Consideriamo due particelle di massa m non interagenti in una scatola posizionata fra 0 ed L.
Una particella si trova nello stato $|n\rangle$ della scatola, l'altra nello stato $|\ell\rangle$ (supporremo $\ell \neq n$).

Calcolare

$$\langle (x_1 - x_2)^2 \rangle$$

nel caso in cui le particelle siano

1/ distinguibili

2/ due fermioni

3/ due bosoni.

Sapendo che, per una particella nella scatola, si ha $\langle x \rangle = \langle n | x | n \rangle = \frac{L}{2}$ e $\langle x^2 \rangle = \langle n | x^2 | n \rangle = L^2 \left[\frac{1}{3} - \frac{1}{2(n\pi)^2} \right]$

Lo stato del sistema si scrive

$|l\, n\rangle$ part. distinguibili

$\frac{1}{\sqrt{2}}(|l\, n\rangle + |n\, l\rangle)$ bosoni

$\frac{1}{\sqrt{2}}(|l\, n\rangle - |n\, l\rangle)$ fermioni

In rappresentazione delle coordinate, le funzioni d'onda sono quindi

$$\psi_D(x_1, x_2) = \psi_l(x_1)\,\psi_n(x_2)$$

$$\psi_S(x_1, x_2) = \frac{1}{\sqrt{2}}\langle x_1\, x_2, S | \left(\frac{1}{\sqrt{2}}[|l\, n\rangle + |n\, l\rangle] \right)$$

$$= \frac{1}{2\sqrt{2}}\Big[\langle x_1 x_2| + \langle x_2 x_1|\Big]\Big[|l\, n\rangle + |n\, l\rangle\Big]$$

$$= \frac{1}{2\sqrt{2}}\Big[\langle x_1 x_2 | l\, n\rangle + \langle x_1 x_2 | n\, l\rangle$$
$$+ \langle x_2 x_1 | l\, n\rangle + \langle x_2 x_1 | n\, l\rangle\Big]$$

$$= \frac{1}{2\sqrt{2}}\Big[\psi_l(x_1)\psi_n(x_2) + \psi_n(x_1)\psi_l(x_2)$$
$$+ \psi_l(x_2)\psi_n(x_1) + \psi_n(x_2)\psi_l(x_1)\Big]$$

$$= \frac{1}{\sqrt{2}}\Big[\psi_l(x_1)\psi_n(x_2) + \psi_n(x_1)\psi_l(x_2)\Big]$$

e $\psi_A(x_1, x_2) = \frac{1}{\sqrt{2}}\left[\psi_\ell(x_1)\psi_n(x_2) - \psi_n(x_1)\psi_\ell(x_2)\right]$

Ora

$$\langle (x_1 - x_2)^2 \rangle = \langle x_1^2 \rangle + \langle x_2^2 \rangle - 2\langle x_1 x_2 \rangle$$

1/ Distinguibili

$$\langle x_1^2 \rangle = \int dx_1\, dx_2\; x_1^2\, |\psi_D(x_1, x_2)|^2$$

$$= \int dx_1\, x_1^2 |\psi_\ell(x_1)|^2 \int dx_2\, |\psi_n(x_2)|^2$$

$$= \int dx_1\, x_1^2 |\psi_\ell(x_1)|^2 = \langle x^2 \rangle_\ell$$

$$\langle x_2^2 \rangle = \langle x^2 \rangle_n$$

$$\langle x_1 x_2 \rangle = \int dx_1\, x_1 |\psi_\ell(x_1)|^2 \int dx_2\, x_2 |\psi_n(x_2)|^2$$

$$= \langle x \rangle_\ell \langle x \rangle_n$$

$\rightarrow \langle x_n^2 \rangle_n + \langle x_\ell^2 \rangle_\ell - 2\langle x \rangle_n \langle x_\ell \rangle$

$$\rightarrow \langle (x_1 - x_2)^2 \rangle_D = L^2\left[\frac{1}{3} - \frac{1}{2\pi^2 \ell^2}\right] + L^2\left[\frac{1}{3} - \frac{1}{2\pi^2 n^2}\right]$$

$$- 2\frac{L^2}{4}$$

$$= \left[\frac{2}{3} - \frac{1}{2}\right] L^2 - \frac{L^2}{2\pi^2}\left[\frac{1}{\ell^2} + \frac{1}{n^2}\right]$$

$$= \frac{4-3}{6} L^2 - \frac{L^2}{2\pi^2}\left(\frac{1}{\ell^2} + \frac{1}{n^2}\right) = \underline{L^2\left[\frac{1}{6} - \frac{1}{2\pi^2}\left(\frac{1}{\ell^2} + \frac{1}{n^2}\right)\right]}$$

Particelle identiche

$$\langle x_1^2 \rangle_{S,A} = \frac{1}{2}\int dx_1\, dx_2 \left[\psi_\ell^*(x_1)\, \psi_n^*(x_2) \pm \psi_n^*(x_1)\psi_\ell^*(x_2)\right]$$

$$x_1^2 \left[\psi_\ell(x_1)\psi_n(x_2) \pm \psi_n(x_1)\psi_\ell(x_2)\right]$$

$$= \frac{1}{2} \Big\{ \int dx_1\ x_1^2\, |\psi_\ell(x_1)|^2 \int dx_2\, |\psi_n(x_2)|^2$$

$$\pm \int dx_1\ x_1^2\ \psi_\ell^*(x_1)\psi_n(x_1) \underbrace{\int dx_2\ \psi_n^*(x_2)\psi_\ell(x_2)}_{0}$$

$$\pm \int dx_1\ x_1^2\ \psi_n^*(x_1)\psi_\ell(x_1) \underbrace{\int dx_2\ \psi_\ell^*(x_2)\psi_n(x_2)}_{0}$$

$$+ \int dx_1\ x_1^2\ |\psi_n(x_1)| \int dx_2\, |\psi_\ell^*(x_2)|^2 \Big\}$$

$$= \frac{1}{2} \left\{ \langle x^2 \rangle_\ell + \langle x^2 \rangle_n \right\}$$

Analogamente

$$\langle x_2^2 \rangle_{S,A} = \frac{1}{2} \left\{ \langle x^2 \rangle_\ell + \langle x^2 \rangle_n \right\}$$

$$\langle x_1 x_2 \rangle_{S,A} = \frac{1}{2} \Big\{ \int dx_1\, x_1 |\psi_e(x_1)|^2 \int dx_2\, x_2 |\psi_n(x_2)|^2$$

$$\pm \int dx_1\, x_1\, \psi_e^*(x_1)\psi_n(x_1) \int dx_2\, x_2\, \psi_n^*(x_2)\psi_e(x_2)$$

$$\pm \int dx_1\, x_1\, \psi_n^*(x_1)\psi_e(x_1) \int dx_2\, x_2\, \psi_e^*(x_2)\psi_n(x_2)$$

$$+ \int dx_1\, x_1 |\psi_n(x_1)|^2 \int dx_2\, x_2 |\psi_e(x_2)|^2 \Big\}$$

$$= \frac{1}{2} \Big\{ \langle x \rangle_e \langle x \rangle_n \pm \underbrace{\langle x \rangle_{en} \langle x \rangle_{ne}}_{|\langle x \rangle_{en}|^2}$$

$$\pm \underbrace{\langle x \rangle_{ne} \langle x \rangle_{en}}_{|\langle x \rangle_{ne}|^2} + \langle x \rangle_n \langle x \rangle_e \Big\}$$

$$= \langle x \rangle_e \langle x \rangle_n \pm |\langle x \rangle_{ne}|^2$$

$$\to \langle (x_1 - x_2)^2 \rangle_{S,A} = \overbrace{\langle x^2 \rangle_e + \langle x^2 \rangle_n - 2\langle x \rangle_e \langle x \rangle_e}^{\langle (x_1 - x_2)^2 \rangle_D} \quad {}_{n?}$$

$$\mp 2 |\langle x \rangle_{ne}|^2$$

L'unico termine da calcolare esplicitamente è l'ultimo

$$\langle x \rangle_{n\ell} = \frac{2}{L}\int_0^L dx\ x \sin\frac{n\pi}{L}x \sin\frac{\ell\pi}{L}x$$

$$= \frac{2}{L}\int_0^L dx\ x\frac{1}{2}\left[\cos\frac{(n-\ell)\pi}{L}x - \cos\frac{(n+\ell)\pi}{L}x\right]$$

$$= \frac{1}{L}\int_0^L dx\ x\cos\frac{(n-\ell)\pi}{L}x - \frac{1}{L}\int_0^L dx\ x\cos\frac{(n+\ell)\pi}{L}x$$

Ora $\displaystyle\int_0^L dx\ x\cos\alpha x = \frac{1}{\alpha}x\sin\alpha x\Big|_0^L - \frac{1}{\alpha}\int_0^L dx\ \sin\alpha x$

$$= \cancel{\frac{1}{\alpha}x\sin\alpha x\Big|_0^L} + \frac{1}{\alpha^2}\cos\alpha x\Big|_0^L$$

$$\langle x \rangle_{n\ell} = \frac{1}{L}\left\{\cancel{\frac{L}{\pi(n-\ell)}x\sin\frac{\pi(n-\ell)}{L}x\Big|_0^L}^{\,0} + \frac{L^2}{\pi^2(n-\ell)^2}\cos\frac{\pi(n-\ell)}{L}x\Big|_0^L\right.$$

$$\left. - \cancel{\frac{L}{\pi(n+\ell)}x\sin\frac{\pi(n+\ell)}{L}x\Big|_0^L}^{\,0} - \frac{L^2}{\pi^2(n+\ell)^2}\cos\frac{\pi(n+\ell)}{L}x\Big|_0^L\right\}$$

$$= \frac{L}{\pi^2}\left\{\frac{1}{(n-\ell)^2}\left[\cos[(n-\ell)\pi] - 1\right] - \frac{1}{(n+\ell)^2}\left[\cos[(n+\ell)\pi] - 1\right]\right\}$$

$$\cos(n\pm\ell)\pi = (-1)^{\ell+n}$$

$n+\ell = 2k+1 \Rightarrow \ell = 2k+1-n$

$n-\ell = n-(2k+1-n)$

$= 2n-2k-1$

$= 2(n+k)-1$

$$= \frac{L}{\pi^2}\left[(-1)^{\ell+n} - 1\right]\left\{\frac{1}{(n-\ell)^2} - \frac{1}{(n+\ell)^2}\right\}$$

$$\langle (x_1-x_2)^2\rangle_S = \langle (x_1-x_2)^2\rangle_D - \left|2\frac{L}{\pi^2}\left[(-1)^{\ell+m}-1\right] \cdot \left\{\frac{1}{(n-\ell)^2} - \frac{1}{(n+m)^2}\right\}\right|^2$$

$$\langle (x_1-x_2)^2\rangle = \langle (x_1-x_2)^2\rangle_D + \left|2\frac{L}{\pi}\left[(-1)^{\ell+n}-1\right]\left\{\frac{1}{(n-\ell)^2} - \frac{1}{(n+m)^2}\right\}\right|^2$$

nb se $\ell+n$ è pari non posso distinguere un caso dall'altro

se $\ell+n$ è dispari i bosoni sono "più vicini" delle particelle distinguibili mentre i fermioni sono "più lontani".

$$\frac{1}{(n-\ell)^2} - \frac{1}{(n+m)^2} = \frac{n^2+m^2+2nm-n^2-m^2+2nm}{(n-\ell)^2(n+m)^2}$$

$$= \frac{4nm}{(\ )^2(\ )^2}$$