|_{q^2=0} =$$
$$= \frac{12}{a^2} \approx 0.66\ \text{fm}^2;$$
$$\sqrt{\langle r^2 \rangle_{dipole}} \approx 0.81\ \text{fm}.$$



Proton structure: comments

$$\left[\frac{d\sigma}{d\Omega}\right]_{\text{Rosenbluth}} \bigg/ \left[\frac{d\sigma}{d\Omega}\right]_{\text{Mott}} =$$

$$= \left(\frac{G_E^2 + \tau G_M^2}{1+\tau} + 2\tau G_M^2 \tan^2\frac{\theta}{2}\right); \quad \left[\tau = \frac{Q^2}{4M^2}\right].$$

therefore $$\lim_{Q^2\to 0}\left(\frac{d\sigma}{d\Omega}\right)_{\text{Rosenbluth}} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}}.$$

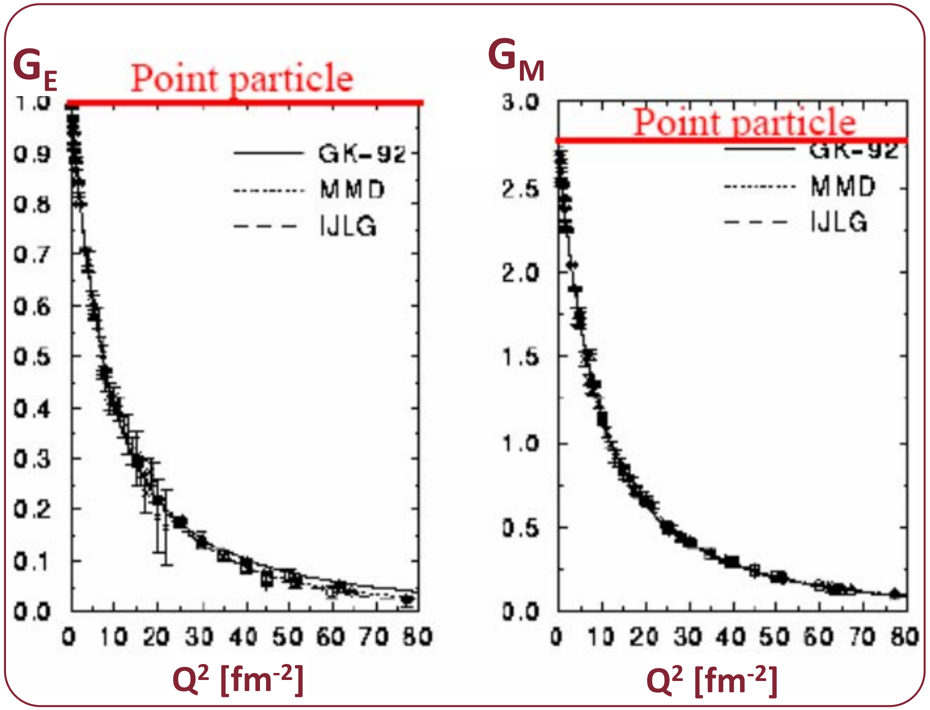
Notice also that, if the proton were point-like, one would find :

$G_E^p(Q^2) = G_M^p(Q^2) = 1$, <u>independent of Q^2</u>

[*and in addition would not understand why "2.79"*].

The form factors of the nucleons show three different ranges :

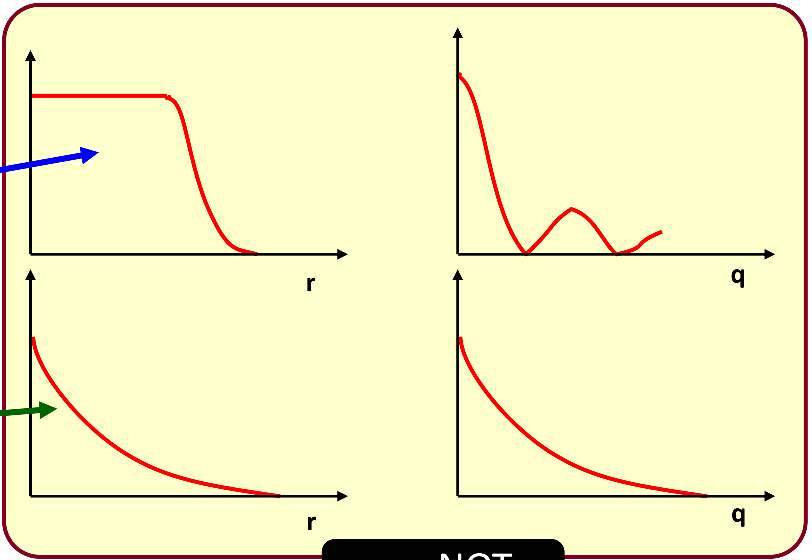
1. $Q^2 << m_p^2$: τ small, G_E dominates the cross section; in this range we measure the average radius of the electric charge : $<r_E> = 0.85 \pm 0.02$ fm;
2. $0.02 \leq Q^2 \leq 3$ GeV^2 :
   G_E and G_M are equally important;
3. $Q^2 > 3$ GeV^2 : G_M dominates.



Proton structure: interpretation

Differences between nuclei and nucleons :

1. nuclei exhibit diffraction maxima/ minima; this fact corresponds to charge distributions similar to homogeneous spheres with thin skin;
2. nucleons have diffused, dipolarly distributed form factors → exp. charge;
3. at this level, it is unclear whether the nucleons have substructure(s) → need experiments at smaller value of distances (i.e. larger values of Q^2);
4. [*maybe that*] the structure of the nucleons in the elastic scattering, described by the Rosenbluth formula, is an average with insufficient resolution;
5. at higher Q^2, one can expect a wider variety of phenomena :

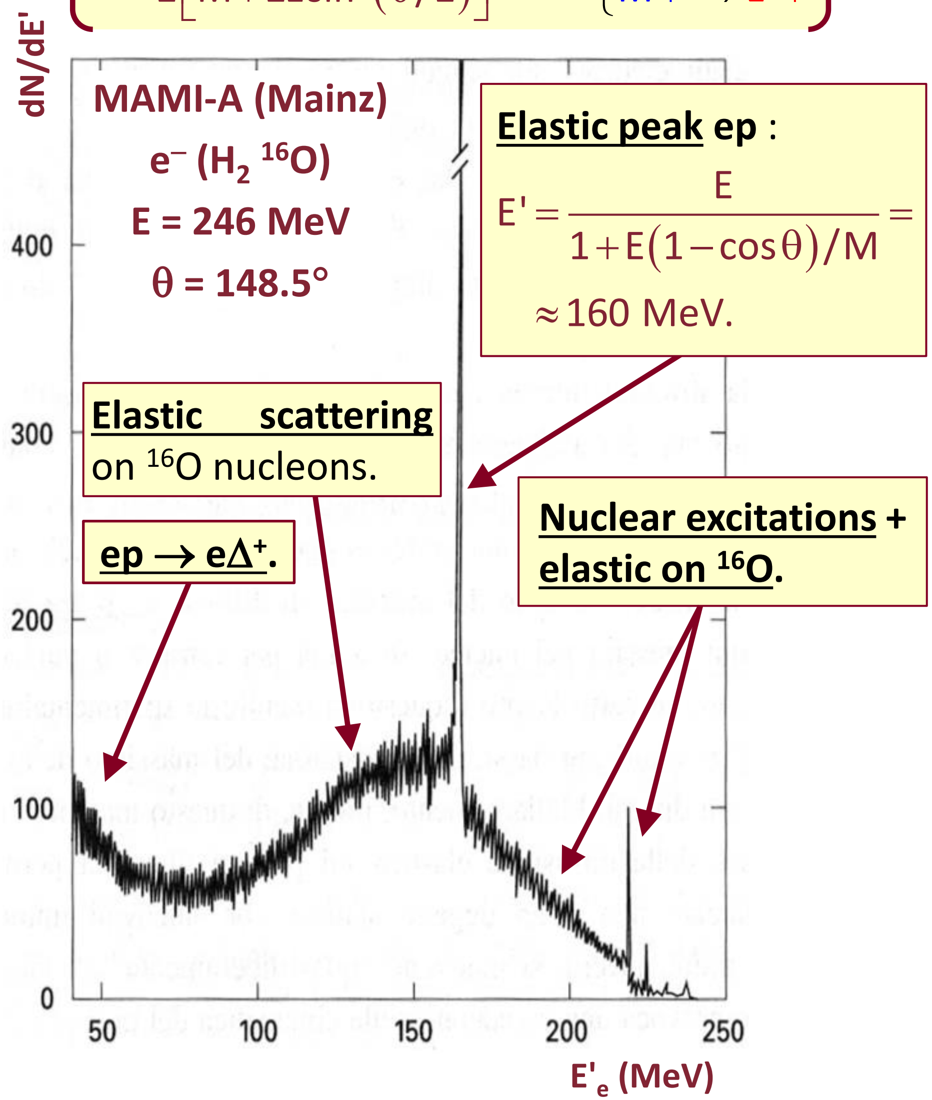


r, q NOT same scale

a. elastic scattering : $ep \to ep$;
b. excitation : $ep \to e$ "p^*" (e.g. $ep \to e\Delta^+$, $\Delta^+ \to p\pi^0$);
c. new states : $ep \to eX^+$ (X^+ = system of many particles).

higher Q^2: H_2O

$$E' = \frac{M^2 + 2ME - W^2}{2\left[M + 2E\sin^2(\theta/2)\right]}, \quad \text{i.e.} \begin{cases} W^2 \uparrow \Rightarrow E' \downarrow \\ M \uparrow \Rightarrow E' \uparrow \end{cases}$$



Send 246 MeV electrons → water vapor.

The scattering shows a complex distribution, with different phenomena in the same plot. At fixed θ of the electron in the final state, with increasing E' :

- e p $\to e\,\Delta^+$ (excitation of p from H);
- e p/n → e p/n ("elastic" on ^{16}O nucleons);
- e p → e p (elastic on H, E'≈160 MeV);
- e p → e X^+ (nuclear excitations);
- e ^{16}O → e ^{16}O (nucl. exc. / elastic)

The distribution depends also on the electron energy E and the final state angle θ.

[Problem: the Δ^+ has m ≈ 1230 MeV, Γ ≈ 120 MeV. In the plot only the tail of ep→eΔ^+ is shown. "Compute" the effect of the Breit-Wigner in mass in the E' variable. Is it sufficient to predict the E' plot ?]

higher Q^2: He^4, $\theta = 45°$

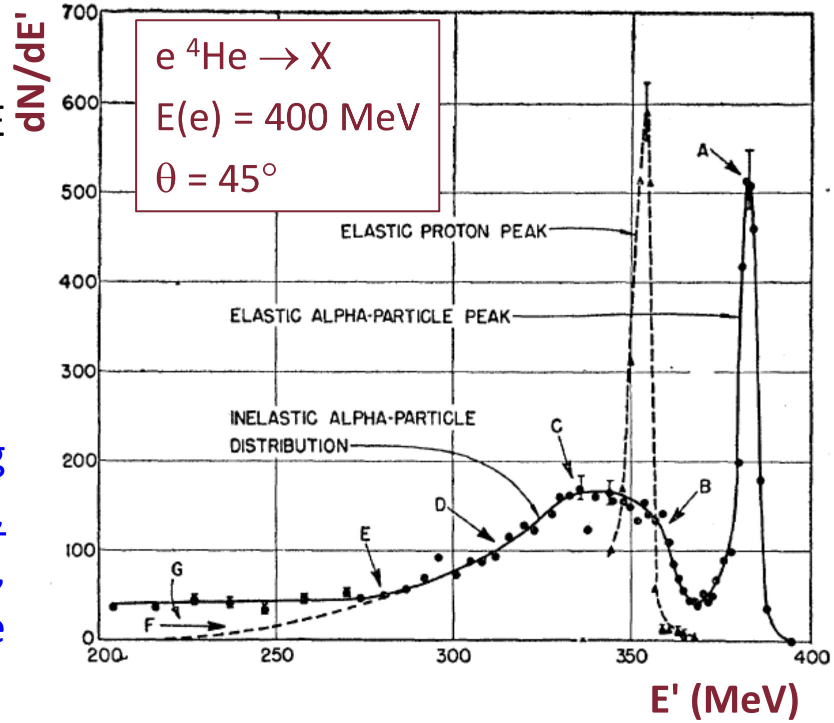
Another of these experiments (Hofstadter 1956, see fig.). Observe :

-- the elastic peak for ep $\to$ ep at the same E and θ, shown for comparison [*no problem*];

A. the elastic scattering e ^{4}He [ok, *expected*];

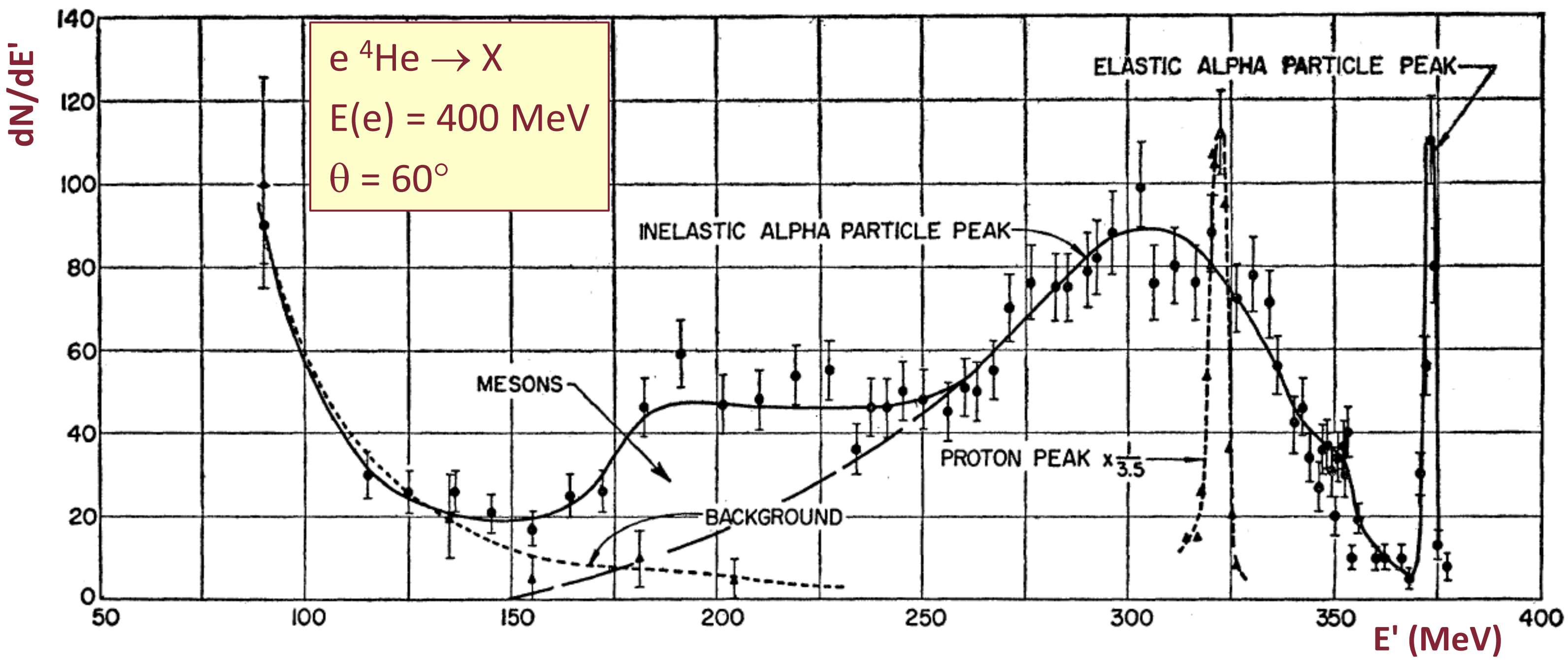
BCDEF. the elastic scattering ep / en (p/n acting as free particles in ^{4}He) [*maybe unexpected, but understandable*]; notice the peak width, due to the Fermi motion of nucleons inside the nucleus;

G. the production of π^- (i.e. of Δ's), which enhances the cross section (otherwise F.); notice : smaller E' $\to$ larger energy transfer [*the new entry in the game*].



$$E' = \frac{M^2 + 2ME - W^2}{2\left[M + 2E\sin^2(\theta/2)\right]}, \quad \text{i.e.} \begin{cases} W^2 \uparrow \Rightarrow E' \downarrow \\ M \uparrow \Rightarrow E' \uparrow \end{cases}$$

higher Q^2: He^4, θ = 60°



Same as before, but θ = 60°, i.e. larger Q^2 [$Q^2 \approx 4EE'\sin^2(\theta/2)$]. Notice :

- smaller elastic peak, both for (e^- ^{4}He) and (e^-p);
- wider ep/en (p/n inside ^{4}He) peak;
- (roughly) constant π production (seems independent from Q^2, as expected for point-like (?) particles;

Possible conclusions [*possibly wrong*] :

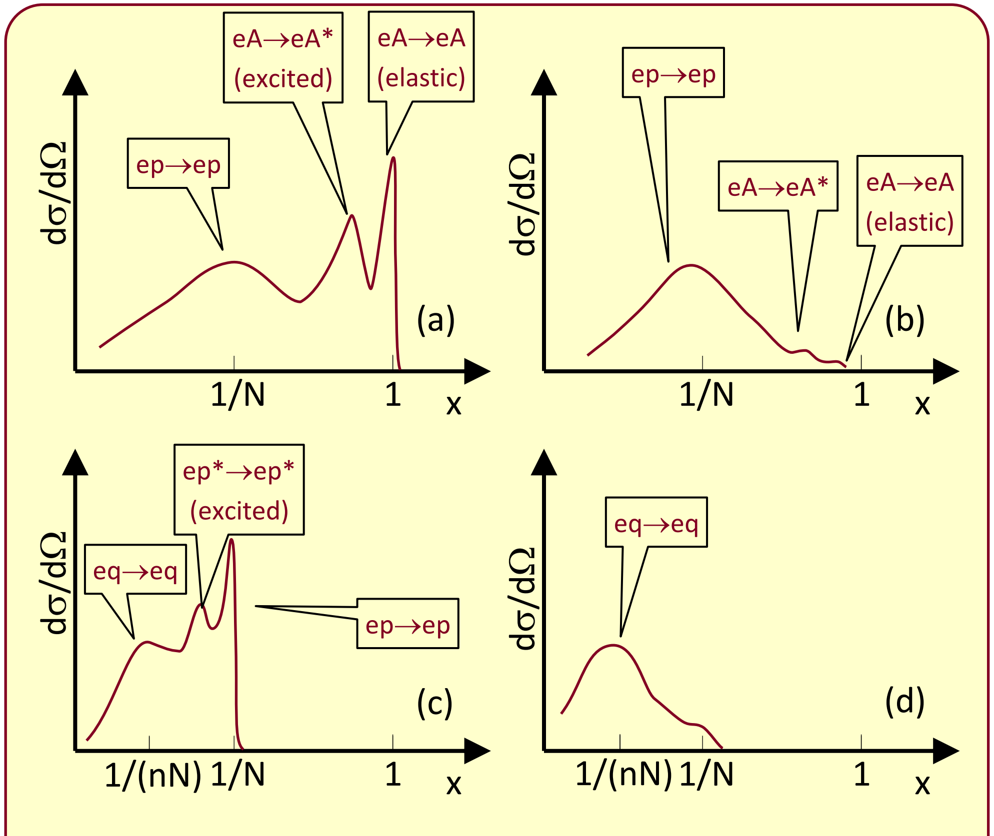
- everything under control for elastic and quasi-elastic data;
- the high-Q^2 part shows no evidence for sub-structures;
- maybe Q^2 is still too small (or maybe there are no substructures … !?);

→ go to even higher Q^2 !!!

higher Q^2: summary

Follow [BJ 444] to understand the dependence of $d\sigma/d\Omega$ on Q^2:

- scattering electron ("e^-") nucleus ("A");
- A with "N" nucleons (use "p", but neutrons similar);
- p with "n" hypothetical components ("q");
- plot vs adimensional variable $x=Q^2/(2M\nu)$, $0 < x < 1$;
- from (a) to (d), Q^2 increases;
  - a) at small Q^2, there are both scatterings with A and p;
  - b) increasing Q^2, the eA scattering disappears, while the ep scattering stays constant;
  - c) increasing Q^2, the constituents (if any) appears as eq → eq;



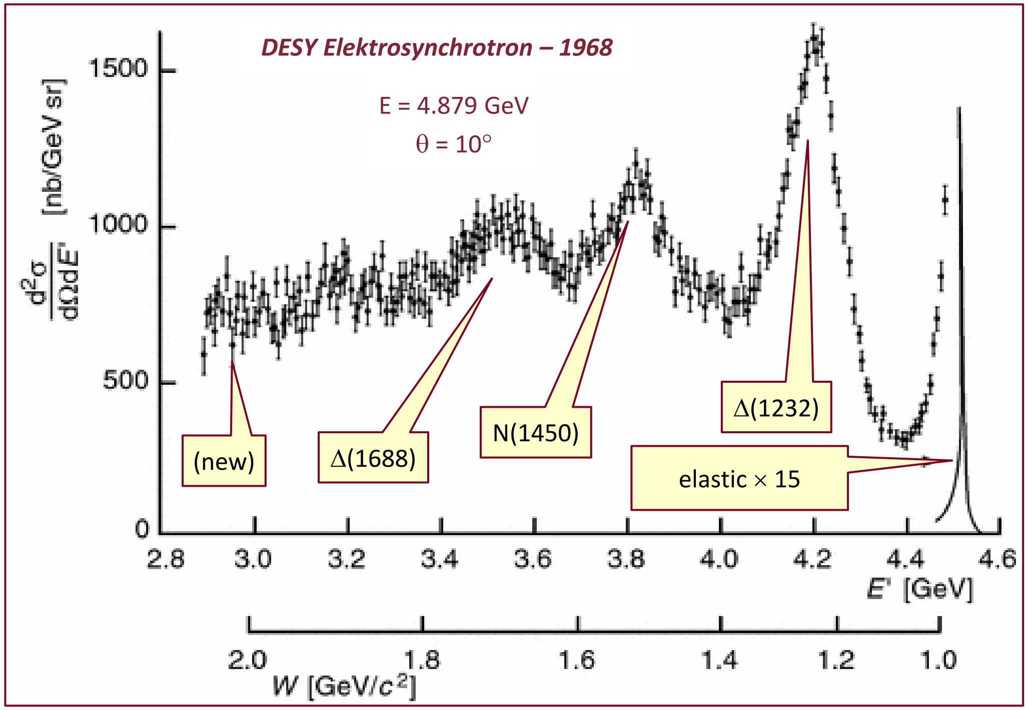
[just a sketch, not a reproduction of real experiments]

d) finally, at very large Q^2, the most (~ only) important process is eq → eq (with all the possible inelastic companions).

higher Q^2: constituents show up

Scattering ep → eX (DESY 1968) :

- Electron energy ≈ 5 GeV (higher than SLAC);
- resonances (R) production ep → eR clearly visible;
- new region at small E' (= high W);
- in this "new" region :
  - ➢ continuum (NO peaks);
  - ➢ rich production of hadrons;
  - ➢ NO new particles, only (p n π's); i.e. the proton breaks, but (different from the nucleus) NO constituent appears;
  - ➢ the constituents, if any, do not show up as free particles;



→ **Do quarks exist ???**
are they confined ??? why ???

[NB in 1968 color was proposed but not really understood, QCD did not exist]

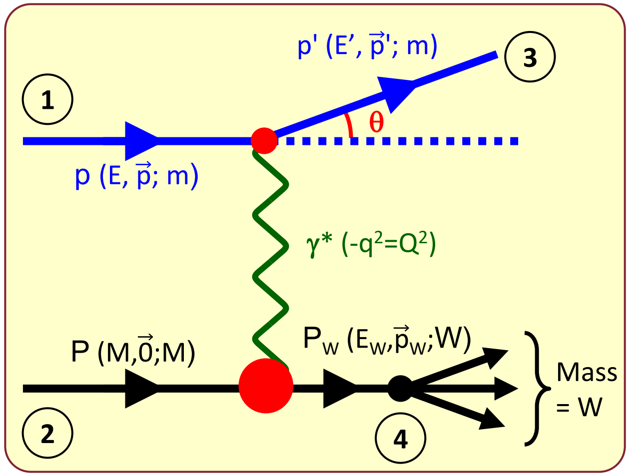
Deep inelastic scattering: structure functions

The usual parameterization of the cross section in the DIS region is the formula:

$$\left[\frac{d^2\sigma}{d\Omega dE'}\right]_{DIS} = \left[\frac{d\sigma}{d\Omega}\right]_{Mott}\left[W_2(Q^2,\nu)+2W_1(Q^2,\nu)\tan^2\frac{\theta}{2}\right] =$$

$$= \frac{4Z^2\alpha^2(\hbar c)^2E'^2}{|qc|^4}\cos^2\frac{\theta}{2}\times\left[W_2(Q^2,\nu)+2W_1(Q^2,\nu)\tan^2\frac{\theta}{2}\right] =$$

$$= \frac{4\alpha^2E'^2}{Q^4}\times\left[W_2(Q^2,\nu)\cos^2\frac{\theta}{2}+2W_1(Q^2,\nu)\sin^2\frac{\theta}{2}\right].$$



- the inelastic cross section requires 2 final-state variables; since Q^2 and ν are L-invariant, they are more convenient;
- W_1 and W_2 are combinations of G_E and G_M for DIS [*next slide*]; sometimes a different normalization is used:

$$F_1(x,Q^2) = MW_1(Q^2,\nu);$$
$$F_2(x,Q^2) = \nu W_2(Q^2,\nu).$$

- the dynamics of the scattering depend on the structure of the target; $W_{1,2}$ $(F_{1,2})$ are the "containers" of this information;
- they are known as ***structure functions*** and must be measured (or computed in a deeper theory);
- [no deep difference $W_{1,2} \leftrightarrow F_{1,2}$; $\rightarrow$ use the most convenient, but modern papers at high $\sqrt{s}$ use only $F_{1,2}$.]

Deep inelastic scattering : $G_{E,M}$ vs $W_{1,2}$

Summary of σ's for p:

- Mott and Rosenbluth σ's;
- the relation $G_{E,M}$ vs $W_{1,2}$ and $F_{1,2}$.
- notice:

  (Q, ν, M) ~ E^1;

  (τ, $G_{E,M}$, $F_{1,2}$) ~ E^0;

  ($W_{1,2}$) ~ E^{-1};

  σ, dσ/dΩ ~ E^{-2}.
- also:

  ($G_{E,M}$, $F_{1,2}$, $W_{1,2}$) = ƒ(Q^2).

$$\left[\frac{d\sigma}{d\Omega}\right]_{\text{Mott}} = \left[\frac{4\alpha^2 E'^2}{Q^4}\right]_{\text{Rutherford}} \left[\cos^2\frac{\theta}{2}\right]_{\to\text{Mott*}} \left[\frac{E'}{E}\right]_{\to\text{Mott}} = \frac{4\alpha^2 E'^3}{EQ^4}\cos^2\frac{\theta}{2};$$

$$\left[\frac{d\sigma}{d\Omega}\right]_{\text{Rosenbluth}} = \left[\frac{4\alpha^2 E'^3}{EQ^4}\cos^2\frac{\theta}{2}\right]_{\text{Mott}} \left[\frac{G_E^2+\tau G_M^2}{1+\tau}+2\tau G_M^2\tan^2\frac{\theta}{2}\right]_{\to\text{Rosenbluth}};$$

$$\left[\frac{d^2\sigma}{d\Omega dE'}\right]_{\text{Rosenbluth}} = \frac{12\alpha^2 E'^2}{EQ^4}\left(\frac{G_E^2+\tau G_M^2}{1+\tau}\cos^2\frac{\theta}{2}+2\tau G_M^2\sin^2\frac{\theta}{2}\right);$$

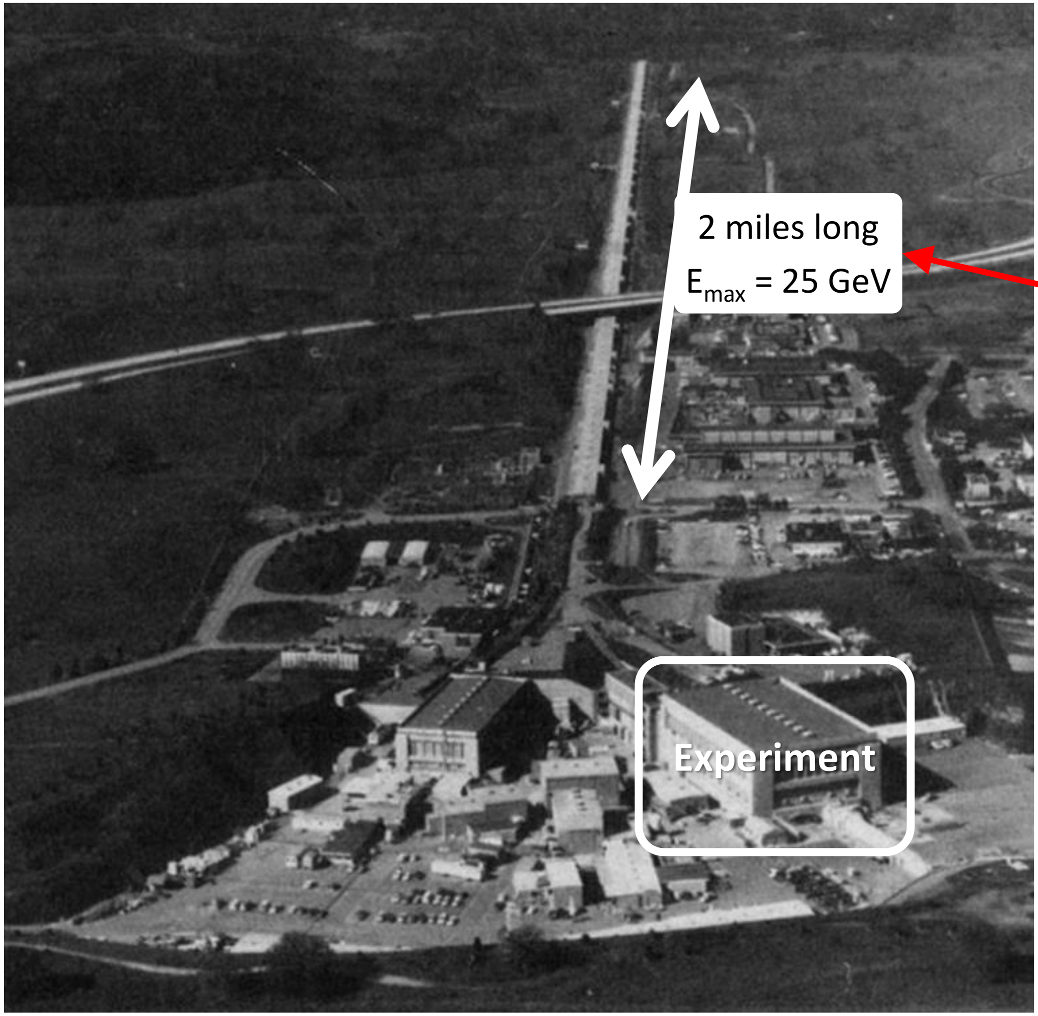
$$\left[\frac{d^2\sigma}{d\Omega dE'}\right]_{\text{DIS}} = \frac{4\alpha^2 E'^2}{Q^4}\times\left[W_2(Q^2,\nu)\cos^2\frac{\theta}{2}+2W_1(Q^2,\nu)\sin^2\frac{\theta}{2}\right];$$

$$W_1(Q^2,\nu) = \frac{F_1(x,y)}{M} = \frac{3}{E}\tau G_M^2 = \frac{3Q^2}{4EM_p^2}G_M^2;$$

$$W_2(Q^2,\nu) = \frac{F_2(x,y)}{\nu} = \frac{3}{E}\left(\frac{G_E^2+\tau G_M^2}{1+\tau}\right) = \frac{3}{E}\left(\frac{4M_p^2G_E^2+Q^2G_M^2}{4M_p^2+Q^2}\right).$$

An interesting question. Do you understand why ?	Rutherford, Mott* and Mott dσ/dΩ's do NOT depend on the proton mass. Rosenbluth dσ/dΩ depends on τ ($Q^2/4M^2$) + any hidden dependence in $G_{E,M}$. $F_{1,2}$ do *NOT* depend: *wait'n see*.

Deep inelastic scattering : SLAC

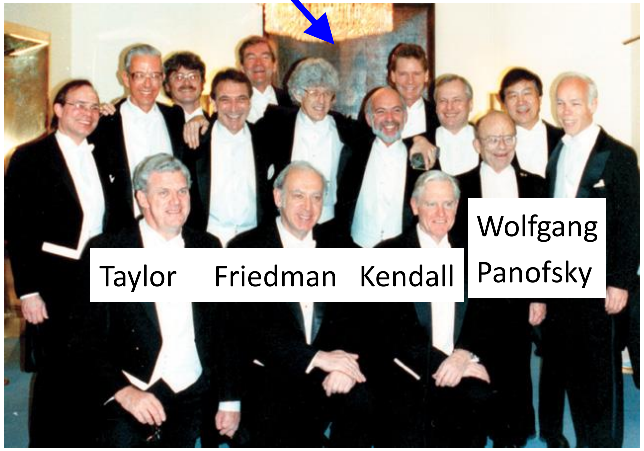


SLAC

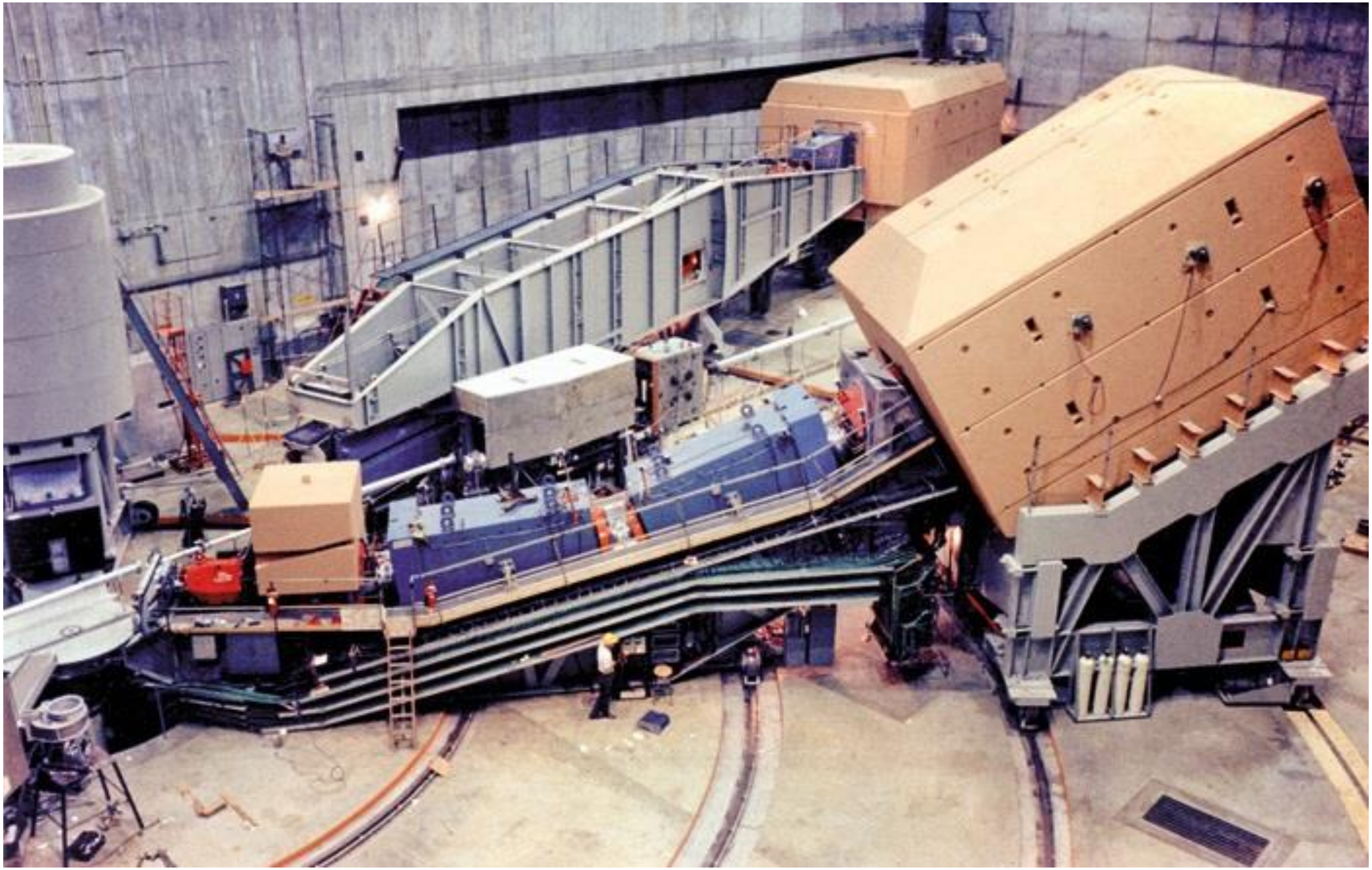
Stanford Linear Accelerator Center

the beginning of the story (1960)

… and this is NOT the end (1990)

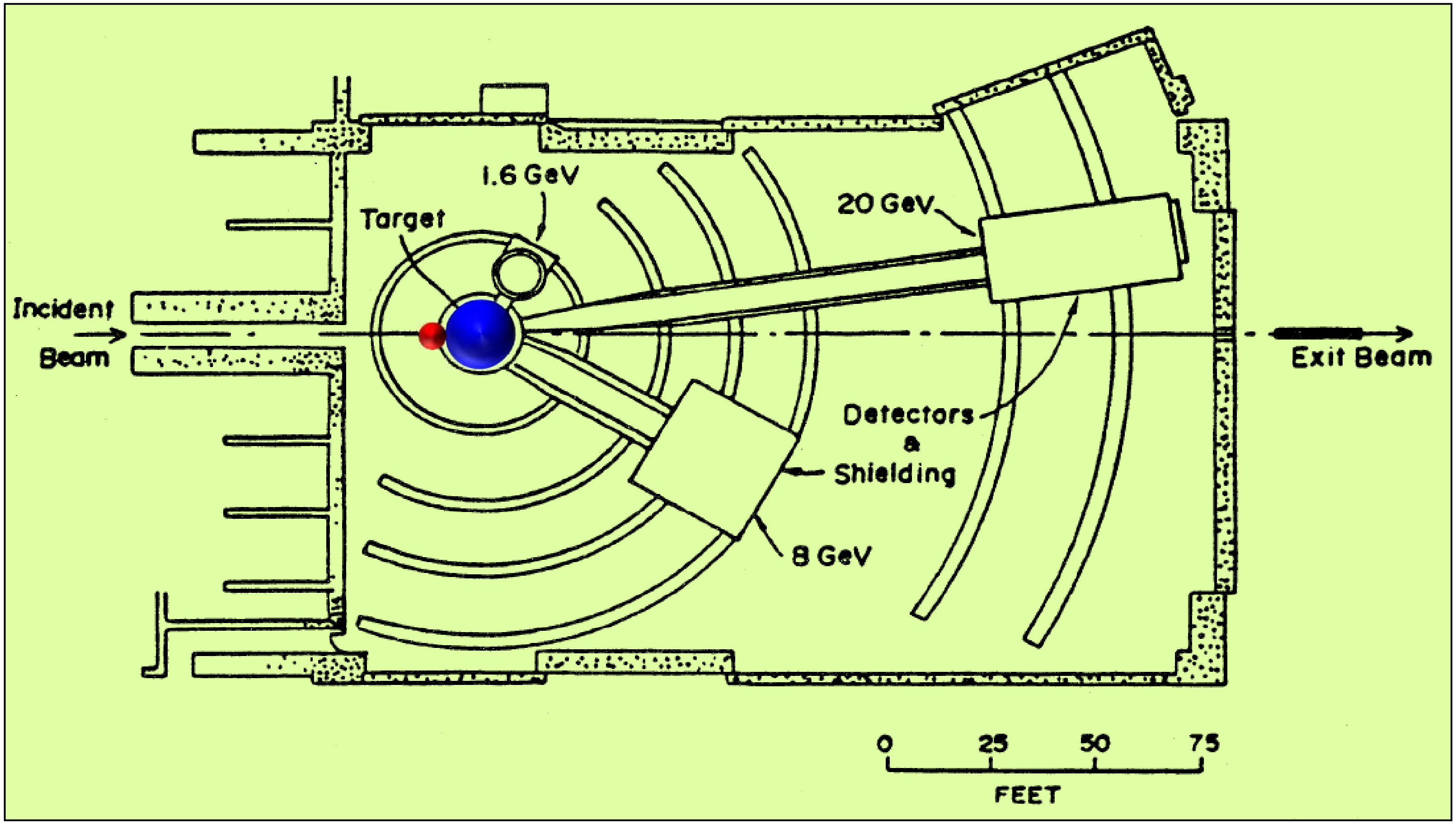


Deep inelastic scattering : SLAC experiment



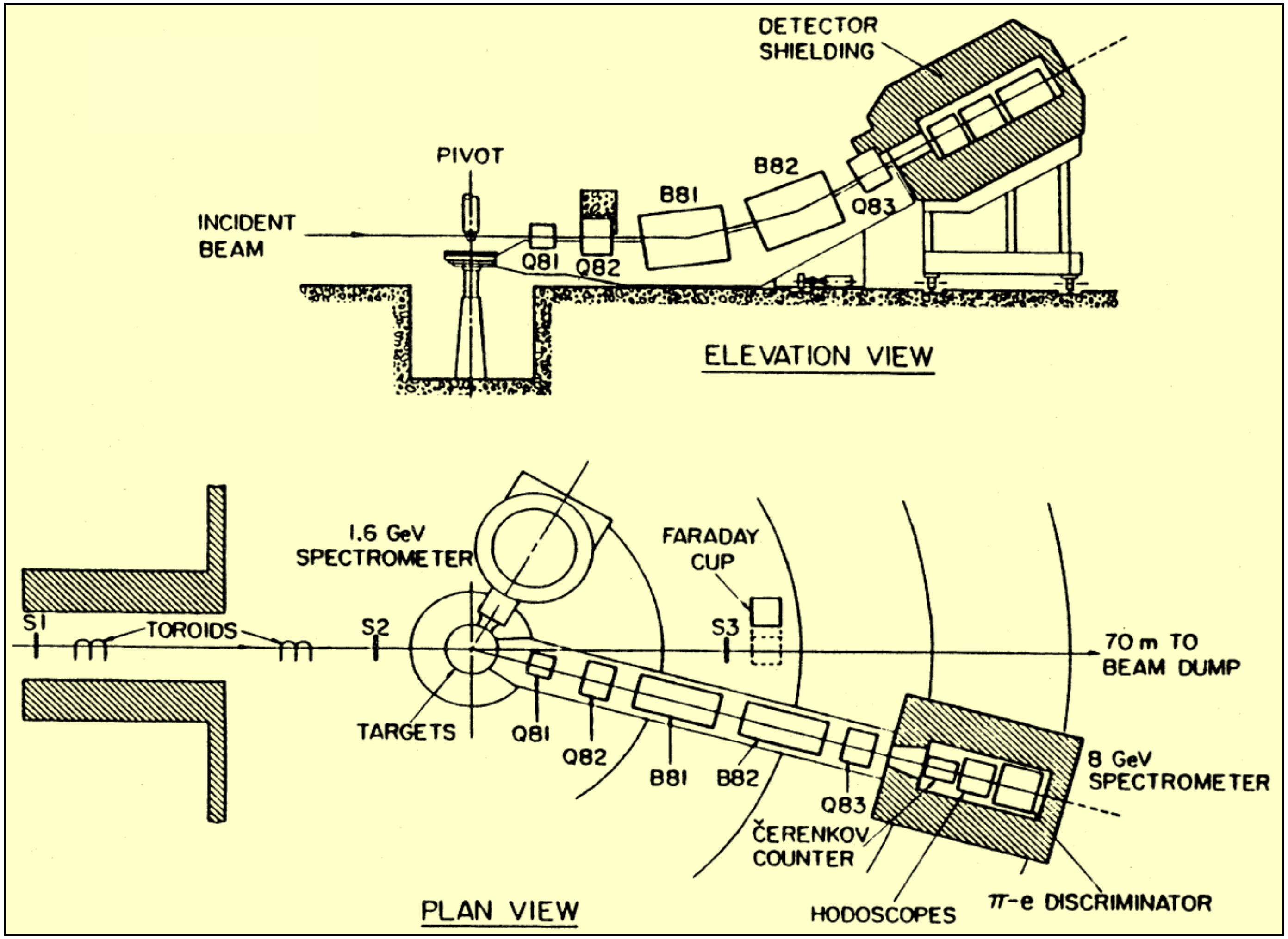
The 8 GeV spectrometer – 1968 (notice the men at the bottom)

Deep inelastic scattering : layout



Layout of the three spectrometers : they can be rotated about their pivot, as shown in the figure. [75 ft $\approx$ 23 m]

Deep inelastic scattering : layout details



Draw of the 8 GeV spectrometer [the 20 GeV is NOT shown]:

B : bending magnets (dipoles);

Q : quadrupoles;

Čerenkov counters;

scintillation hodoscopes,

shower counters for e-π discrimination;

dE/dx counters.

a big effort for physics and engineering of 50 years ago !!!
not to be compared with modern experiments …

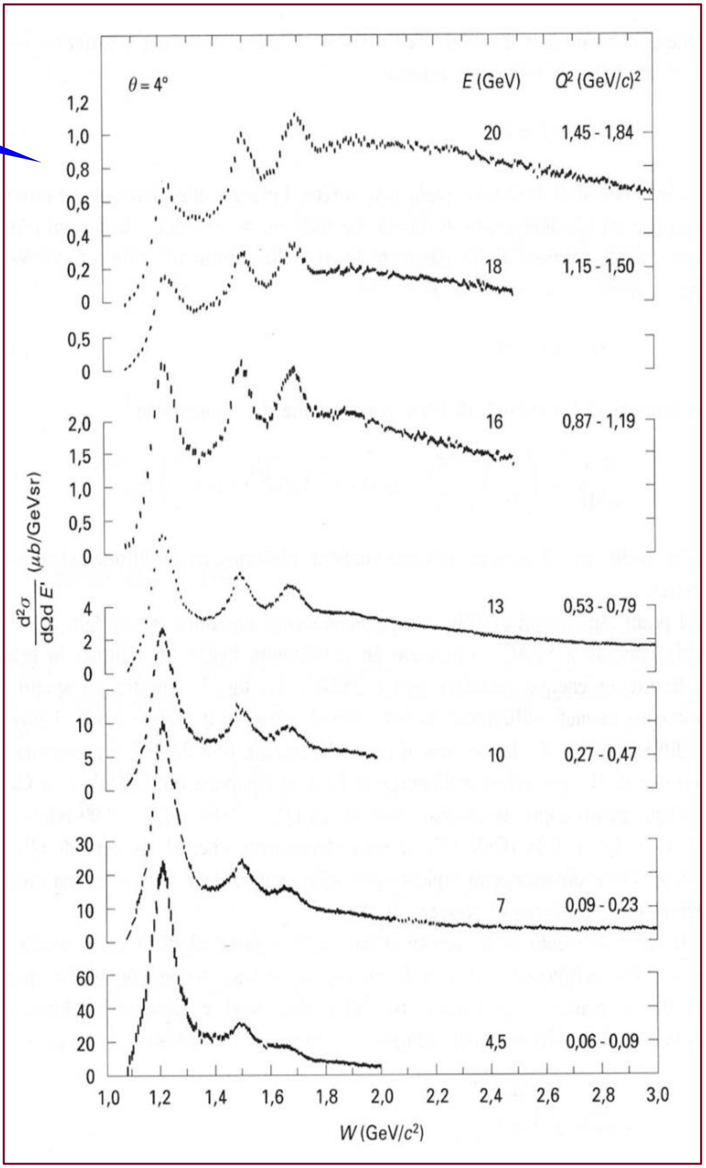
Deep inelastic scattering : $d^2\sigma/d\Omega dE'$

ep → eX, $\theta = 4°$, $d^2\sigma/d\Omega dE'$ vs W (= hadr. mass)

Notice :

- the intervals in W and Q^2, due to fixed E and θ;
- the elastic scattering ($W = M_p$) is out of scale;
- the decrease in cross section (the vertical scale) when E increases;
- the presence of excited states of the nucleon (resonances → peaks), e.g. $\Delta^+(1232)$;
- the "fading out" of resonances, when W increases at fixed E and θ;
- the continuum at high W, with ~const σ (1-2 μb / GeV sr, independent from E and Q^2).

???



Deep inelastic scattering : $d\sigma/d\theta$ vs $d\sigma/d\theta_{Mott}$

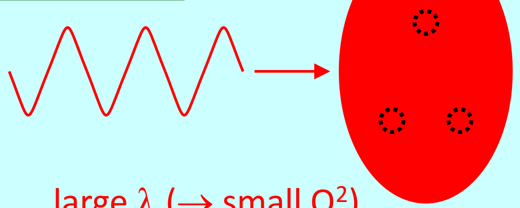
Ratio R = exp./Mott = $W_2 + 2\, W_1 \tan^2\theta/2 = R(Q^2)$.

Notice that the structure functions appear to be nearly independent of Q^2. Instead, the elastic scattering for a non-pointlike target has a strong Q^2 dependence !!!

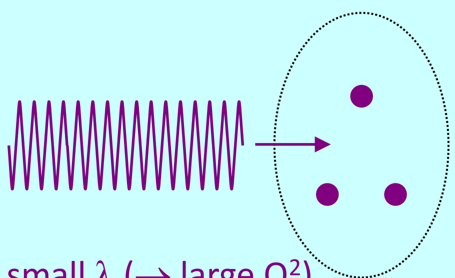
I.e., for DIS, the target (whatever it be), behaves like a point-like particle [$\mathcal{F}(Q^2)$=const] , cfr the Rutherford formula] !!! [NB constant, but << 1 $\rightarrow$ charge < 1]

This Q^2 independence is another confirmation that the DIS "breaks" the proton : the scattering happens with one of its constituents. The constituents looks "quasi-free" and "quasi-pointlike", at least at this scale of Q^2.

$\lambda \approx 2\pi/|q|$
$\approx 4\pi Mx/Q^2$



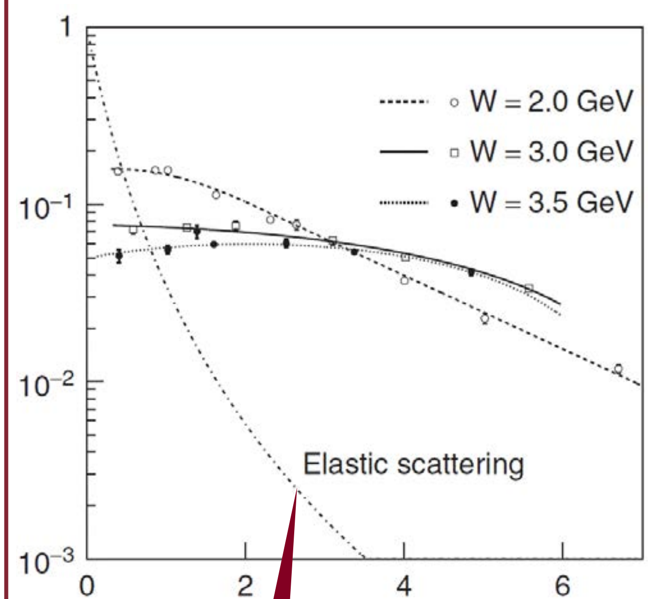
large λ ($\rightarrow$ small Q^2)
coherent scattering ep



small λ ($\rightarrow$ large Q^2)
scattering eq

M.Breidenbach et al., PRL 23, 935

$R=(d\sigma_{exp}/d\theta)/(d\sigma_{Mott*}/d\theta)|_{\theta=10°}$



Q^2 (GeV)^2

dipole form factor:
- $R(Q^2=0) = 1$;
- $R(Q^2) \propto Q^{-8}$

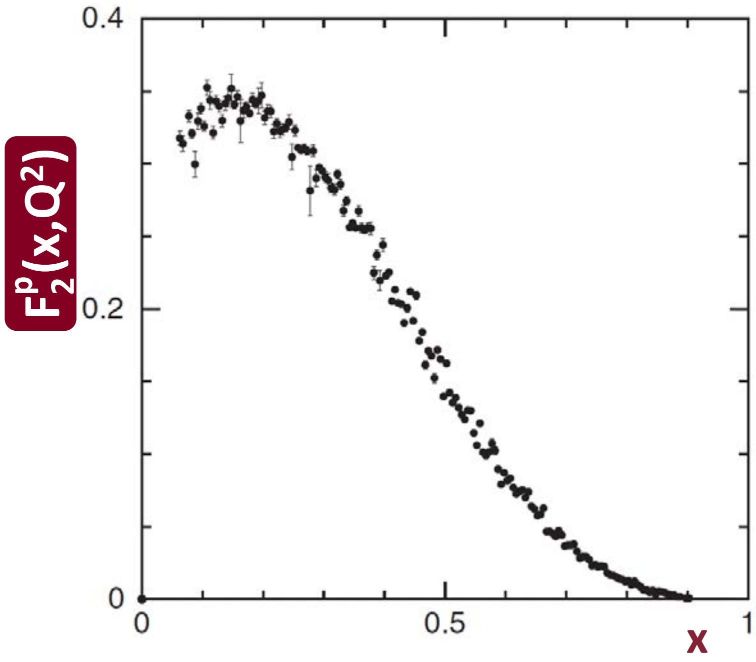
Bjorken scaling: (F_1, F_2) vs (x, Q^2)

Plot the data as F_1 and F_2 vs x and Q^2:

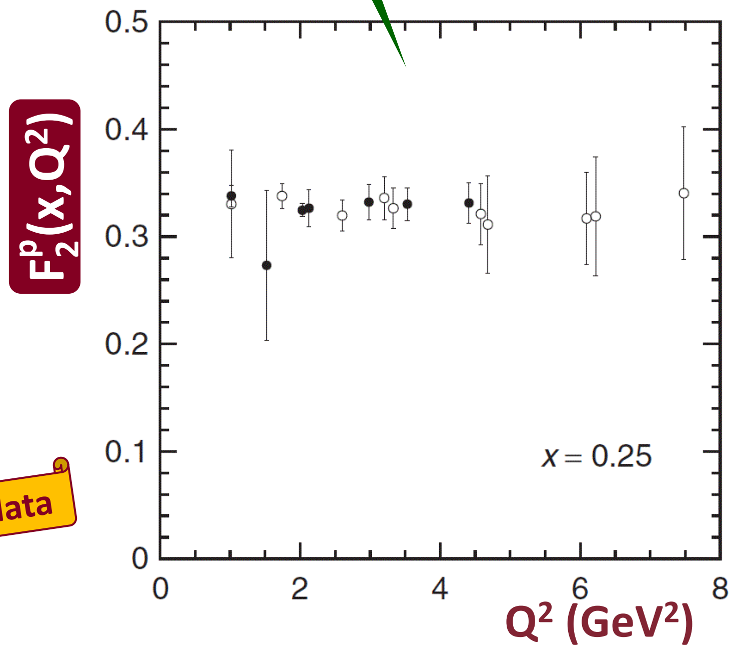
- F_2 depends on x, but NOT on Q^2;
- are F_1 and F_2 correlated ? if the nucleons are made by point-like, spin ½ objects, from the DIS formula the **Callan-Gross relation** can be derived [*next slide*] :

$$2xF_1(x) = F_2(x)$$

Seen as functions of x and Q^2, $F_{1,2}$ appear NOT to depend on Q^2 for a large range of it.



SLAC ep data



Bjorken scaling : Callan-Gross formula

a) the cross sections of pointlike spin ½ particle of mass m (à la Rosenbluth with $G_E=G_M=1$) :

$$\left[\frac{d^2\sigma}{d\Omega dE'}\right]_{\text{point-like, spin1/2}} = \frac{12\alpha^2 E'^2}{EQ^4}\left[\cos^2\frac{\theta}{2}+2\tau\sin^2\frac{\theta}{2}\right];$$

$$\left[\frac{d^2\sigma}{d\Omega dE'}\right]_{\text{DIS}} = \frac{4\alpha^2 E'^2}{Q^4}\left[W_2\cos^2\frac{\theta}{2}+2W_1\sin^2\frac{\theta}{2}\right];$$

$$W_2\cos^2\frac{\theta}{2}+2W_1\sin^2\frac{\theta}{2}=\frac{3}{E}\left[\cos^2\frac{\theta}{2}+2\tau\sin^2\frac{\theta}{2}\right];$$

$$W_1=\frac{3\tau}{E};\qquad W_2=\frac{3}{E};\qquad \frac{W_1}{W_2}=\frac{F_1(x)}{F_2(x)}\frac{\nu}{M}=\tau=\frac{Q^2}{4m^2};$$

b) from the kinematics of elastic scattering of point-like constituents of mass m :

$$Q^2=2m\nu=2M\nu x\rightarrow m=xM;$$

$$\frac{F_1(x)}{F_2(x)}=\frac{Q^2}{4m^2}\frac{M}{\nu}=\frac{2m\nu}{4m^2}\frac{M}{\nu}=\frac{M}{2m}=\frac{1}{2x};\quad\rightarrow$$

$$\boxed{2xF_1(x)=F_2(x).\ \text{Callan-Gross}}$$

Assume the nucleon (mass M, spin ½) be made of pointlike costituents q (mass m, spin ½).

Warnings :

- don't confuse the inelastic scattering ep with the elastic scattering eq;
- x refers to the inelastic case;
- an hypothetical [*nobody uses it*] variable ξ, analogous to x but for the constituent scattering; in this case, $Q^2=2m\nu\xi$, ξ = **1;**
- we learn that x = m/M [REMEMBER].

Bjorken scaling : parton model

Assume that the nucleon be made of ***partons*** (point-like, spin ½, mass m_i), which scatter elastically in the ep process.

Then the DIS cross section

$$\frac{d^2\sigma}{d\Omega dE'} = \frac{4\alpha^2 E'^2}{Q^4}\left[W_2 \cos^2\frac{\theta}{2} + 2W_1 \sin^2\frac{\theta}{2}\right];$$

reduces to an incoherent sum of constituent cross sections, $q_{electron}e_i$ being the charge of each of them :

$$\left.\frac{d^2\sigma}{d\Omega dE'}\right|_{m_i} = \frac{4\alpha^2 E'^2}{Q^4}\sum_i \left[e_i^2 \left(\cos^2\frac{\theta}{2} + \frac{Q^2}{2m_i^2}\sin^2\frac{\theta}{2}\right) \delta\left(\nu - \frac{Q^2}{2m_i}\right)\right];$$

where the $\delta()$ means that, at the constituent level, the scattering is elastic, i.e. $Q^2 = 2m_i\nu$.

For such partons [*next 2 slides*]:

$$\begin{cases} F_1\left[x = \dfrac{Q^2}{2m\nu}\right] = MW_1(Q^2,\nu) = \dfrac{1}{2}\sum_j e_j^2 f_j(x) \\ F_2\left[x = \dfrac{Q^2}{2m\nu}\right] = \nu W_2(Q^2,\nu) = x\sum_j e_j^2 f_j(x) \end{cases}$$

i.e. F_1 and F_2 do NOT depend on Q^2 and ν separately, but only on their ratio. F_1 and F_2 are also related by the Callan-Gross equation.

This mechanism (the **Bjorken scaling**) was interpreted by Feynman in 1969 as the dominance of partons in the nucleon dynamics (the **parton model**).

Richard Feynman James Bjorken

Bjorken scaling : $\sigma_{DIS} \rightarrow W_{1,2}$

[BJ, 446-460]

2

if $B(x=x_0)=0 \rightarrow$

$$\int A(x)\delta[B(x)]dx = A(x_0)/|B'(x_0)|,$$

$$B(x)=\nu-\frac{Q^2}{2Mx} \rightarrow x_0=\frac{Q^2}{2M\nu};$$

$$\rightarrow B'(x_0)=\left.\frac{Q^2}{2Mx^2}\right|_{x=x_0}=\frac{2M\nu^2}{Q^2}.$$

3

DIS formula for ep, p NOT pointlike, mass=M:

$$\left[\frac{d^2\sigma}{d\Omega dE'}\right]_{DIS}=\frac{4\alpha^2E'^2}{Q^4}\left[W_2(Q^2,\nu)\cos^2\left(\frac{\theta}{2}\right)+2W_1(Q^2,\nu)\sin^2\left(\frac{\theta}{2}\right)\right]$$

Elastic scattering e"q", pointlike, spin ½, charge e, mass m=Mx:

$$\left[\frac{d^2\sigma}{d\Omega dE'}\right]_{e"q"}=\frac{4\alpha^2E'^2}{Q^4}\left[e^2\cos^2\left(\frac{\theta}{2}\right)+e^2\frac{Q^2}{2m^2}\sin^2\left(\frac{\theta}{2}\right)\right]\delta\left(\nu-\frac{Q^2}{2m}\right)$$

4

$$\left.W_1\right|_x=\frac{e^2Q^2}{4m^2}\delta\left(\nu-\frac{Q^2}{2m}\right)=\frac{e^2Q^2}{4M^2x^2}\delta\left(\nu-\frac{Q^2}{2Mx}\right);$$ [at fixed x]

f(x) : x-distribution of (a single) substructure;

$$W_1=\int\frac{e^2Q^2}{4M^2x^2}\delta\left(\nu-\frac{Q^2}{2Mx}\right)f(x)dx=\frac{e^2Q^2}{4M^2}\int\frac{f(x)dx}{x^2}\delta\left(\nu-\frac{Q^2}{2Mx}\right)=$$

$$=\frac{e^2Q^2}{4M^2}\left.f(x)\right|_{x=\frac{Q^2}{2M\nu}}\left(\frac{Q^2}{2M\nu}\right)^{-2}\frac{Q^2}{2M\nu^2}=$$

$$=e^2f(x)\left(2^{-2+2-1}\right)\left(M^{-2+2-1}\right)\left(Q^{2-4+2}\right)\left(\nu^{2-2}\right)=\frac{e^2f(x)}{2M}.$$

1

here ONLY ONE parton "q", with m, e, x=m/M.

5

[similarly:]

$$\left.W_2\right|_x=e^2\delta\left(\nu-\frac{Q^2}{2Mx}\right);$$

$$W_2=\int e^2\delta\left(\nu-\frac{Q^2}{2Mx}\right)f(x)dx=$$

$$=e^2\left.f(x)\right|_{x=\frac{Q^2}{2M\nu}}\frac{Q^2}{2M\nu^2}=\frac{e^2xf(x)}{\nu}.$$

Bjorken scaling : $W_{1,2} \to F_{1,2}$

previous page

$$\left[\frac{d^2\sigma}{d\Omega dE'}\right]_{DIS} = \frac{4\alpha^2 E'^2}{Q^4}\left[W_2(Q^2,\nu)\cos^2\left(\frac{\theta}{2}\right) + 2W_1(Q^2,\nu)\sin^2\left(\frac{\theta}{2}\right)\right];$$

$$\left[\frac{d^2\sigma}{d\Omega dE'}\right]_{e"q"} = \frac{4\alpha^2 E'^2}{Q^4}\left[e^2\cos^2\left(\frac{\theta}{2}\right) + e^2\frac{Q^2}{2m^2}\sin^2\left(\frac{\theta}{2}\right)\right]\delta\left(\nu - \frac{Q^2}{2m}\right);$$

a single substructure $\{e, m=Mx\} \to W_1 = \frac{e^2 f(x)}{2M}; \quad W_2 = \frac{e^2 x f(x)}{\nu}.$

this form ("Σ...") is actually very important (why ?)

Many sub-structures: for each $\{e, x, f(x)\} \to \{e_j, x_j, f_j(x)\}$:

$$W_1 = \frac{e^2 f(x)}{2M} \quad \to W_1 = \sum_j \frac{e_j^2 f_j(x)}{2M} \quad \to \boxed{MW_1 = F_1(x) = \frac{1}{2}\sum_j e_j^2 f_j(x)};$$

$$W_2 = \frac{e^2 x f(x)}{\nu} \quad \to W_2 = \sum_j \frac{e_j^2 x f_j(x)}{\nu} \quad \to \boxed{\nu W_2 = F_2(x) = x\sum_j e_j^2 f_j(x)};$$

$$\to \boxed{\text{Callan} - \text{Gross} \ : \ 2xF_1(x) = F_2(x)}.$$

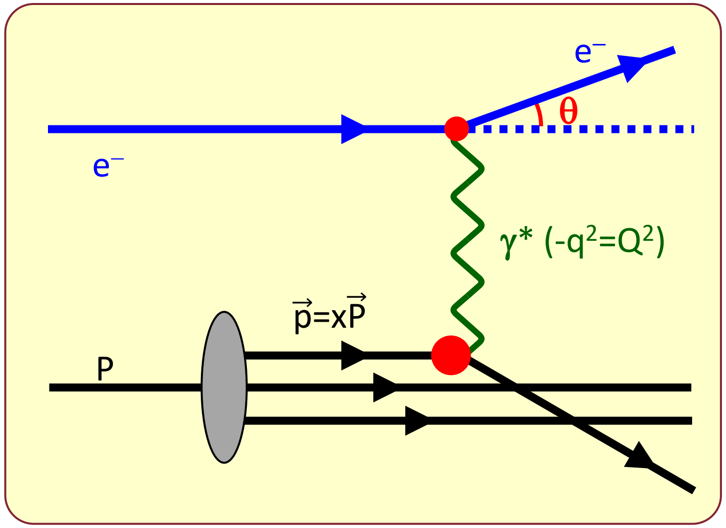
The parton model

Summary: the nucleons are made up of partons, later identified with quarks; partons are:

- point-like (at least at the scale of Q^2 accessible to the experiments, *both then and now*);
- spin ½ fermions;
- define : $x_{Feynman} = x_F = |\vec{p}_{parton}| / |\vec{p}_{nucleon}| =$

  $\approx |p^{long}_{parton}| / |\vec{p}_{nucleon}|$

  [cfr. $x_{Bjorken} = x_B = Q^2/(2M\nu) = m/M$];
- the interaction e-parton is so fast and violent, that they behave like free particles (similar, *mutatis mutandis*, to the collision approximation in classical mechanics);
- the other partons [*at least in 1st approx.*] do NOT take part in the interaction ("*spectators*");
- it follows $x_F = x_B = x$ [*next slide*];
- the DIS is an incoherent sum of processes on the partons; at high Q^2 the nucleons as such are mere containers, with no role [$F_{1,2} = \Sigma \ldots$].



Despite the formal identity between x_F and x_B, they have a different dynamical origin :

- x_F is defined in the hadronic system (= fraction of the nucleon momentum);
- x_B comes from the lepton part (momentum transfer and lepton energies).

The parton model : $x_F \leftrightarrow x_B$

Show : $x_{Feynman} \equiv x_F = x_{Bjorken} \equiv x_B$

In the "infinite momentum frame" (IMF), where all the masses are negligible :

$$p^{init}_{nucleon}\Big|_{IMF} = (p,p,0,0);$$

$$p^{init}_{parton}\Big|_{IMF} = x_F p^{init}_{nucleon} = (x_F p, x_F p, \sim 0, \sim 0);$$

$$p^{fin}_{parton}\Big|_{IMF} = p^{init}_{parton} + q_{transf};$$

$$\left(p^{fin}_{parton}\right)^2 = 0 = \left(p^{init}_{parton} + q_{transf}\right)^2 =$$

$$= 0 + q^2_{transf} + 2\left(p^{init}_{parton} \cdot q_{transf}\right);$$

$(p^{init}_{parton} \cdot q_{transf})$ is L-invariant; compute it in the lab frame:

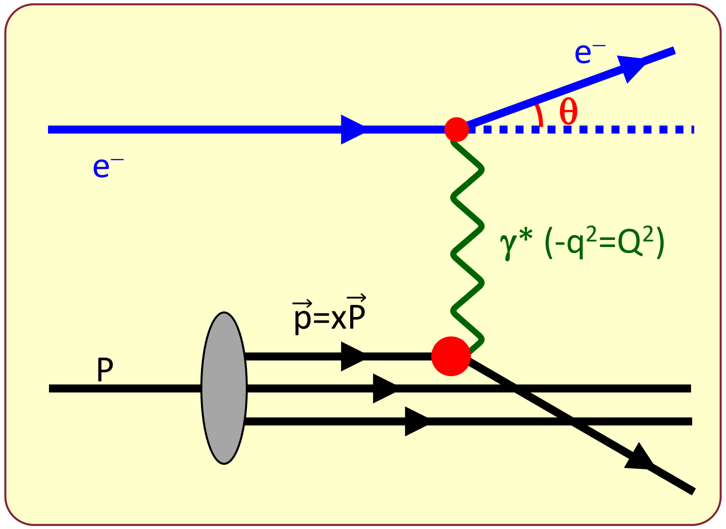
$$p^{init}_{proton}\Big|_{LAB} = (M, \vec{0}); \quad p^{init}_{parton}\Big|_{LAB} = (Mx_F, \vec{0});$$

$$q_{transf}\Big|_{LAB} = (E - E' = \nu, \vec{q});$$

$$-q^2_{transf} = Q^2 = 2\left(p^{init}_{parton} \cdot q_{transf}\right) = 2Mx_F\nu \rightarrow$$

$$x_F = Q^2 / (2M\nu) \equiv x_B.$$

Warning : the equality holds only in the IMF. It is also a reasonable approx. in the "ultra-relativistic" case, when the masses are negligible wrt momenta.



In the following (also next chapters):

- drop the subscript $x_F = x_B = x$;
- usually interpret x à la Feynman, as the fraction of the nucleon 4-mom. carried by the parton.

The parton model : sum rules

Remarks and comments (discuss the proton, the neutron is similar):

- experimentally, it is enough to control the initial state (E_{e-}, M) + measure the leptonic final state (E', θ);
- the model implies that $\sum_i x_i = 1$, when the sum runs over ALL the partons;
- at the time there was no clue about the nature of the partons, nor if they are charged or neutral (i.e. not interacting with the electrons); therefore:

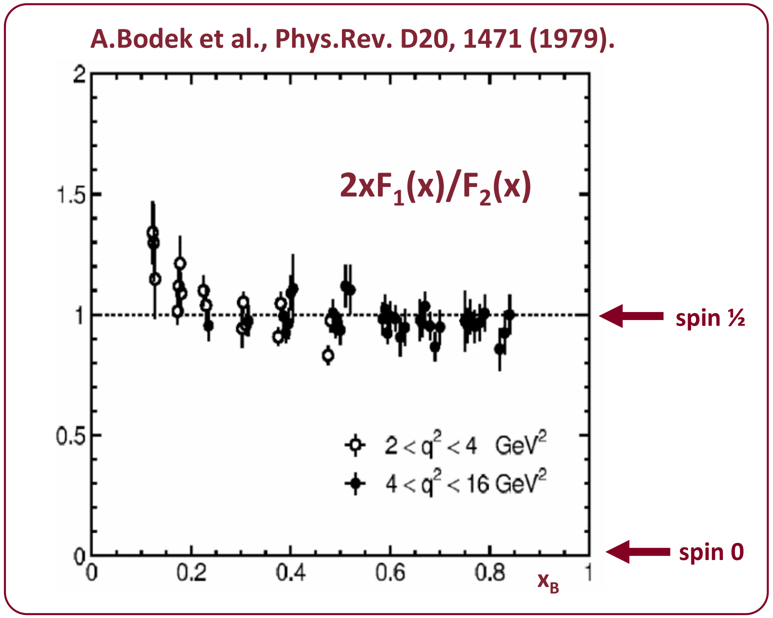
  $$\boxed{\sum{}'_i \, x_i \le 1}$$

  (the sum is only over those partons, which interact with the electron);
- given the intrinsic q.m. structure of the nucleon, the values x_i are not fixed, but described by a distribution $f_j^p(x)$ for partons of type "j" in the proton:

$$\rightarrow \sum_j \int dx \, [x f_j^p(x)] = \sum{}'_j < |\vec{p}_j| > / \, |\vec{p}_p| \le 1,$$

with the same caveats over the sum.

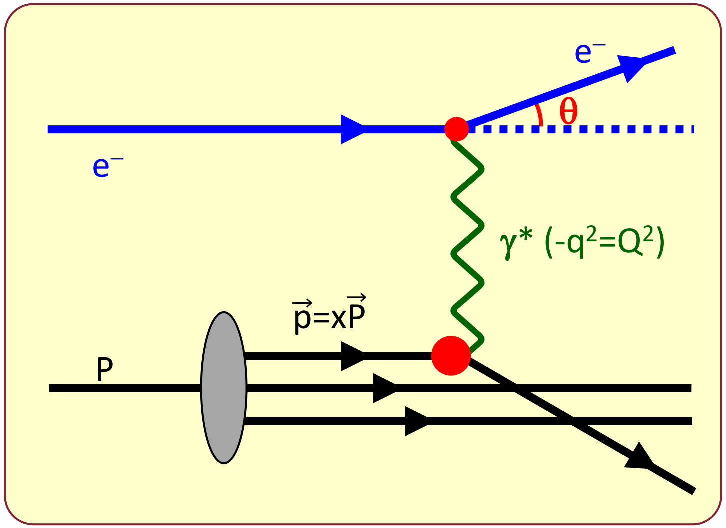
- if partons are spin ½, then the Callan-Gross relation $\boxed{2xF_1(x) = F_2(x)}$ holds;
- instead, spin = 0 $\rightarrow \tau = 0 \rightarrow F_1(x) = 0$;
- but … can we measure it ? YES, it's OK !!!



The parton model : summary

A summary of the model, with final formulæ [*box and next slide*]:

- at high Q^2, a hadron (p/n) behaves as a mixture of small components, the partons.
- partons are pointlike, spin ½;
- each parton in each interaction is described by its fraction of the 4-momentum of the hadron, i.e. $|\vec{p}_i^{\,parton}| \,/\, |\vec{p}^{\,hadron}| = x_i$;
- the x_i are qm variables, described by their distribution functions $f_i^p(x)$ [called "***PDF***"];
- in principle the PDF are different for each parton and each hadron;
- $\sum_j \int dx\, x\, f_j^p(x) \leq 1$;
- parton spin = ½ $\rightarrow$ Callan-Gross $2xF_1(x) = F_2(x)$.



$$\frac{d^2\sigma}{d\Omega dE'} = \frac{4\alpha^2 E'^2}{Q^4}\left[W_2\left(Q^2,\nu\right)\cos^2\frac{\theta}{2} + 2W_1\left(Q^2,\nu\right)\sin^2\frac{\theta}{2}\right];$$

$$\frac{d^2\sigma}{dxdy} = \frac{4\pi\alpha^2 s}{Q^4}\left[xy^2F_1\left(x,\cancel{Q^2}\right) + (1-y)F_2\left(x,\cancel{Q^2}\right)\right];$$

$$F_1\left(x,\cancel{Q^2}\right) = MW_1(Q^2,\nu) = \frac{1}{2}\sum_j e_j^2 f_j(x);$$

$$F_2\left(x,\cancel{Q^2}\right) = \nu W_2(Q^2,\nu) = x\sum_j e_j^2 f_j(x).$$

next slide

The parton model : $d^2\sigma/dxdy$

$$s = 2EM; \quad \nu = E - E'; \quad y = \frac{\nu}{E} = 1 - \frac{E'}{E}; \quad E' = E(1-y); \quad Q^2 = 4EE'\sin^2\left(\frac{\theta}{2}\right) = 4E^2(1-y)\sin^2\frac{\theta}{2};$$

$$x = \frac{Q^2}{2M\nu} = 4E^2(1-y)\sin^2\left(\frac{\theta}{2}\right)\frac{1}{2MEy} = \frac{2E(1-y)}{My}\sin^2\left(\frac{\theta}{2}\right);$$

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{Mxy}{2E(1-y)}; \quad \cos^2\left(\frac{\theta}{2}\right) = 1 - \sin^2\left(\frac{\theta}{2}\right) \approx 1.$$

$$\frac{d^2\sigma}{d\Omega dE'} = \frac{4\alpha^2 E'^2}{Q^4}\left(W_2\cos^2\frac{\theta}{2} + 2W_1\sin^2\frac{\theta}{2}\right).$$

$$\frac{\partial x}{\partial\cos\theta} = \frac{\partial x}{\partial\sin^2(\theta/2)}\frac{\partial\sin^2(\theta/2)}{\partial\cos\theta} = \frac{2E}{-2M}\frac{1-y}{y} = -\frac{E}{M}\frac{1-y}{y}; \quad \frac{\partial y}{\partial E'} = -\frac{1}{E};$$

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial\cos\theta} & \dfrac{\partial x}{\partial E'} \\ \cancel{\dfrac{\partial y}{\partial\cos\theta}} & \dfrac{\partial y}{\partial E'} \end{vmatrix} = \frac{1-y}{My}.$$

$$\frac{\partial}{\partial\cos\theta}\left(\sin^2\frac{\theta}{2}\right) = \frac{\partial}{\partial\cos\theta}\left(\frac{1-\cos\theta}{2}\right) = -\frac{1}{2}$$

Jacobian
$\cos\theta, E'$
$\rightarrow x, y$

[YN1], probl. 17.7 :
page 697, 698, 911.

L-inv : s, M, ν, x, y, Q^2.
Labo : E, E', θ, Ω.

link

result

$$\frac{d^2\sigma}{dxdy} = \frac{2\pi}{|J|}\frac{d^2\sigma}{d\cos\theta dE'} = \frac{2\pi My}{1-y}\frac{4\alpha^2E^2(1-y)^2}{Q^4}\times\left[\frac{F_2(x,y)}{\nu}\cos^2\left(\frac{\theta}{2}\right) + \frac{2F_1(x,y)}{M}\frac{Mxy}{2E(1-y)}\right] =$$

$$= \frac{s\pi y 4\alpha^2 \cancel{E}(1-y)}{Q^4}\left[\frac{F_2(x,y)}{\cancel{E}y} + \cancel{2}F_1(x,y)\frac{xy}{\cancel{2}\cancel{E}(1-y)}\right] = \frac{4\pi\alpha^2}{Q^4}s\left[(1-y)F_2(x,y) + xy^2F_1(x,y)\right].$$

The quark-parton model

partons = quarks ???

In the SM the answer is YES: the quark-parton model.

Which is the dynamical meaning of $F_{1,2}$? Can we measure them ? [*yes, of course*]

- in principle the proton and the neutron have different structure functions;
- also a given process could result in a different structure [*e.g. the electron scattering could "see" different* $F_{1,2}$ *from neutrino- or hadron-hadron interactions*];
- in this picture, e.g. we will refer to "$F_1^{ep}(x)$", meaning $F_1(x)$ for the proton, when probed in DIS by an electron;
- similarly "$F_2^{ep}(x)$", "$F_2^{en}(x)$", "$F_2^{\nu p}(x)$", …
- however, these functions are NOT independent : if they parametrize the actual structure of nucleons, they must be correlated.

- assume that the nucleons are made by three quarks [*Nature is much more complicated, but wait …*];
- call them "**valence quarks**" [*why ???*];
- each of them is described by a x distribution, identified with "$f_j^p(x)$" [e.g. "$u^p(x)$" = the x distribution for u-quarks in the proton];
- e.g. $u^p(x)dx$ = number of u quarks in the proton, with x in the interval (x, x+dx);
- then $d^p(x)$, $\bar{u}^p(x)$, $\bar{u}^{\bar{p}}(x)$, $u^n(x)$, $\bar{u}^{\bar{n}}(x)$,…;

→ the $q^N(x)$ [q=u,d,ū,…; N=p,n], the PDF (parton distribution functions), tell the structure of nucleons at high Q^2.

(continue …)

The q-p model: u^p, u^n, d^p, d^n, …

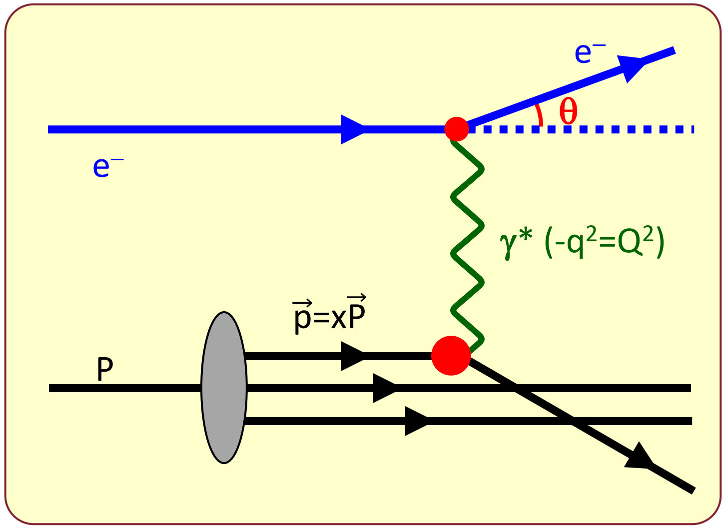
(… *continue*)

Some obvious relations hold [*the green ones with a* (*) *are provisional, we'll modify them*] :

- particle-antiparticle symmetry : $u^p(x) = \bar{u}^{\bar{p}}(x)$;
- quark model + isospin invariance : $u^p(x) \approx d^n(x)$;
- ditto : $u^p(x) \approx 2\, u^n(x)$;
- ditto : $d^n(x) \approx 2\, d^p(x)$;
- (*) for valence quarks only, $\bar{u}^p(x) = 0$;
- (*) for valence quarks only, $s^p(x) = 0$;
- (*) therefore, e.g.

$$F_2^{ep}(x) = x \sum_j e_j^2 f_j(x) = x \left(\frac{4u^p(x) + d^p(x)}{9} \right);$$

… many more formulæ, all quite intuitive.



see also §7 : Structure functions in ν DIS.

The q-p model : valence and sea

- According to the uncertainty principle, for short intervals q.m. allows quark-antiquark pairs to exist in the nucleons;
- in the hadrons some neutral particles exist, called gluons [*??? ... wait*].

Therefore, let us modify the scheme:

- in the nucleons, 3 types of particles :
  - **valence quarks** [*already seen*] with distribution $q_V(x)$ *[e.g* $u_V^p(x)$ *[already defined with the simpler notation* $u^p(x)$*]*;
  - **sea quarks**, i.e. the quark-antiquark pairs, described by distributions $q_S(x)$ [e.g $u_S^p(x)$, $s_S^p(x)$, $\bar{u}_S^p(x)$, $\bar{s}_S^p(x)$];
  - **gluons**, described by the distributions $g^p(x)$ and $g^n(x)$.

Obviously only sums can be measured:

$$u^p(x) \equiv u_V^p(x) + u_S^p(x);$$
$$d^p(x) \equiv d_V^p(x) + d_S^p(x);$$
$$\bar{u}^p(x) \equiv \bar{u}_V^p(x) + \bar{u}_S^p(x) = \bar{u}_S^p(x);$$
$$s^p(x) \equiv s_V^p(x) + s_S^p(x) = s_S^p(x);$$

Relations (*final, no further refinement*) :

- particle-antiparticle constraint :
$$u^p(x) = \bar{u}^{\bar{p}}(x);$$
- from quark model + isospin invariance :
$$u_V^p(x) \approx d_V^n(x) \equiv u_V(x);$$
$$d_V^p(x) \approx u_V^n(x) \equiv d_V(x);$$
- from quark model : $u_V^p(x) \approx 2\, u_V^n(x)$;
- from quark model : $d_V^n(x) \approx 2\, d_V^p(x)$;
- from quantum mechanics and isospin invariance [*and neglecting quark masses*] :
$$u_S^p(x) = \bar{u}_S^p(x) \approx d_S^p(x) = \bar{d}_S^p(x) \approx$$
$$\approx s_S^p(x) = \bar{s}_S^p(x) \equiv q_S^p(x) \approx q_S^n(x);$$
- ... many more, all quite intuitive.

the "valence-ness" is not an observable, i.e. a u-quark "does not know" whether (s)he is v or s.

The q-p model : $F^{proton}(x)$ vs $F^{neutron}(x)$

Putting everything together, we have [neglecting heavier quarks] :

$$F_2^{ep}(x) = x\left\{\frac{4}{9}\left[u^p(x)+\bar{u}^p(x)\right]+\frac{1}{9}\left[d^p(x)+\bar{d}^p(x)\right]+\frac{1}{9}\left[s^p(x)+\bar{s}^p(x)\right]\right\} =$$

$$= x\left\{\frac{4}{9}\left[u_V(x)+2q_S(x)\right]+\frac{1}{9}\left[d_V(x)+2q_S(x)\right]+\frac{1}{9}\left[2q_S(x)\right]\right\} =$$

drop the "p"

$$= x\left\{\frac{4}{9}u_V(x)+\frac{1}{9}d_V(x)+\frac{4}{3}q_S(x)\right\};$$

use iso-spin

$$F_2^{en}(x) = x\left\{\frac{1}{9}u_V(x)+\frac{4}{9}d_V(x)+\frac{4}{3}q_S(x)\right\};$$

i.e. $u_V = u_V^p \approx d_V^n$

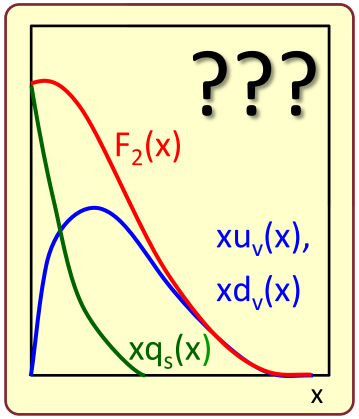
$$F_2^{en}/F_2^{ep} = R_{np} = \begin{cases} 1 & \text{(a);} \\ \left[4d_V(x)+u_V(x)\right]/\left[4u_V(x)+d_V(x)\right] & \text{(b).} \end{cases}$$

(a) if sea dominates (see little sketch);

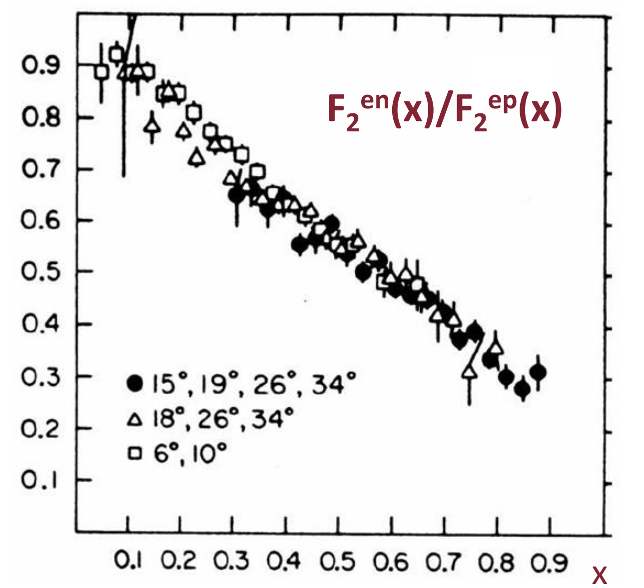
(b) if valence dominates [if (u_V>>d_V) → $R_{np} \approx$ ¼].

The measurement shows that case (a) happens at low x, while (b) dominates at high x.

In other words, there are plenty of $q\bar{q}$ pairs at small momentum, while valence is important at high x….



A.Bodek et al., PL 51B (1974) 417.



The q-p model : toy models for $F_2(x)$

$$\begin{cases} \sum_{\text{partons}} \int_0^1 x f_j(x)\,dx < 1; \\ \sum_{\text{partons}} \int_0^1 f_j(x)\,dx = \text{undefined (but large).} \end{cases}$$

Sum rules (from momentum conservation) :

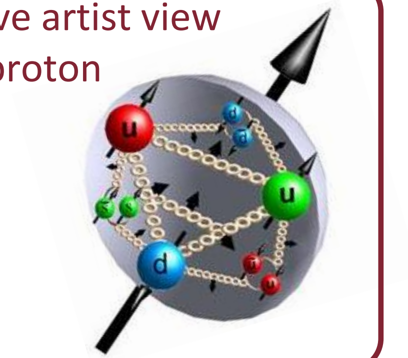
$$\int_0^1 dx\left[u^p(x) - \bar{u}^p(x)\right] = \int_0^1 dx\, u_V^p(x) = 2;$$

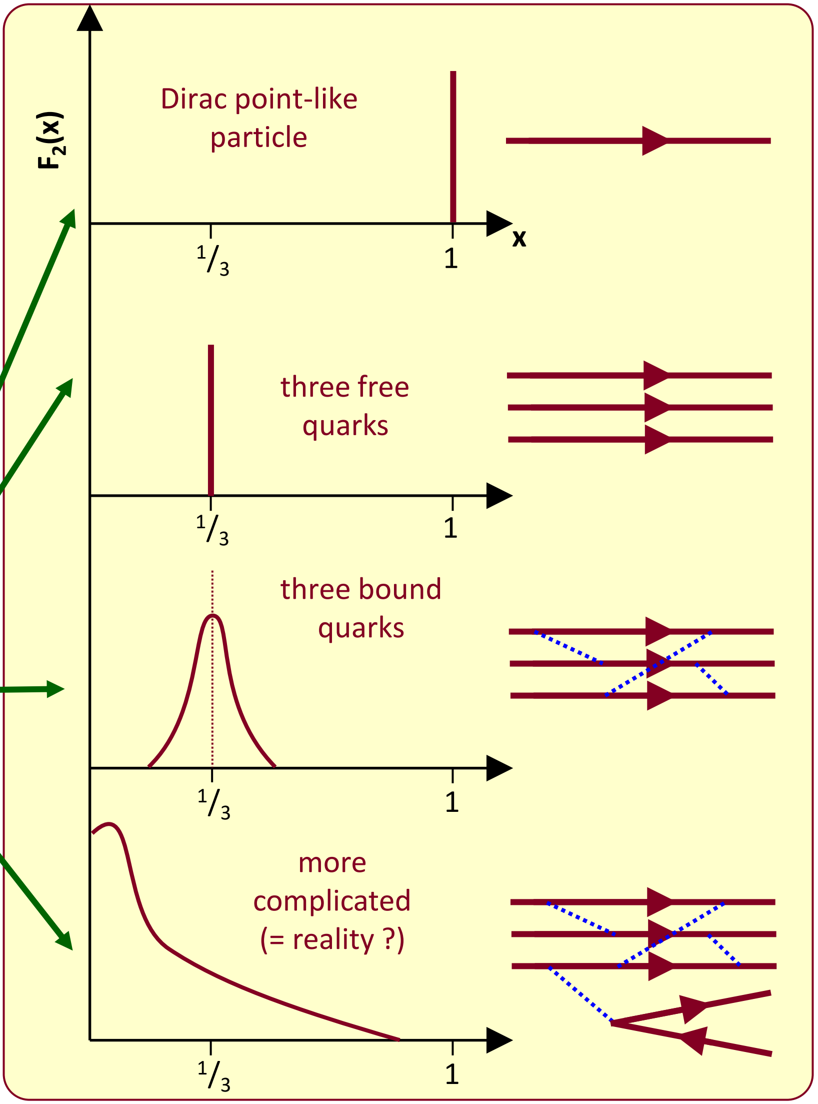
$$\int_0^1 dx\left[d^p(x) - \bar{d}^p(x)\right] = \int_0^1 dx\, d_V^p(x) = 1;$$

$$\int_0^1 dx\left[s^p(x) - \bar{s}^p(x)\right] = 0.$$

Hypothetical (**NOT CORRECT**) shapes of $F_2(x)$ from naïve dynamical models :

a naive artist view of a proton





The q-p model : $F_2^{ep}(x) - F_2^{en}(x)$

From :

$F_2^{ep}(x) = x\,[4u_V(x) + \;\; d_V(x) + 12\,q_S(x)]\,/\,9;$

$F_2^{en}(x) = x\,[\;\; u_V(x) + 4d_V(x) + 12\,q_S(x)]\,/\,9;$

we get

$F_2^{ep}(x) - F_2^{en}(x) = x\,[u_V(x) - d_V(x)]\,/\,3;$

If, moreover, from the naïve quark model

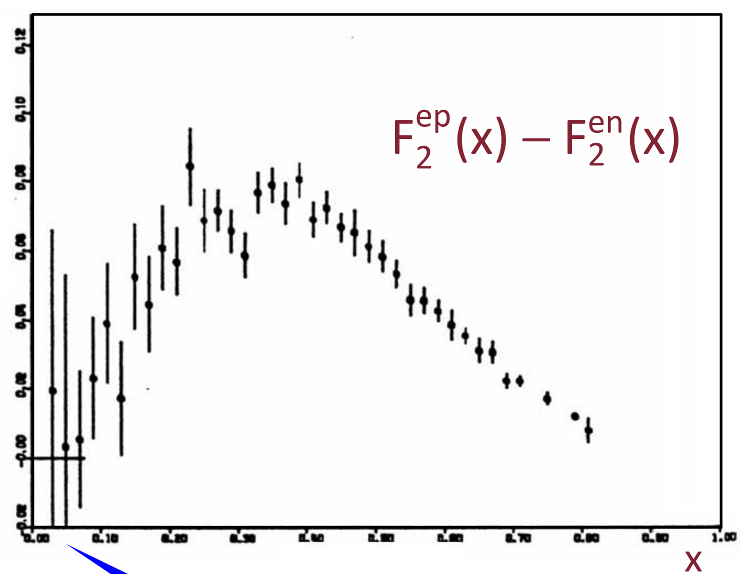
$u_V(x) \approx 2\,d_V(x)$

we get

$F_2^{ep}(x) - F_2^{en}(x) = x\,d_V(x)\,/\,3;$

i.e. this difference, which is an observable, roughly corresponds to ⅓x × [the x-distribution of the "lone" valence quark (d_V^p or u_V^n)].

Friedman, Kendall - Ann.Rev.Nucl.Sci. 22, 203 (1972)



no valence at x=0

(!!!)

The q-p model : the gluon

The integrals of $F_2(x)$ are both calculable and measurable. By neglecting the small contribution of $s\bar{s}$:

$$\int_0^1 dxF_2^{ep}(x) = \frac{4}{9}\int_0^1 x\left[u^p(x)+\bar{u}^p(x)\right]dx + \frac{1}{9}\int_0^1 x\left[d^p(x)+\bar{d}^p(x)\right]dx = \frac{4}{9}f_u + \frac{1}{9}f_d;$$

$$\int_0^1 dxF_2^{en}(x) = \frac{4}{9}\int_0^1 x\left[d^p(x)+\bar{d}^p(x)\right]dx + \frac{1}{9}\int_0^1 x\left[u^p(x)+\bar{u}^p(x)\right]dx = \frac{4}{9}f_d + \frac{1}{9}f_u;$$

where $f_{u,d}$ are the fractions of the proton momentum carried by the quark u,d (+ the respective $\bar{q}$).

From direct measurement, we get :

meas.

$$\left.\begin{array}{l}\int_0^1 dxF_2^{ep}(x) = \frac{4}{9}f_u + \frac{1}{9}f_d \approx \boxed{0.18;} \\ \int_0^1 dxF_2^{en}(x) = \frac{4}{9}f_d + \frac{1}{9}f_u \approx \boxed{0.12;}\end{array}\right\} \rightarrow \left\{\begin{array}{l}f_u \approx 0.36; \\ f_d \approx 0.18; \\ f_u + f_d \approx \boxed{0.54.}\end{array}\right.$$

Result (important) :

$$f_u + f_d \approx 50\ \%.$$

Only ≈ ½ of the nucleon momentum is carried by quarks and antiquarks.

The rest is "invisible" in the DIS by a charged lepton.

This was one of the first (and VERY convincing) evidences for the existence of the **gluons**, the carriers of the hadronic force.

The gluons are neutral and do not "see" the e.m. interactions.

The q-p model : e⁻p vs νp DIS

Compute $F_2^{eN}(x)$ for an *isoscalar target* ***N***, i.e. a target with $n_{protons}$ = $n_{neutrons}$, both *quasi-free* (*Fermi-gas approx*) :

$$F_2^{ep}(x) = x\left\{\frac{4}{9}\left[u^p(x)+\bar{u}^p(x)\right]+\frac{1}{9}\left[d^p(x)+\bar{d}^p(x)\right]+\frac{1}{9}\left[s^p(x)+\bar{s}^p(x)\right]\right\};$$

$$F_2^{en}(x) = x\left\{\frac{4}{9}\left[d^p(x)+\bar{d}^p(x)\right]+\frac{1}{9}\left[u^p(x)+\bar{u}^p(x)\right]+\frac{1}{9}\left[s^p(x)+\bar{s}^p(x)\right]\right\};$$

$$F_2^{eN}(x) \equiv \frac{F_2^{ep}(x)+F_2^{en}(x)}{2} =$$

$$= x\left\{\frac{5}{18}\left[u^p(x)+\bar{u}^p(x)+d^p(x)+\bar{d}^p(x)\right]+\frac{1}{9}\left[s^p(x)+\bar{s}^p(x)\right]\right\} \xrightarrow[\text{neglect s}]{}$$

$$\rightarrow \frac{5x}{18}\left[u^p(x)+\bar{u}^p(x)+d^p(x)+\bar{d}^p(x)\right].$$

Notice that in neutrino DIS (see) the dynamics is different, but the effective structure function for an isoscalar target **turns out to be very similar**, up to a factor, as in the purely e.m. case :

$$F_2^{\nu N}(x) = x\left[u^p(x)+\bar{u}^p(x)+d^p(x)+\bar{d}^p(x)\right] = F_2^{eN}(x)\Big/\frac{5}{18}.$$

The experimental value (see) is $F_2^{eN} / F_2^{\nu N} = 0.29 \pm 0.02$, very compatible with this prediction (5/18 = 0.278).

why "isoscalar" ?

because (especially in ν scattering) the target has to be heavy, i.e. made of heavy nuclei, well reproduced by this approximation.

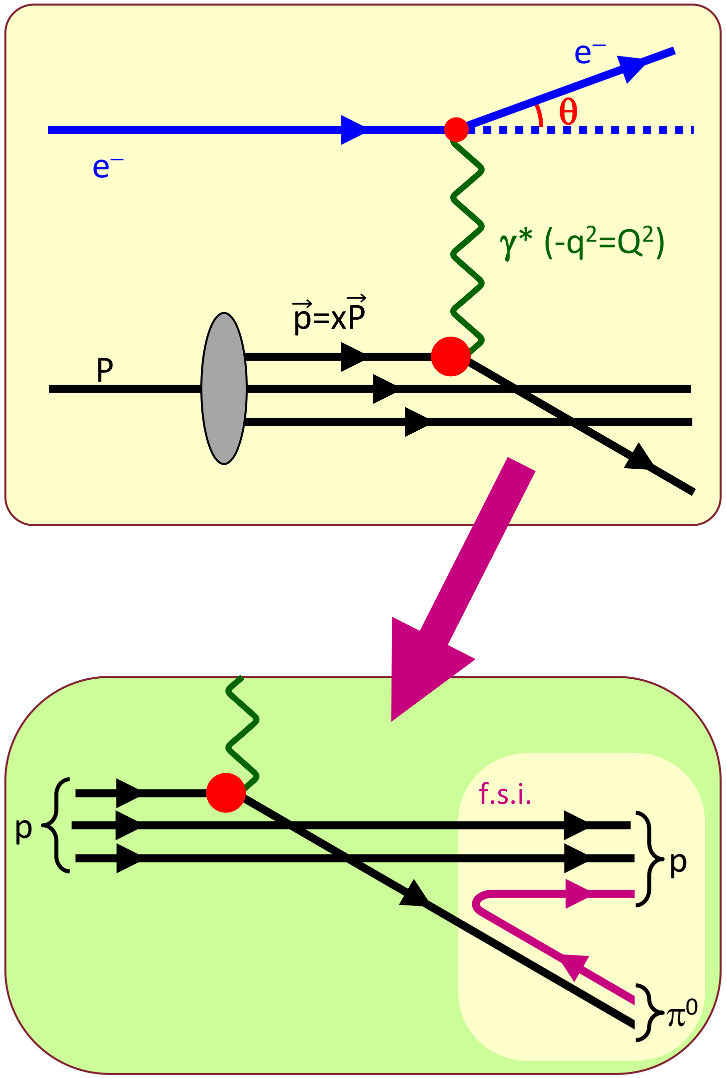
i.e. the structure functions depend on real properties of the nucleon structure, and are not dependent on the interaction.

The q-p model : hadrons in the final state

Consider the hadrons on the on the bottom right: is it possible ?

- free quarks do NOT exists (§ 2 and § 6);
- only (qqq) ($\bar{q}\bar{q}\bar{q}$) ($q\bar{q}$) hadrons observable (§ 6);
- therefore some "recombination" must occur [*see a possible example, in general it is more complicated*];
- these effects are called "final state interactions" [f.s.i.];
- usually f.s.i. are factorized, i.e. they are treated as a "phase 2" process, which does NOT interfere with "phase 1" (i.e. the DIS);
- at higher energy and higher Q^2, quarks in the final state *fragment* into **hadron jets**.

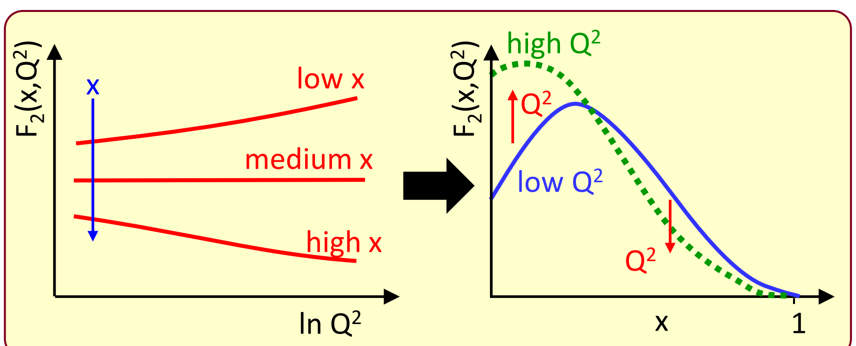
[*all that – and much more – for next semester, e.g. in the "Collider Physics" course: see you there*].

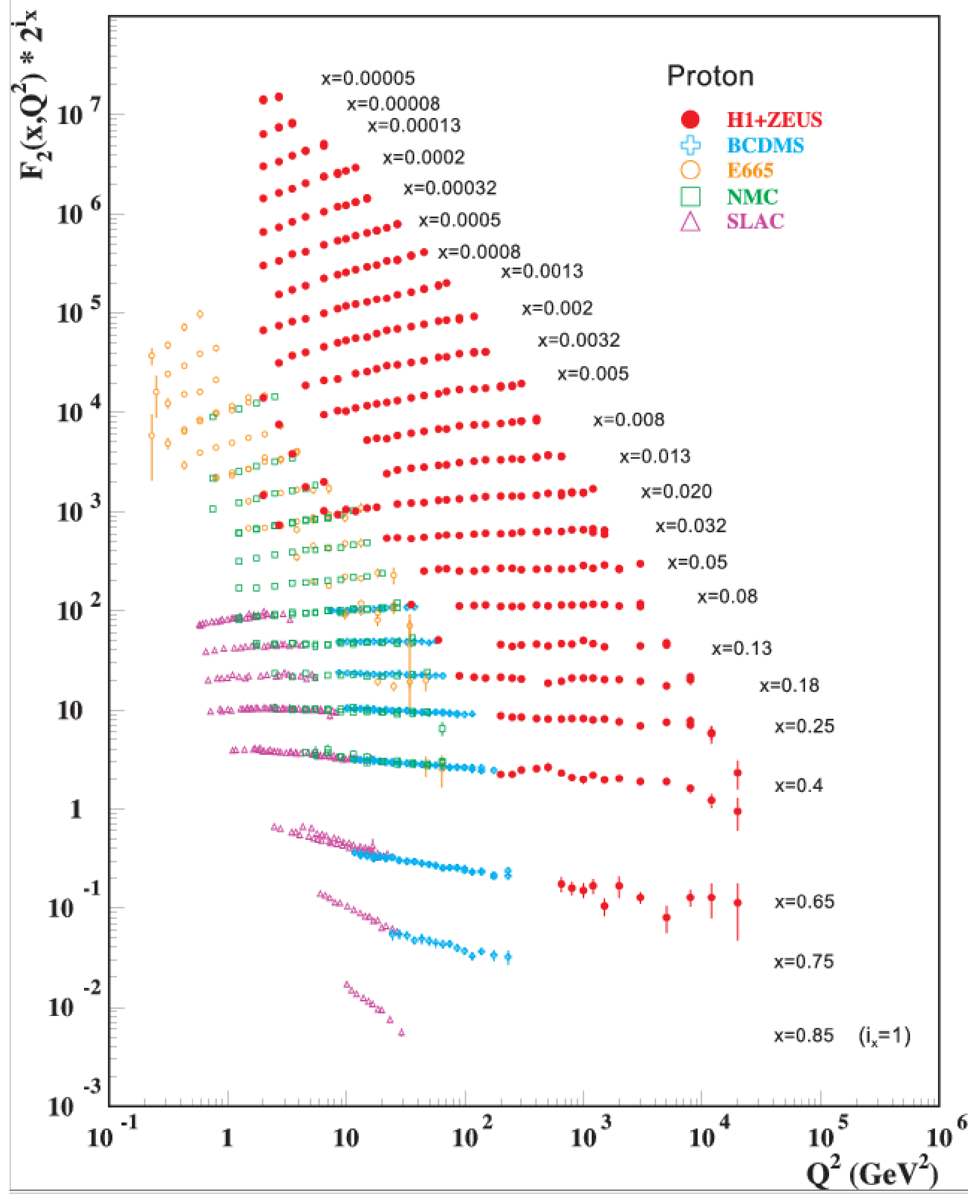


$F_2(x,Q^2)$: Scaling violations

Modern experiments have probed the nucleon to very high values of Q^2. Now electrons are often replaced with muons, which have the advantage of intense beams of higher momenta. Or, even better, the experiments are carried out at e^-p Colliders (HERA).

There are data up to $Q^2 \approx 10^5$ GeV^2: when plotting F_2 as function of Q^2 at fixed x, some Q^2-dependence appears, incompatible with Bjorken scaling [*see plot and sketch, and the next slides*].





$F_2(x,Q^2)$: Q^2 evolution

However, this effect (*scaling violations)*, is NOT attributed to sub-structures or other novel physics, but to a dynamical change in F_2, well understood in QCD.

$$\begin{cases} \sum_{\text{partons}} \int_0^1 x f_j(x)dx < 1; \\ \sum_{\text{partons}} \int_0^1 f_j(x)dx = \text{undefined (but large)}. \end{cases}$$

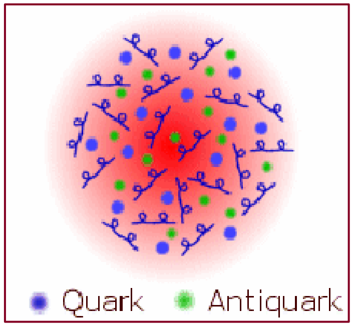
In QCD :

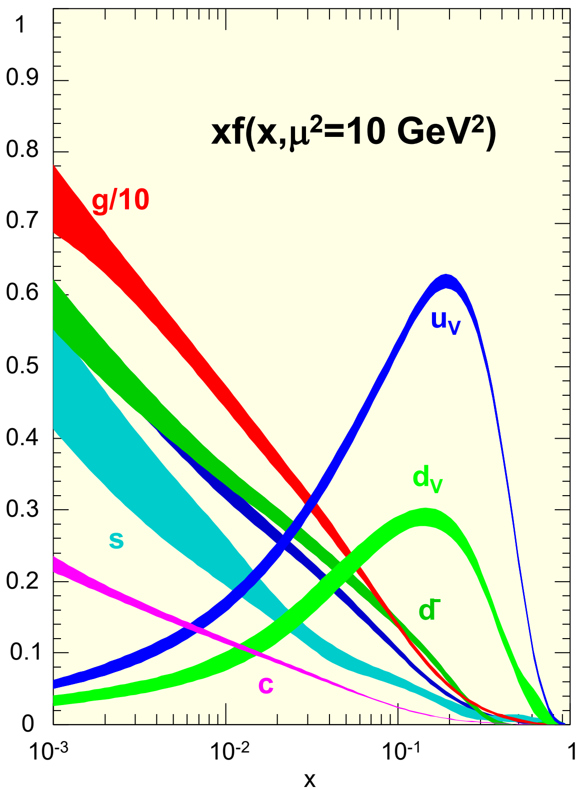
- higher Q^2

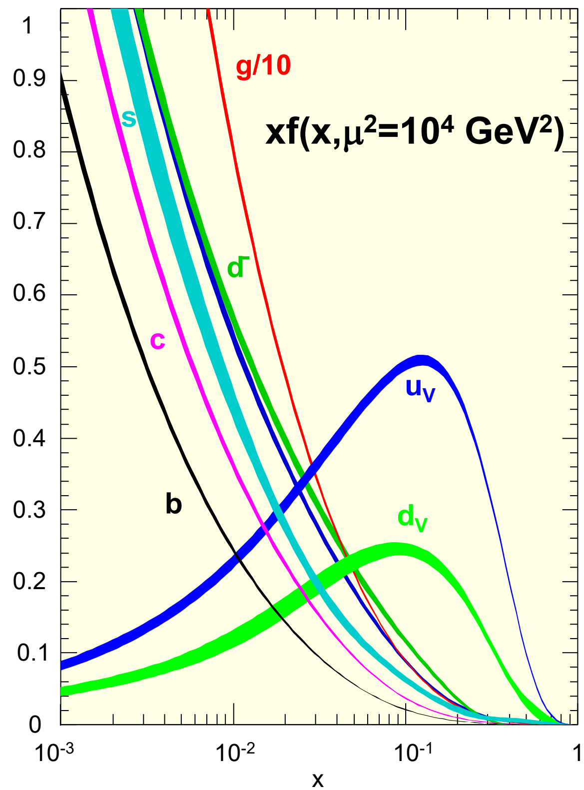
→ smaller size probed

→ more $q\bar{q}$ and gluons

→ less valence quarks.







a modern parameterization of the PDF [*NNPDF3.0-(NNLO)*] shows clearly the difference in the PDF when $Q^2 = 10 \div 10^4$ GeV^2:

- u_V, d_V → down;
- $\bar{u}$, $\bar{d}$, [= u_S, d_S,] g → up;
- s, c, b → up (more phase space)

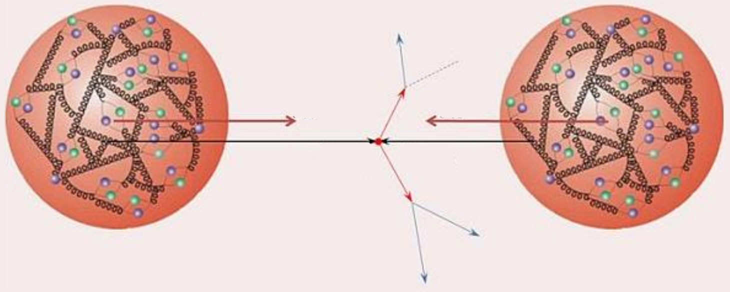
$F_2(x,Q^2)$: parton distribution functions

For modern experiments with hadrons the knowledge of $F_2^{p,n}(x)$ is a necessary ingredient of the data analysis.

- The structure functions are an effect of the hadronic forces. However, being a complicated result of an ill-defined number of bodies in non-perturbative regime, they cannot be reliably computed with today's technology (lattice QCD is still a hope).
- *Similar to the chemistry of complicated molecules, which is a difficult subject, although the fundamental interactions are* [*supposed to be*] *well understood*.
- When studying hadron interactions at large Q^2, the initial state is parameterized by its structure function, as an incoherent sum of all the PDF's, including the gluon.

- In practice, all the computations (e.g. the Higgs production) must use a numerical parameterization of the PDF's, and take into account their uncertainties.

- the PDF's are probabilistic, i.e. the value of x is different for each event !!!
- *consequence: the 4-mom conservation at parton level is a difficult constraint in the computation !!! (see later*)



An artist's view of the pp interaction [from the CERN ATLAS www site]

Summary of cross-sections

$$\left[\frac{d\sigma}{d\Omega}\right]_{\text{Rutherford}} = \frac{4Z^2\alpha^2E'^2}{|q|^4};$$

$$\left[\frac{d\sigma}{d\Omega}\right]^{*}_{\text{Mott}} = \left[\frac{d\sigma}{d\Omega}\right]_{\text{Rutherford}} \times \left(1-\beta^2\sin^2\frac{\theta}{2}\right) \xrightarrow{\beta\to 1} \left[\frac{d\sigma}{d\Omega}\right]_{\text{Rutherford}} \times \cos^2\frac{\theta}{2};$$

$$\left[\frac{d\sigma}{d\Omega}\right]_{\text{Mott}} = \left[\frac{d\sigma}{d\Omega}\right]^{*}_{\text{Mott}} \times \frac{E'}{E}; \qquad \left[\frac{d\sigma}{d\Omega}\right]_{\text{non-point.}} = \left[\frac{d\sigma}{d\Omega}\right]^{(*)}_{\text{Mott}} \times \left|F(q^2)\right|^2.$$

$$\left[\frac{d\sigma}{d\Omega}\right]_{\text{point-like spin1/2}} = \left[\frac{d\sigma}{d\Omega}\right]_{\text{Mott}} \times \left(1+2\tau\tan^2\frac{\theta}{2}\right); \qquad \left[\tau = \frac{Q^2}{4M^2c^2}\right];$$

$$\left[\frac{d\sigma}{d\Omega}\right]_{\text{Rosenbluth}} = \left[\frac{d\sigma}{d\Omega}\right]_{\text{Mott}} \times \left(\frac{G_E^2(Q^2)+\tau G_M^2(Q^2)}{1+\tau} + 2\tau G_M^2(Q^2)\tan^2\frac{\theta}{2}\right);$$

$$\left[\frac{d^2\sigma}{d\Omega dE'}\right]_{\text{DIS}} = \frac{4\alpha^2E'^2}{Q^4} \times \left[W_2(Q^2,\nu)\cos^2\frac{\theta}{2} + 2W_1(Q^2,\nu)\sin^2\frac{\theta}{2}\right];$$

$$\left[\frac{d^2\sigma}{dxdy}\right]_{\text{DIS}} = \frac{4\pi\alpha^2 s}{Q^4} \times \left[xy^2F_1(x,Q^2) + (1-y)F_2(x,Q^2)\right].$$

Q^2 scale

References

1. [Povh, 6,7,8]
2. [BJ, 12]
3. M. N. Rosenbluth, Phys.Rev. 79 (1950) 615.
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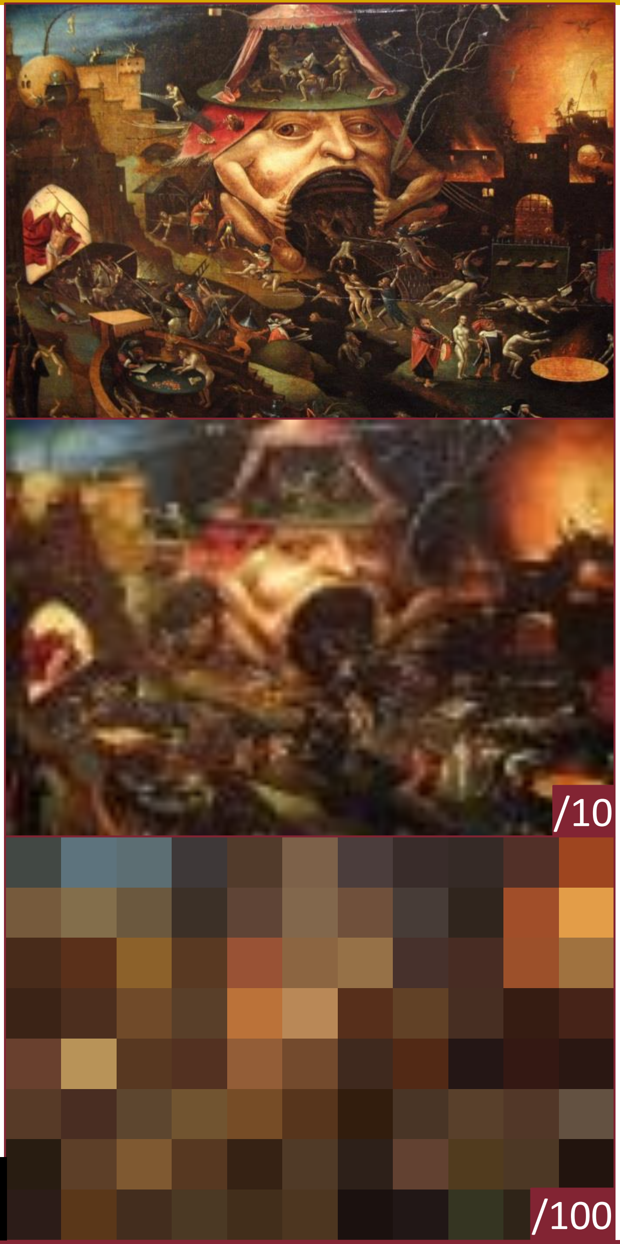
5. J.Friedman, H.Kendall, Ann. Rev. Nucl. Sci. 22 (1972) 203.



Follower of Hieronymus Bosch – Christ in Limbo (particular) – Indianapolis Museum of Art

End of chapter 2