A defense of Columbo (and of the use of Bayesian inference in forensics) – A multilevel introduction to probabilistic reasoning –

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Abstract

Triggered by a recent interesting New Scientist article on the too frequent incorrect use of probabilistic evidence in courts, I introduce the basic concepts of probabilistic inference with a toy model, and discuss several important issues that need to be understood in order to extend the basic reasoning to real life cases. In particular, I emphasize the often neglected point that degrees of beliefs are updated not by 'bare facts' alone, but by all available information pertaining to them, including how they have been acquired. In this light I show that, contrary to what claimed in that article, there was no "probabilistic pitfall" in the Columbo's episode pointed as example of "bad mathematics" yielding "rough justice". Instead, such a criticism could have a 'negative reaction' to the article itself and to the use of Bayesian reasoning in courts, as well as in all other places in which probabilities need to be assessed and decisions need to be made. Anyway, besides introductory/recreational aspects, the paper touches important questions, like: role and evaluation of priors; subjective evaluation of Bayes factors; role and limits of intuition; 'weights of evidence' and 'intensities of beliefs' (following Peirce) and 'judgments leaning' (here introduced), including their uncertainties and combinations; role of relative frequencies to assess and express beliefs; pitfalls due to 'standard' statistical education; weight of evidences mediated by testimonies. A small introduction to Bayesian networks, based on the same toy model (complicated by the possibility of incorrect testimonies) and implemented using Hugin software, is also provided, to stress the importance of formal, computer aided probabilistic reasoning.

> *"Use enough common sense to know*
> *when ordinary common sense does not apply"*
> (I.J. Good's *guiding principle of all science*)

Figure 1: Peter Galesco, played by Dick Van Dyke, taking the picture of its wife before killing her.

1 Introduction

A recent New Scientist article [1] deals with errors in courts due to "bad mathematics", advocating the use of the so-called Bayesian methods to avoid them. Although most examples of resulting "rough justice" come from real life cases, the first "probabilistic pitfall" is taken from crime fiction, namely from a "1974 episode of the cult US television series" *Columbo*, in which a "society photographer has killed his wife and disguised it as a bungled kidnapping."

The pretended mistake happens in the concluding scene, when "the hangdog detective [...] induces the murderer to grab from a shelf of 12 cameras the exact one used to snap the victim before she was killed." According to the article author (or to experts on which scientific journalists often rely on) the question is that "killer or not, anyone would have a 1 in 12 chance of picking the same camera at random. That kind of evidence would never stand up in court." Then a sad doubt is raised, "Or would it? In fact, such probabilistic pitfalls are not limited to crime fiction."

Being myself not particularly fond of this kind of entertainment (perhaps with a little exception of the Columbo series, that I watch casually), I cannot tell how much crime fiction literature and movies are affected by "probabilistic pitfalls". Instead, I can give firm witness that scientific practice is plenty of mistakes of the kind reported in Ref.[1], that happen even in fields the general public would hardly suspect, like frontier physics, whose protagonists are supposed to have a skill in mathematics superior to police officers and lawyers.

But it is not just a question of math skill (complex calculations are usually done without mistakes), but of *probabilistic reasoning* (what to calculate!). This is a quite old story. In fact, as David Hume complained 260 years ago [2],

> *"The celebrated Monsieur Leibniz has observed it to be a defect in the common systems of logic, that they are very copious when they explain the operations of the understanding in the forming of demonstrations, but are too concise when they treat of probabilities, and those other measures of evidence on which life and action entirely depend, and which are our guides even in most of our philosophical speculations."*

It seems to me that the general situation has not improved much. Yes, 'statistics' (a name that, meaning too much, risks to mean little) is taught in colleges and universities to students of several fields, but distorted by the 'frequentistic approach', according to which one is not allowed to speak of probabilities of causes. This is, in my opinion, the original sin that gives grounds for a large number of probabilistic mistakes even by otherwise very valuable scientists and practitioners (see e.g. chapter 1 of Ref. [3]).

Going back to the "shambling sleuth Columbo", being my wife and my daughter his fans, it happens we own the DVD collections of the first seven seasons. It occurred then I watched with them, not much time ago (perhaps last winter), the 'incriminated', superb episode *Negative Reaction* [4], one of the best performances of Peter Falk playing the role of the famous lieutenant. However, reading the mentioned New Scientist article, I did not remember I had a 'negative reaction' from the final scene, although I use and teach Bayesian methods for a large variety of applications. Did I overlook something?

I watched again the episode and I was again convinced Columbo's last move was a conclusive checkmate.[1] Then I have invited some friends, all with physics or mathematics degree and somewhat knowledgeable of the Bayesian approach, to enjoy an evening together during the recent end of year holidays in order to let them make up their minds whether Columbo had good reasons to take Paul Galesco, magnificently impersonated by Dick Van Dyke, in front of the court (Bayes or not, we had some fun...).

The verdict was unanimous: Columbo was fully absolved or, more precisely, there was nothing to reproach the story writer, Peter S. Fischer. The convivial after dinner jury also requested me to write a note on the question, possibly with a short, self-contained introduction to the 'required math'. Not only to 'defend Columbo' or, more properly, his writer, but, and more seriously, to defend the Bayesian approach, and in particular its applications in forensic science. In fact, we all deemed the beginning paragraphs of the New Scientist article could throw a bad light on the rest of the contents.

Imagine a casual reader of the article, possibly a lawyer, a judge or a student in forensic science, to which the article was virtually addressed, and who might have seen *Negative Reaction*. Most likely he/she considered legitimate the charges of the policemen against the

[1] Just writing this note, I have realized that the final scene is directed so well that, not only the way the photographer loses control and commits his fatal mistake looks very credible, but also spectators forget he could play valid countermoves, not depending on the negative of the pretended destroyed picture (see footnote 32). Therefore, rather than chess, the name of the game is *poker*, and Columbo's bluff is able to induce the murderer to provide a crucial piece of evidence to finally incriminate him.

photographer. The ‘negative reaction’ would be that the reader would consider the rest of the article a support of dubious validity to some ‘strange math’ that can never substitute the human intuition in a trial.[2] Not a good service to the ‘Bayesian cause’. (Imagine somebody trying to convince you with arguments you hardly understand and who begins asserting something you consider manifestly false.)

In the following section I introduce the basic elements of Bayesian reasoning (subsection 2.4 can be skipped on first reading), using a toy model as guiding example in which the analysis of ref. [1] (“1 in 12”, or, more precisely “1 in 13”) holds. Section 4 shows how such a kind of evidence would change Columbo’s and jury’s opinion. Then I discuss in section 5 why a similar argument does not apply to the clip in which Columbo finally frames Galesco, and why all witnesses of the crucial actions (including TV watchers, with the exception of the author of Ref. [1] and perhaps a few others) and an hypothetical court jury (provided the scene had been properly reported) had to be absolutely positive the photographer killed his wife (or at least he knew who did it in his place).

The rest of the paper might be marginal, if you are just curious to know why I have a different opinion than Ref. [1], although I agree on the validity of Bayesian reasoning. In fact, at the end of the work, this paper is not the ‘short note’ initially planned. The reason is that the past months I had many discussions on some of the questions treated here with people from several fields. I have realized once more that it is not easy to put the basic principles at work if some important issues are not well understood. People are used to solving their statistical problems with ‘ad hoc’ formulae (see Appendix H) and therefore tend to add some ‘Bayesian recipes’ in their formularium. It is then too high the risk that one looks at simplified methods – Bayesian methods require a bit more thinking and computation that others! – that are even advertised as ‘objective’. Or one just refuses to use any math, on the defense of pure intuition. (By the way, this is an important point and I will take the opportunity to comment on the apparent contradictions between intuition and formal evaluation of beliefs, defending . . . both, but encouraging the use of the latter, superior to the former in complex situations – see in particular Appendix C).

So, to conclude the introduction, this document offers several levels of reading:

- If you are only interested to Columbo’s story, you can just jump straight to section 5.
- If you also (or regardless of Columbo) want to have an opportunity to learn the basic rules of Bayesian inference, subsections 2.1, 2.2 and 2.3, based on a simple master example, have been written on the purpose. Then you might appreciate the advantage of logarithmic updating (section 2.4) and perhaps see how it applies to the AIDS example of Appendix F.

[2] This kind of objection, in defense of what is often nothing but “the capricious ipse dixit of authority”[5], from which we should instead “emancipate”[5], is quite frequent. It is raised not only by judges, who tend to claim their job is ”to evaluate evidence not by means of a formula... but by the joint application of their individual common sense.”[1], but also by other categories of people who take important decisions, like doctors, managers and politicians.

- If you already know the basics of the probabilistic reasoning, but you wonder how it can be applied into real cases, then section 3 should help, together with some of the appendices.

- If none of the previous cases is yours (you might even be an expert of the field), you can simply browse the document. Perhaps some appendices or subsections might still be of your interest.

- Finally, there is the question of the many footnotes, which can break the pace of the reading. They are not meant to be necessarily read sequentially along with the main text and could be skipped on a first fast reading (in fact, this document is closer to an hypertext than to a standard article.)

Enjoy!

2 One in thirteen – Bayesian reasoning illustrated with a toy model

Let us leave aside Columbo's cameras for a while and begin with a different, simpler, stereotyped situation easier to analyze.

Imagine there are two types of boxes, B_1, that only contain white balls (W), and B_2, that contain one white balls and twelve black (incidentally, just to be precise, although the detail is absolutely irrelevant, we have to infer from Columbo's words, *"You didn't touch any of these twelve cameras. You picked up that one"*, the cameras were thirteen).

You take at random a box and extract a ball. The resulting color is *white*. You might be interested to evaluate the probability that the box is of type B_1, in the sense of stating in a quantitative way how much you believe this hypothesis. In formal terms we are interested in $P(B_1\,|\,W,I)$, knowing that $P(W\,|\,B_1,I)=1$ and $P(W\,|\,B_2,I)=1/13$, a problem that can be sketched as

$$\left\{\begin{array}{lcl} P(W\,|\,B_1,I) & = & 1 \\ P(W\,|\,B_2,I) & = & 1/13 \end{array}\right. \qquad \Rightarrow \qquad P(B_1\,|\,W,I) = \textbf{?} \tag{1}$$

[Here '|' stands for 'given', or 'conditioned by'; I is the general ('background') status of information under which this probability is assessed; 'W,I' or 'B_i,I' after '|' indicates that both conditions are relevant for the evaluation of the probability.]

A typical mistake at this point is to confuse $P(B_1\,|\,W,I)$ with $P(W\,|\,B_1,I)$, or, more often, $P(B_2\,|\,W,I)$ with $P(W\,|\,B_2,I)$, as largely discussed in Ref. [1]. Hence we need to learn how to turn properly $P(W\,|\,B_1,I)$ into $P(B_1\,|\,W,I)$ using the rules of probability theory.

2.1 Bayes theorem and Bayes factor

The 'probabilistic inversion' $P(W\,|\,B_1,I) \rightarrow P(B_1\,|\,W,I)$ can *only*[3] be performed using the so-called *Bayes' theorem*, a simple consequence of the fact that, given the *effect* E and some *hypotheses* H_i concerning its possible cause, the joint probability of E and H_i, conditioned by the *background information*[4] I, can be written as

$$P(E\cap H_i\,|\,I) \;=\; P(E\,|\,H_i,I)\cdot P(H_i\,|\,I) = P(H_i\,|\,E,I)\cdot P(E\,|\,I)\,, \tag{2}$$

where '$\cap$' stands for a logical 'AND'. From the second equality of the last equation we get

$$P(H_i\,|\,E,I) \;=\; \frac{P(E\,|\,H_i,I)}{P(E\,|\,I)}\cdot P(H_i\,|\,I)\,, \tag{3}$$

that is one of the ways to express Bayes' theorem.[5]

Since a similar expression holds for any other hypothesis H_j, dividing member by member the two expressions we can restate the theorem in terms of the relative beliefs, that is

$$\underbrace{\frac{P(H_i\,|\,E,I)}{P(H_j\,|\,E,I)}}_{\text{updated odds}} \;=\; \underbrace{\frac{P(E\,|\,H_i,I)}{P(E\,|\,H_j,I)}}_{\substack{\text{updating factor}\\ \text{(`}\textit{Bayes factor}\text{')}}} \times \underbrace{\frac{P(H_i\,|\,I)}{P(H_j\,|\,I)}}_{\text{initial odds}} \;: \tag{4}$$

the initial ratio of beliefs ('odds') is updated by the so-called *Bayes factor*, that depends on how likely *each* hypothesis can produce that effect.[6] Introducing $O_{i,j}$ and $\mathrm{BF}_{i,j}$, with obvious meanings, we can rewrite Eq. (4) as

$$O_{i,j}(E,I) \;=\; \mathrm{BF}_{i,j}(E,I)\times O_{i,j}(I)\,. \tag{5}$$

[3] Beware of methods that provide 'levels of confidence', or something like that, without using Bayes' theorem! See also footnote 9 and Appendix H.

[4] The background information I represents all we know about the hypotheses and the effect considered. Writing I in all expressions could seem a pedantry, but it isn't. For example, if we would just write $P(E)$ in these formulae, instead of $P(E\,|\,I)$, one might be tempted to take this probability equal to one, "because the observed event is a well established fact', that has happened and is then certain. But it is not this certainty that enters these formulae, but rather the probability 'that fact could happen' in the light of 'everything we knew' about it ('I').

[5] Bayes' theorem can be often found in the form

$$P(H_i\,|\,E,I) \;=\; \frac{P(E\,|\,H_i,I)\cdot P(H_i\,|\,I)}{\sum_i P(E\,|\,H_i,I)\cdot P(H_i\,|\,I)}\,,$$

valid if we deal with a class of incompatible hypotheses [i.e. $P(H_i\cap H_j\,|\,I)=0$ and $\sum_i P(H_i\,|\,I)=1$]. In fact, in this case a general rule of probability theory [Eq. (35) in Appendix A] allows us to rewrite the denominator of Eq. (3) as $\sum_i P(E\,|\,H_i,I)\cdot P(H_i\,|\,I)$. In this note, dealing only with two hypotheses, we prefer to reason in terms of probability ratios, as shown in Eq. (4).

[6] Note that, while in the case of only two hypotheses entering the inferential game their initial probabilities are related by $P(H_2\,|\,I) = 1 - P(H_1\,|\,I)$, the probabilities of the effects $P(E\,|\,H_1,I)$ and $P(E\,|\,H_2,I)$ have usually nothing to do with each other.

Note that, if the initial odds are unitary, than the final odds are equal to the updating factor. Then, *Bayes factors can be interpreted as odds due only to an individual piece of evidence, if the two hypotheses were considered initially equally likely.*[7] This allows us to rewrite $\text{BF}_{i,j}(E,I)$ as $\tilde{O}_{i,j}(E,I)$, where the tilde is to remind that *they are not properly odds*, but rather '*pseudo-odds*'. We get then an expression in which all terms have *virtually* uniform meaning:

$$O_{i,j}(E,I) \;\; = \;\; \tilde{O}_{i,j}(E,I) \times O_{i,j}(I)\,. \tag{6}$$

If we have only two hypotheses, we get simply $O_{1,2}(E,I) = \tilde{O}_{1,2}(E,I) \times O_{1,2}(I)$. If the updating factor is unitary, then the piece of evidence does not modify our opinion on the two hypotheses (no matter how small can numerator and denominator be, as long as their ratio remains finite and unitary! – see Appendix G for an example worked out in details); when $\tilde{O}_{1,2}(E,I)$ vanishes, then hypothesis B_1 becomes impossible ("*it is falsified*"); if instead it is infinite (i.e. the denominator vanishes), then it is the other hypothesis to be impossible. (The undefined case 0/0 means that we have to look for other hypotheses to explain the effect.[8])

2.2 Role of priors

Applying the updating reasoning to our box game, the Bayes factor of interest is

$$\tilde{O}_{1,2}(W,I) \;\; = \;\; \frac{P(W\,|\,B_1,I)}{P(W\,|\,B_2,I)} = \frac{1}{1/13} = 13\,. \tag{7}$$

As it was remarked, this number would give the required odds *if* the hypotheses were initially equally likely. But how strong are the initial relative beliefs on the two hypotheses? 'Unfortunately', we cannot perform a probabilistic inversion if we are unable to assign somehow *prior probabilities* to the hypotheses we are interested in.[9] Indeed, in the formulation

[7] Those who want to base the inference only on the probabilities of the observations given the hypotheses, in order to "let the data speak themselves", might be in good faith, but their noble intention does dot save them from dire mistakes [3]. (See also footnotes 9 and 43, as well as Appendix H.)

[8] Pieces of evidence modify, in general, relative beliefs. When we turn relative beliefs into absolute ones in a scale ranging from 0 to 1, we are *always* making the implicit assumption that the possible hypotheses are only those of the class considered. If other hypotheses are added, the relative beliefs do not change, while the absolute ones do. This is the reason why an hypothesis can eventually be falsified, if $P(E\,|\,H_i,I) = 0$, but an absolute truth, i.e. $P(E\,|\,H_j,I) = 1$, depends on which class of hypotheses is considered. Stated in other words, in the realm of probabilistic inference *falsities can be absolute, but truths are always relative.*

[9] You might be reluctant to adopt this way of reasoning, objecting "I am unable to state priors!", or "I don't want to be influenced by prior!", or even "I don't want to state degrees of beliefs, but only real probabilities". No problem, provided you stay away from probabilistic inference (for example you can enjoy fishing or hiking – but I hope you are aware of the large amount of prior beliefs involved in these activities too!). Here I can only advice you, provided you are interested in evaluating probabilities of 'causes' from effects, not to overlook prior information and not to blindly trust statistical methods and software packages advertised as prior-free, unless you don't want to risk to arrive at very bad conclusions. For more comments on the question see Ref. [3], footnote 43 and Appendix H.

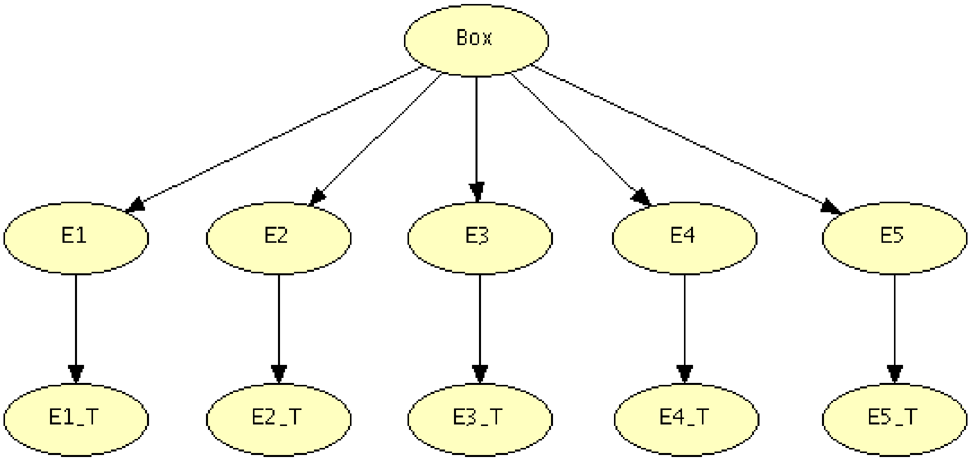

Figure 2: Graphical representation of the causal connections Box→ E_i, where E_i are the effects (White/Black at each extraction). These effects are then causes of other effects (E_i_T), the reported colors, where 'T' stands for 'testimony'. The arrows connecting the various 'nodes' represent conditional probabilities. The model will be fully exploited in Appendix J.

of the problem I on purpose passed over the relevant pieces of information to evaluate the prior probabilities (it was said that "there are two types of boxes", not "there are two boxes"!). If we specify that we had n_1 boxes of type B_1 and n_2 of the other kind, then the initial odds are n_1/n_2 and the final ones will be

$$\begin{aligned} O_{1,2}(W,I) &= \tilde{O}_{1,2}(W,I) \times O_{1,2}(I) & (8)\\ &= 13 \times \frac{n_1}{n_2}, & (9) \end{aligned}$$

from which we get (just requiring that the probability of the two hypotheses have to sum up to one[10])

$$P(B_1\,|\,W, I_0) \;=\; \frac{13}{13 + n_2/n_1}\,. \tag{10}$$

If the two hypotheses were *initially* considered equally likely, then the evidence W makes B_1 13 times more believable than B_2, i.e. $P(B_1\,|\,W, I_0) = 13/14$, or approximately 93%. On the other hand, if B_1 was *a priori* much less credible than B_2, for example by a factor

[10] If H_1 and H_2 are generic, complementary hypotheses we get, calling b the Bayes factor of H_1 versus H_2 and x_0 the initial odds to simplify the notation, the following convenient expressions to evaluate the probability of H_1:

$$P(H_1\,|\,x_0, b) \;=\; \frac{b\,x_0}{1+b\,x_0} = \frac{b}{b+1/x_0} = \frac{x_0}{x_0+1/b}\,.$$

13, just to play with round numbers, the same evidence made B_1 and B_2 equally likely. Instead, if we were initially in strong favor of B_1, considering it for instance 13 times more plausible than B_2, that evidence turned this factor into 169, making us 99.4% confident – *highly confident*, some would even say 'practically sure'! – that the box is of type B_1.

2.3 Adding pieces of evidence

Imagine now the following variant of the previous toy experiment. After the white ball is observed, you put it again in the box, shake well and make a second extraction. You get white the second time too. Calling W_1 and W_2 the two observations, we have now:[11]

$$\begin{aligned}
\frac{P(B_1\,|\,W_1,W_2,I_0)}{P(B_2\,|\,W_1,W_2,I_0)} &= \frac{P(W_1,W_2\,|\,B_1,I_0)}{P(W1,W_2\,|\,B_2,I_0)}\times\frac{P(B_1\,|\,I_0)}{P(B_2\,|\,I_0)} && (11)\\
&= \frac{P(W_2\,|\,W_1,B_1,I_0)\cdot P(W_1\,|\,B_1,I_0)}{P(W_2\,|\,W_1,B_2,I_0)\cdot P(W_1\,|\,B_2,I_0)}\times\frac{P(B_1\,|\,I_0)}{P(B_2\,|\,I_0)} && (12)\\
&= \frac{P(W_2\,|\,B_1,I_0)}{P(W_2\,|\,B_2,I_0)}\times\frac{P(W_1\,|\,B_1,I_0)}{P(W_1\,|\,B_2,I_0)}\times\frac{P(B_1\,|\,I_0)}{P(B_2\,|\,I_0)} && (13)\\
&= \frac{P(W_2\,|\,B_1,I_0)}{P(W_2\,|\,B_2,I_0)}\times\frac{P(B_1\,|\,W_1,I_0)}{P(B_2\,|\,W_1,I_0)}\,, && (14)
\end{aligned}$$

that, using the compact notation introduced above, we can rewrite in the following enlighting forms. The first is [Eq. (14)]

$$O_{1,2}(W_1,W_2,I) \;=\; \tilde{O}_{1,2}(W_2,I)\times O_{1,2}(W_1,I)\,, \qquad (15)$$

that is, *the final odds after the first inference become the initial odds of the second inference* (and so on, if there are several pieces of evidence). Therefore, beginning from a situation in which B_1 was thirteen times more credible than B_2 is exactly equivalent to having started from unitary odds updated by a factor 13 due to a piece of evidence.

The second form comes from Eq. (13):

$$\begin{aligned}
O_{1,2}(W_1,W_2,I) &= \tilde{O}_{1,2}(W_1,I)\times\tilde{O}_{1,2}(W_2,I)\times O_{1,2}(I) && (16)\\
&= \tilde{O}_{1,2}(W_1,W_2,I)\times O_{1,2}(I)\,, && (17)
\end{aligned}$$

i.e.[12]

$$\tilde{O}_{1,2}(W_1,W_2,I) \;=\; \tilde{O}_{1,2}(W_1,I)\times\tilde{O}_{1,2}(W_2,I)\;: \qquad (18)$$

[11] Note that we are still using Eq. (4), although we are dealing now with more complex events and complex hypotheses, logical AND of simpler ones. Moreover, Eq. (12) is obtained from Eq. (11) making use of the formula (2) of joint probability, that gives $P(W_1,W_2\,|\,B_1,I) = P(W_2\,|\,W_1,B_1,I)\times P(W_1\,|\,B_1,I)$ and an analogous formula for B_2. Note also that, going from Eq. (12) to Eq. (13), $P(W_2\,|\,W_1,B_i,I_0)$ has been rewritten as $P(W_2\,|\,B_i,I_0)$ to emphasize that the probability of a second white ball, conditioned by the box composition and the result of the first extraction, depends indeed only on the box content and not on the previous outcome ('extraction after re-introduction').

[12] Eq. (17) follows from Eq. (16) because a Bayes factor can be defined as the ratio of final odds over the

Bayes factors due to *independent*[13] pieces of evidence *multiply.* That is, two independent pieces of evidence (W_1 and W_2) are equivalent to a single piece of evidence ('$W_1 \cap W_2$'), whose Bayes factor is the product of the individual ones. In our case $\tilde{O}_{1,2}(W_1 \cap W_2, I) = 169$.

In general, if we have several hypotheses H_i and several *independent* pieces of evidence, E_1, E_2, ..., E_n, indicated all together as $\boldsymbol{E}$, then Eq. (4) becomes

$$O_{i,j}(\boldsymbol{E}, I) \;=\; \left[\prod_{k=1}^{n} \tilde{O}_{i,j}(E_k, I)\right] \times O_{i,j}(I)\,, \qquad (19)$$

i.e.

$$\tilde{O}_{i,j}(\boldsymbol{E}, I) \;=\; \prod_{k=1}^{n} \tilde{O}_{i,j}(E_k, I)\,, \qquad (20)$$

where $\prod$ stand for 'product' (analogous to $\sum$ for sums).

2.4 How the independent arguments sum up in our judgement – logarithmic updating and its interpretation

The remark that Bayes factors due to independent pieces of evidence multiply together and the overall factor finally multiplies the initial odds suggests a change of variables in order to play with additive quantities.[14] This can be done taking the logarithm of both sides of Eq. (19), that then become

$$\log_{10}[O_{i,j}(\boldsymbol{E}, I)] \;=\; \sum_{k=1}^{n} \log_{10}[\tilde{O}_{i,j}(E_k, I)] + \log_{10}[O_{i,j}(I)]\,, \qquad (21)$$

initial odds, depending on the evidence. Therefore

$$\tilde{O}_{1,2}(W_1, W_2, I) \;=\; \frac{O_{1,2}(W_1, W_2, I)}{O_{1,2}(I)} = \tilde{O}_{1,2}(W_1, I) \times \tilde{O}_{1,2}(W_2, I)\,.$$

[13] Probabilistic, or 'stochastic', independence of the observations is related to the validity of the relation $P(W_2 \,|\, W_1, B_i, I) = P(W_2 \,|\, B_i, I)$, that we have used above to turn Eq. (12) into Eq. (13) and that can be expressed, in general terms as

$$P(E_2 \,|\, E_1, H_i, I) = P(E_2 \,|\, H_i, I)\,,$$

i.e., *under the condition of a well precise hypothesis* (H_i), the probability of the effect E_2 does not depend on the knowledge of whether E_1 has occurred or not. Note that, in general, although E_1 and E_2 are independent given H_i (they are said to be *conditionally independent*), they might be otherwise *dependent*, i.e. $P(E_2 \,|\, E_1, I_0) \neq P(E_2 \,|\, I_0)$. (Going to the example of the boxes, it is rather easy to grasp, although I cannot enter in details here, that, if we do not know the kind of box, the observation of W_1 changes our opinion about the box composition and, as a consequence, the probability of W_2 – see the examples in Appendix J)

[14] The idea of transforming a multiplicative updating into an additive one via the use of logarithms is quite natural and seems to have been firstly used in 1878 by Charles Sanders Peirce [6] and finally introduced in the statistical practice mainly due to the work of I.J. Good [7]. For more details see the Appendix E.

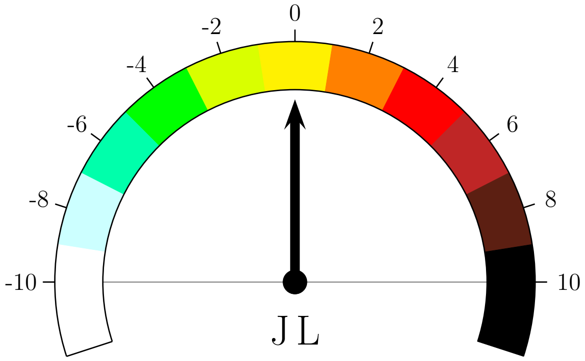

Figure 3: Judgement leaning.

respectively, where the base 10 is chosen for practical convenience because, as we shall discuss later, what substantially matters are powers of ten of the odds.

Introducing the new symbol JL, we can rewrite Eq. (21) as

$$\begin{aligned} \mathrm{JL}_{i,j}(\boldsymbol{E},I) &= \mathrm{JL}_{i,j}(I)+\sum_{k=1}^{n}\Delta\mathrm{JL}_{i,j}(E_k,I) \quad (22)\\ &= \mathrm{JL}_{i,j}(I)+\Delta\mathrm{JL}_{i,j}(\boldsymbol{E},I) \quad (23)\end{aligned}$$

or

$$\Delta\mathrm{JL}_{i,j}(\boldsymbol{E},I) = \mathrm{JL}_{i,j}(\boldsymbol{E},I)-\mathrm{JL}_{i,j}(I)\,, \quad (24)$$

where

$$\begin{aligned} \mathrm{JL}_{i,j}(\boldsymbol{E},I) &= \log_{10}\left[O_{ij}(\boldsymbol{E},I)\right] \quad (25)\\ \mathrm{JL}_{i,j}(I) &= \log_{10}\left[O_{i,j}(I)\right] \quad (26)\\ \Delta\mathrm{JL}_{i,j}(E_k,I) &= \log_{10}\left[\tilde{O}_{i,j}(E_k,I)\right] \quad (27)\\ \Delta\mathrm{JL}_{i,j}(\boldsymbol{E},I) &= \sum_{k=1}^{n}\Delta\mathrm{JL}_{i,j}(E_k,I)\,. \quad (28)\end{aligned}$$

The letter ‘L’ in the symbol is to remind *logarithm.* But it has also the mnemonic meaning of *leaning*, in the sense of ‘inclination’ or ‘propension’. The ‘J’ is for *judgment.* Therefore ‘JL’ stands for *judgement leaning*, that is an inclination of the judgement, an expression I have taken the liberty to introduce, using words not already engaged in probability and statistics, because in these fields many controversies are due to different meanings attributed to the same word, or expression, by different people (see Appendices B and G for further comments). JL can then be visualized as the indicator of the ‘justice balance’[15] (figure

[15] I have realized only later that JL sounds a bit like ‘jail’. That might be not so bad, if H_1 to which $\mathrm{JL}_{1,2}(E_k)$ refers stands for ‘guilty’.

3), that displays zero if there is no unbalance, but it could move to the positive or the negative side depending on the weight of the several arguments pro and con. The role of the evidence is to vary the JL indicator by quantities ΔJL's equal to base 10 logarithms of the Bayes factors, that have then a meaning of *weight of evidence*, an expression due to Charles Sanders Peirce [6] (see Appendix E).

But the judgement is rarely initially unbalanced. This the role of $\text{JL}_{i,j}(I)$, that can be considered as a a kind of *initial weight of evidence* due to our prior knowledge about the hypotheses H_i and H_j [and that could even be written as $\Delta\text{JL}_{i,j}(E_0, I)$, to stress that it is related to a 0-th piece of evidence]

To understand the rationale behind a possible uniform treatment of the prior as it would be a piece of evidence, let us start from a case in which you now *absolutely nothing.* For example You have to state your beliefs on which of my friends, Dino or Paolo, will first run next Rome marathon. It is absolutely reasonable you assign to the two hypotheses equal probabilities, i.e. $O_{1,2} = 1$, or $\text{JL}_{1,2} = 0$ (your judgement is perfectly balanced). This is because in Your brain these names are only possibly related to Italian males. Nothing more. (But nowadays search engines over the web allow to modify your opinion in minutes.)

As soon as you deal with *real* hypotheses of your interest, things get quite different. It is in fact very rare the case in which the hypotheses tell you not more than their names. It is enough you think at the hypotheses 'rain' or 'not rain', the day after you read these lines in the place where you live. In general the information you have in your brain related to the hypotheses of your interest can be considered the initial piece of evidence *you* have, usually different from that somebody else might have (this the role of I in all our expressions). It follows that prior odds of 10 (JL = 1) will influence your leaning towards one hypothesis, exactly like unitary odds (JL = 0) followed by a Bayes factor of 10 (ΔJL = 1). This the reason they enter on equal foot when "balancing arguments" (to use an expression à la Peirce – see the Appendix E) pro and against hypotheses.

Finally, table 1 compares judgements leanings, odds and probabilities, to show that the human sensitivity to belief (that is something like Peirce's *intensity of belief* – see Appendix E) is not linear with probability. For example, if we assign probabilities of 44%, 50% or 56% to events E_1, E_2 and E_3 we do not expect one of them really more strongly than the others, in the sense that we are not much *surprised* of any of the three occurs. But the same differences in probability produce quite different *sentiment of surprise* if we shift the probability scale (if they were, instead, 1%, 7% and 13%, we would be highly surprised if E_1 occurs).

Similarly 99.9% probability on H is substantially different from 99.0%, although the difference in probability is 'only' 0.9%. This is well understood, and in fact it is known that the best way to express the perception of probability values very close to 1 is to think to the opposite hypothesis $\overline{H}$, that is 0.1% probable in the first case and 1% probable in the second – we could be quite differently surprised if H does not result to be true in the two cases![16]

[16] The 'switch of perspective' from E to $\overline{H}$ is done in a way somewhat automatic if, instead of the

Judg. leaning	Odds(1:2)	$P(H_1)$		Judg. leaning	Odds(1:2)	$P(H_1)$
[$\text{JL}_{1,2}$]	[$O_{1,2}$]	(%)		$\text{JL}_{1,2}$]	[$O_{1,2}$]	(%)
0	1.0	50				
−0.1	0.79	44		0.1	1.3	56
−0.2	0.63	39		0.2	1.6	61
−0.3	0.50	33		0.2	2.0	67
−0.4	0.40	28		0.4	2.5	71
−0.5	0.32	24		0.5	3.2	76
−0.6	0.25	20		0.6	4.0	80
−0.7	0.20	17		0.7	5.0	83
−0.8	0.16	14		0.8	6.3	86
−0.9	0.13	11		0.9	7.9	89
−1.0	0.10	9.1		1.0	10	91
−1.1	0.079	7.4		1.1	13	92.6
−1.2	0.063	5.9		1.2	16	94.1
−1.3	0.050	4.7		1.0	20	95.2
−1.4	0.040	3.8		1.4	25	96.2
−1.5	0.032	3.1		1.5	32	96.9
−1.6	0.025	2.5		1.6	40	97.5
−1.7	0.020	2.0		1.7	50	98.0
−1.8	0.016	1.6		1.8	63	98.4
−1.9	0.013	1.2		1.9	80	98.8
−2.0	0.010	1.0		2.0	100	99.0

Table 1: A comparison between probability, odds and judgement leanings

From the table we can see that the *human resolution* is about 1/10 of the JL, although this does not imply that a probability value of 53.85% (JL = 0.0670) cannot be stated. It all depends how this value has been evaluated and what is the purpose of it.[17]

probability, we take the logarithm of the odds, for example our JL (obviously the base of the logarithm is irrelevant). Since $\text{JL}_H(I) = \log_{10}[P(H\,|\,I)/P(\overline{H}\,|\,I)]$, in the limit $P(H\,|\,I) \to 0$ we have that $\text{JL}_H(I) \approx \log_{10}[P(H\,|\,I)]$, while the limit $P(H\,|\,I) \to 1$ it is $\text{JL}_H(I) \approx -\log_{10}[P(\overline{H}\,|\,I)]$.

[17] This is more or less what happens in measurements. Take for example the probabilities that appears in the E_1 'monitor' of figure 11: 53.85% for white and 46.15% for black. This is like to say that two bodies weigh 53.85 g and 46.15 g, as resulting from a measurement with a precise balance (the Bayesian network tool described in Appendix J applied to the box toy model is the analogue of the precise balance). For some purposes two, three and even four significant digits can be important. But, anyhow, as far as our perception is concerned, not only the least digits are absolutely irrelevant but we can hardly distinguish between 54 g and 46 g.

2.5 Recap of the section

This section had the purpose of introducing the so-called Bayesian reasoning (that is, in reality, nothing more than just probabilistic reasoning) with an aseptic, simple example, that shows however the ingredients needed to update our opinion on the light of new observations. At this point the role of the priors and of the evidence in forming our opinion about the hypotheses of interest should be clear. Note also how I have used on purpose several expressions to mean essentially the same thing, expressions that involve words such as ‘probability’, ‘belief’, ‘plausibility’, ‘credibility’, ‘confidence’, and so on.

3 Weight of priors and weight of evidence in real life

The box example used to introduce the Bayesian reasoning was particularly simple for two reasons. First, the updating factor was calculated from elementary probability rules in an ‘objective way’ (in the sense that everybody would agree on a Bayes factor of 13, corresponding to a ΔJL of 1.1). Second, also the prior odds n_1/n_2 were univocally determined by the formulation of the problem.

In real life the situations are never so simple. Not only priors can differ a lot from a person to another. Also the probabilities that enter the Bayes factor might not be the same for everybody. Simply because they are probabilities, and probabilities, meant as degree of belief, have an intrinsic *subjective* nature [10]. The very reason for this trivial remark (although not accepted by everybody, because of ideological reasons) is that probability depends on the available information and – fortunately! – there are no two identical brains in the world, made exactly the same way and sharing exactly the same information. Therefore, the same event is not expected with the same security by different subjects, and the same hypothesis is not considered equally credible.[18]

At most degrees of belief can be *inter-subjective*, because in many cases there are people or entire communities that share the same initial beliefs (the same *culture*), reason more or less the same way (similar *brains* and similar *education*) and have access to the *same data.*

[18] The following quotes can be rather enlighting, especially for those who think they think, just for educational reasons, ‘they have to be frequentist’:

> *“Given the state of our knowledge about everything that could possibly have any bearing on the coming true of a certain event (thus in dubio: of the sum total of our knowledge), the numerical probability p of this event is to be a real number by the indication of which we try in some cases to set up a quantitative measure of the strength of our conjecture or anticipation, founded on the said knowledge, that the event comes true.*
>
> ...
>
> *Since the knowledge may be different with different persons or with the same person at different times, they may anticipate the same event with more or less confidence, and thus different numerical probabilities may be attached to the same event. ... Thus whenever we speak loosely of the ‘probability of an event,’ it is always to be understood: probability with regard to a certain given state of knowledge.”* [11]

Finally, there are stereotyped 'games' in which probabilities can even be *objective*, in the sense that everybody will agree on its value. But these situations have to be considered the exceptions rather than the rule (and even when we state with great security that the probability of head tossing a regular coin is exactly 1/2, we forget it could remain vertically, a possibility usually excluded but that I have personally experienced a couple of times in my life.)

Therefore, although educational games with boxes and balls might be useful to learn the grammar and syntax of probabilistic reasoning, at a given point we need to move to real situations.

3.1 Assessing subjective degrees of beliefs – virtual bets

A good way to force experts to provide the initial beliefs they have formed in their minds, elaborated somehow by their 'educated intuition' (see Appendix C), is to propose them a virtual lottery, in which they can choose the event on which to bet to win a rich prize. One is the event of interest (let us call it A), the other one is a simpler one, based on coins, urns, dice or card games. The latter can be considered a kind of 'standard', or a 'reference' (as it is done in measurements to calibrate instruments), whose probability is the same for everyone. We can ask ourselves (or the experts), for example, if we (or they) prefer to bet on A rather than on head resulting from a regular coin; or on white extracting a ball from a box containing 100 balls, 90 of which white; and so on.

Obviously, none can state initial odds with very high precision.[19] But this does not matter (table 1 can help to get the point). We want to understand if they are of the order of 1 (equally likely), of the order of a few units (one is a bit more likely than the other one), or of suitable powers of 10 (much more or much less likely than the other one). If one has doubts about the final result, one can make a 'sensitivity analysis', i.e. vary the value inside a wide but still believable range and check how the result changes. The sensitivity (or insensitivity) will depend also on the other pieces of evidence to draw the final conclusion. Take for example two different evidences, characterized by Bayes factors of H_1 versus H_2 very high (e.g. 10^4) or very small (e.g. 10^{-4}), corresponding to ΔJL's of $+4$ or -4, respectively (for the moment we assume all subjects agree on the evaluation of these factors). Given these values, it is easy to check that, for many practical purposes, the

[19] Those who are not familiar with this approach have understandable initial difficulties and risk to be at lost. A formula, they might argue, can be of practical use only if we can replace the symbols by numbers, and in pure mathematics a number is a well defined object, being, for example, 49.999999 different from 50. Therefore, they might conclude that, being unable to choose *the* number, the above formulae, that seem to work nicely in die/coin/ball games, are useless in other domains of applications (the most interesting of all, as it was clear already centuries ago to Leibniz and Hume). But in the realm of uncertainty things go quite differently, as everybody understands, apart from hypothetical Pythagorean monks living in a ivory monastery. For practical purposes not only 49.999999% is 'identical' to 50%, but also 49% and 51% give to our mind essentially the same expectations of what it could occur. In practice we are interested to understand if somebody else's degrees of belief are low, very low, high, very very high, ad so on. And the same is what other people expect from us.

conclusions will be the same even if the initial odds are in the range 1/10 to 10, i.e. a JL between -1 and $+1$, that can be stated as $\mathrm{JL}_{1,2}(E_0) = 0 \pm 1$. Adding 'weights of evidence' of $+4$ or -4, we get final JL's of 4 ± 1 or -4 ± 1, respectively.[20]

The limit case in which the Bayes factor is zero or infinity (i.e. ΔJL's $-\infty$ or $+\infty$) makes the conclusion absolutely independent from priors, as it seems obvious.

3.2 Beliefs versus frequencies

At this point a remark on the important (and often misunderstood) issue of the relation between degrees of beliefs and relative frequencies is in order.

The concept of subjective probability does not preclude the use of *relative frequencies* in the reasonings. In particular, beliefs can be evaluated from the relative frequencies of other events, analogous to the one of interest, have occurred in the past. This can be done roughly (see Hume's quote in Appendix B) or in a rigorous way, using probability theory under well defined assumptions (Bayes' theorem applied to the inference of the parameter p of a binomial distribution).

Similarly, if we believe that a given event will occur with 80% probability, it is absolutely correct to say that, if we think at a large number of analogous independent events that we consider equally probable, we expect that in about 80% of the cases the event will occur. This also comes from probability theory (Bernoulli theorem).

This means that, contrary to what one reads often in the literature and on the web, evaluating probabilities from past frequencies and expressing beliefs by expected frequencies does not imply to adhere to *frequentism*.[21] The importance of this remark in the context of this paper is that people might find natural, for their reasons, to evaluate and to express beliefs this way, although they are perfectly aware that the event about they are reasoning is unique. For further comments see Appendix B.

3.3 Subjective evaluation of Bayes factors

As we have mentioned above, and as we shall see later, not always the evaluation of updating factors can be done with the help of mathematical formulae like in the box example. However, we can make use of the virtual bet in this case too, remembering that a Bayes factor can be considered as the odds due a single piece of evidence, provided the two hypotheses are considered otherwise equally likely (hence, let us remember, the symbol $\tilde{O}$ used here to indicate Bayes factors).

[20] That is, the final probability of H_1 would range between 99.90% and 99.999% in the first case, between 0.001% and 0.1% in the second one, making us 'practically sure' of either hypothesis in the two cases.

[21] Sometimes frequency is even confused with 'proportion' when it is said, for example, that the probability is evaluated thinking how many persons in a given population would behave in a given way, or have a well defined character.

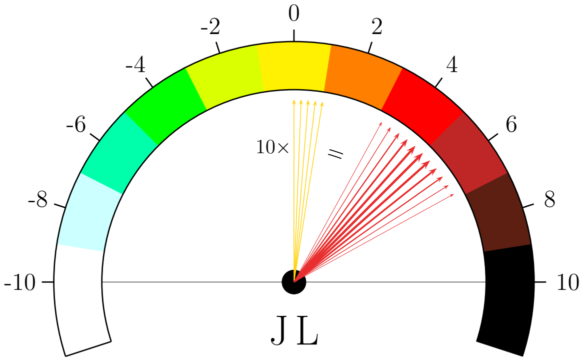

Figure 4: Combined effect of 10 weak and 'vague' pieces of evidence, each yielding a ΔJL of 0.5 ± 0.5 (see text).

3.4 Combining uncertain priors and uncertain weights of evidence

When we have set up our problem, listed the pieces of evidence pro and con, including the 0-th one (the prior), and attributed to each of them a weight of evidence, quantified by the corresponding ΔJL's, we can finally sum up all contributions.

As it is easy to understand, if the number of pieces of evidence becomes large, the final judgment can be rather precise and far from being perfectly balanced, even if each contribution is weak and even uncertain. This is an effect of the famous 'central limit theorem' that dumps the weight of the values far from the average.[22] Take for example the case of 10 JL's each uniformly[23] ranging between 0 and 1, i.e. $\Delta\text{JL}_{1,2}(E_i, I) = 0.5 \pm 0.5$.

[22] The reason behind it is rather easy to grasp. When we have uncertain beliefs it is like if *our mind oscillates* among possible values, without being able to choose an exact value. Exactly as it happens when we try to guess, just by eye, the length of a stick, the weight of an object or a temperature in a room: extreme values are promptly rejected, and our judgement oscillates in an interval, whose width depends on our estimation ability, based on previous experience. Our guess will be somehow the center of the interval. The following minimalist example helps to understand the rule of combination of uncertain evaluations. Imagine that the (not better defined) quantities x and y might each have, in our opinion, the values 1, 2 or 3, among which we are unable to choose. If we now think of a $z = x + y$, its value can then range between 2 and 6. But, if our mind oscillates uniformly and independently over the three possibilities of x and y, the oscillation over the values of z is not uniform. The reason is that $z = 2$ can is only related to $x = 1$ and $y = 1$. Instead, we think at $z = 3$ if we think at $x = 1$ and $y = 2$, or at $x = 2$ and $y = 1$. Playing with a cross table of possibilities, it is rather easy to prove that $z = 4$ gets a weight three times larger than that of $z = 2$. We can add a third quantity v, similar to x and y, and continue the exercise, understanding then the essence of what is called in probability theory *central limit theorem*, which then applies also to the weight of our JL's. [Solution and comment: if $w = z + v$, the weights of the 7 possibilities, from 3 to 9 are in the following proportions: 1:3:6:7:6:3:1. Note that, contrary to z, the weights do not go linearly up and down, but there is a non-linear concentration at the center. When many variables of this kind are combined together, then the distribution of weights exhibits the well known bell shape of the Gaussian distribution. The widths of the red arrows in figure 4 tail off from the central one according to a Gaussian function.]

[23] It easy to understand that if the judgement would be uniform in the odds, ranging then from 1 to 10, the conclusion could be different. Here it is assumed that the 'intensity of belief'[6] is proportional to the logarithm of the odds, as extensively discussed in Appendix E.

Each piece of evidence is marginal, but the sum leads to a combined $\Delta\mathrm{JL}_{1,2}(\boldsymbol{E}, I)$ of 5.0±1.8, where "$[3.2, 6.8]$" defines now an effective range of leanings[24], as depicted in figure 4. Note that in this graphical representation the 5 yellow arrows (the lighter ones if you are reading the text in black/white) do not represent individual values of JL, but its interval. These arrows have all the same width to indicate that the exact value is indifferent to us. The red arrow have instead different widths to indicate that we prefer the values around 5 and the preference goes down as we move far form it. The 12 arrows only indicate an effective range, because the full range goes from 0 to 10, although ΔJL values very far from 5 must have negligible weight in our reasoning.

3.5 Agatha Christie's "three pieces of evidence"

As we have seen, a single evidence, yielding a Bayes factor of the order of 10, or a ΔJL around 1, is not a strong evidence. But many individual, independent pieces of evidence of that weight should have much a greater consideration in our judgement.

This is, somehow, the rational behind Agatha Christie's "three pieces of evidence". However it is worth remarking that something is to say there is a rational behind this expression, that can be used as a rough rule of thumb, something else is to take it as a 'principle', as it is often supposed in the Italian dictum "tre indizi fanno una prova". First, pieces of evidence are usually not 'equally strong', in the sense they do not carry the same weight of evidence and sometimes even several pieces of evidence are not enough.[25] Second, the prior – that is our '0-th evidence – can completely balance the weight of evidence. Finally, we have also to remember that sometimes they are not even completely independent, in which case the product rule is not any longer valid.[26]

A final remark on the combination of pieces of evidence is still in order. From a mathematical point of view there is no difference between a single piece of evidence yielding a tremendous Bayes factor of 10^{10} ($\Delta\mathrm{JL} = 10$) and ten independent pieces of evidence, each having the more modest Bayes factor of 10 ($\Delta\mathrm{JL} = 1$). However, I have somehow the impression (mainly got from media and from fiction, since I have no direct experience of courts) that the first is considered as *the* incriminating evidence (the 'smoking gun'), while the ten weak pieces of evidence are just taken as some *marginal indications*, that all together are not as relevant as the single incriminating 'proof'. Not only this reasoning is

[24] Using the language of footnote 22, this is the range in which the minds oscillate in 95% of the times when thinking of $\Delta\mathrm{JL}_{1,2}(\boldsymbol{E}, I)$.

[25] I wish judges state Bayes factors of *each* piece of evidence, as vaguely as they like (much better than telling nothing! – Bruno de Finetti was used to say that "*it is better to build on sand that on void*"), instead of saying that somebody is guilty "behind any reasonable doubt" – and I am really curious to check to what degree of belief that level of doubt corresponds!

[26] What to do in this case? As it easy to imagine, when the structure of dependencies among evidences is complex, things might become quite complicated. Anyway, if one is able to isolate two o more pieces of evidence that are correlated with themselves (let they be E_1 and E_2), then, one can consider the joint event $E_{1\&2} = E_1 \cap E_2$ as the effective evidence to be used. In the extreme case in which E_1 implies logically E_2 (think at the events 'even' and '2' rolling a die), then $P(E_2 \,|\, E_1, I) = 1$, from which it follows that $P(E_1 \cap E_2 \,|\, I) = P(E_1 \,|\, I)$: the second evidence E_2 is therefore simply superfluous.

mathematically incorrect, as we have learned, but, if I were called to state my opinion on the two sets of evidence, I had no doubt to consider the ten weak pieces of evidence more incriminating than the single 'strong' one, although they seem to be formally equivalent. Where is the point? In all reasonings done until now we have focused on the weight of evidence, assuming each evidence is a true and not a fake one, for instance incorrectly reported, or even fabricated by the investigators. In real cases one has to take into account also this possibility.[27] As a consequence, if there is any slight doubt on the validity of each piece of evidence, it is rather simple to understand that the single evidence is somewhat weaker than the ten ones all together (Agatha Christie's three pieces of evidence are in qualitative agreement with this remark). For further details see Appendix I.

3.6 Critical values for guilt/innocence – Assessing beliefs versus making decisions

At this point a natural question raises spontaneously. What is the possible threshold of odds or of JL's to condemn or to absolve somebody? This is a problem of a different kind. It is not just a question of believing something, but on deciding which action to take.

Decision issues are a bit more complicate than probability ones. Not only they inherit all probabilistic questions, but they need careful considerations of all possible benefits and losses resulting from the action. I am not a judge and fortunately I have never been called to join a popular jury, on the validity of which I have, by the way, quite some doubts.[28] So I do not know exactly how they make their decisions, but personally, being 99% confident that somebody is guilty (that is a JL of 2), I would not behave the same way if the person is accused of a 'simple' crime of passion, or of being a Mafia or a serial killer.

3.6.1 I have a dream

I hope judges know what they do, but I wish one day they will finally state somehow, in a quantitative way, with all possible uncertainties, the beliefs they have in their mind, the individual contributions they have considered and the society benefits and losses taken into account to behave the way they did.

[27] When we are called to make critical decisions even very remote hypotheses, although with very low probability, should be present to our minds – that is Dennis Lindley's *Cromwell's rule* [18]. [The very recent news from New York offer material for reflection [19].]

[28] Again, my impression comes from media, literature and fiction, but I cannot see how 'casual judges' can be better than professional ones to evaluate all elements of a complex trial, or how to distinguish sound arguments from pure rhetoric of the lawyers. This is particularly true when the 'network of evidences' is so intricate that even well trained human minds might have difficulties, and *artificial intelligence* tools would be more appropriated (see Appendices C and J).

4 Columbo's priors versus jury's priors

Going back to Columbo's episode, the prior of interest here is the probability that Peter Galesco killed his wife, taking into account 'all' pieces of evidence *but* those deriving from the last scene.

It is interesting to observe how *probabilities* change as the story goes on. Different characters develop different opinions, depending on their previous experience, on the information they get and on their capability to process the information quickly. Also each spectator forms his/her own opinion, although all of them get virtually the same 'external' pieces of information (that however are combined with internal pre-existing ones, whose combination and rapidity of combination depend on many other internal things and environmental conditions) – and this is part of the fun of watching a thriller with friends.

4.1 Columbo's priors

By definition a person suspected by a detective is not just anybody, whose name was extracted at random from the list of citizens in the region where the crime was committed. Police does not like to lose time, money and reputation, if it does not have valid suspicions, and investigations proceed in different directions, with priorities proportional to the chance of success. The probabilities of the various hypotheses go up and down as the story goes on, and an alibi or a witness could drop a probability to zero (but policemen are aware of fake alibis or lying witnesses).

If we see Columbo loosing sleep following some hints, we understand he has strong suspicions. Or, at least, he is not convinced of the official version of the facts, swallowed instead by his colleagues: some elements of the puzzle do not fit nicely together or, told in probabilistic terms, the *network of beliefs*[29] he has in mind[30] makes him **highly confident** that the suspected person is guilty.

4.2 Court priors

But a policeman is not the court that finally returns the verdict. Judges tend, by their experience, to trust policemen, but they cannot have exactly the same information the direct investigators have, that is not limited to what appears in the official reports.

Columbo might have formed his opinion on instinctive reactions of Galesco, on some photographer's hints of smile or on nervous replies to fastidious questions, and so on, all little things the lieutenant knows they cannot enter in the formal suit.[31] We can form

[29] See Appendices C and J.

[30] Obviously, saying Columbo has a network of beliefs in his head, I don't mean he is thinking at these mathematical tools. On the other way around, these tools try to model the way we reason, with the advantage they can better handle complex situations (see Appendices C and J).

[31] There is, for example, the interesting case of the clochard who was on the scene of the crime and, although still drunk, tells, among other verifiable things, to have heard two gun shots with a remarkable time gap in between, something in absolute contradiction with Galesco reconstruction of the facts, in which

ourself an idea about the prior probability that the court can assign to the hypothesis that the photographer is guilty from the reaction of Columbo's colleagues and superiors, who try to convince him the case is settled: **quite low**.

4.3 Effect of a Bayes factor of 13

To evaluate how a new piece of evidence modifies these levels of confidence, we need to quantify somehow the different priors. Since, as we have seen above, what really matters in these cases are the powers of ten of the odds, we could place Columbo's ones in the region 10^2-10^3, the hypothetical jury ones around 10^{-2}, perhaps up to 10^{-1}. Multiplying these values by 13 we see that, while the lieutenant would be practically sure Galesco is guilty, the jury component could hardly reach the level of a sound suspicion.

Using the expressions of subsection 2.4, a Bayes factor of 13 corresponds to $\Delta\text{JL} = 1.1$, that, added to initial leanings of $\approx 2.5 \pm 0.5$ (Colombo) and $\approx -1.5 \pm 0.5$ (jury), could lead to combined JL's of 3.6 ± 0.5 or -0.4 ± 0.5 in the two cases.

However, although such a small weight of evidence is not enough, by itself, to condemn a person, I do not agree that *"that kind of evidence would never stand up in court"*[1] for the reasons expounded in section 3.4.

Nevertheless, my main point in this paper is not that even such a modest piece of evidence *should* stand up in court (provided it is not the only one), but rather that the weight of evidence provided by the rash Galesco's act is not 1.1, but much higher, infinitely higher.

5 The weight of evidence of the full sequence of actions

In the previous section we have done the exercise of assuming a Bayes factor of 13, that is a weight of evidence of 1.1, as if taking that camera would be the same as extracting a ball from a box, as in the introductory example. But does this look reasonable?

5.1 The 'negative reaction'

Let us summarize what happens in the last scene of the episode.

- Galesco is suddenly taken to the police station, where he is waited for by Columbo, who receives him not in his office but in a kind of repository containing shelfs full of

he states to have killed Alvin Deschler, that he pretends to be the kidnapper and murderer of his wife, for self-defense, thus shooting practically simultaneously with him. Unfortunately, days after, when the clochard is interviewed by Columbo, he says, apparently honestly, to remember nothing of what happened the day of the crime, because he was completely drunk. He confesses he doesn't even remember what he declared to the police immediately after. Therefore he could never be able to testify in a court. However, it is difficult an investigator would remove such a piece of evidence from his mind, a piece of evidence that fits well with the alternative hypothesis that starts to account better for many other major and minor details. He knows he cannot present it to the court, but it pushes him to go further, looking for more 'presentable' pieces of evidence, and possibly for conclusive proofs.

objects (see figure 6), including the cameras in question, a few of which are visible behind Columbo, although nobody mentions them.

- Columbo starts arguing about the rest of the newspaper found in Deschler's motel room and used to cut out the words glued in the kidnapper's note. The missing bits and pieces support the hypothesis that the collage was not done by Deschler. Galesco, usually very prompt in suggesting explanations to Columbo's doubts and insinuations, is surprised by the frontal attack of the lieutenant, who until that moment only expressed him a series of doubts. He gets then quite upset.

- Immediately after, Columbo announces his final proof, meant to destroy Galesco's alibi. He has prepared a giant enlargement of the picture of Mrs Galesco taken by the murderer just before she was killed. The photograph shows clearly a clock indicating exactly ten (A.M.), time at which the lady had to be with her husband, while Deschler had a very solid alibi, that morning doing the driving test to get his licence.

- The expert photographer refuses this new reconstruction, on the ground that, he claims, there is a mistake in the enlargement, in which, he says, the picture has been erroneously reversed, thus transforming the original 2:00 (P.M.) into 10:00 (obviously, the analog clock had no digits, but just marks to indicate the hours). He asks then Columbo to check on the original.

- But Columbo acts very well in pretending he destroyed the original by accident when he was in the dark room to supervise the work (his often goofy way to behave makes the thing plausible).

- This clever move is able to stress the otherwise always lucid Galesco, who suddenly thinks he is going to fall into a trap, based on a false, incriminating evidence fabricated by the police. He gets then so nervous to loose control and, with a kind of desperate jump of a feline who sees itself lost, does his fatal mistake.

- Suddenly he has a kind of inspiration. He says the negative can prove the picture has been reversed.[32] Then he rapidly goes to the shelf, displaces one camera and, with

[32] In reality he has several ways out, not depending on that negative (this could be a weak point of the story, but it is plausible, and the dramatic force of the action induces also TV watchers to neglect this particular, as my friends and I have experienced):

1. He knew Columbo owns a second picture, discarded by the killer because of minor defects and left on the crime scene. (That was one of the several hints against Galesco, because only a maniac photographer – and certainly not Alvin Deschler – would care of the artistic quality of a picture shot just to prove a person was in his hands – think at the very poor quality pictures from real kidnappers and terrorists).
2. As an expert photographer, he had to think that the asymmetries in the picture would save him. In particular
   (a) The picture shows an asymmetric disposition of the furniture. Obviously he cannot tell which one is the correct one, but he could simply say that he was so sure it was 2:00 PM that, for

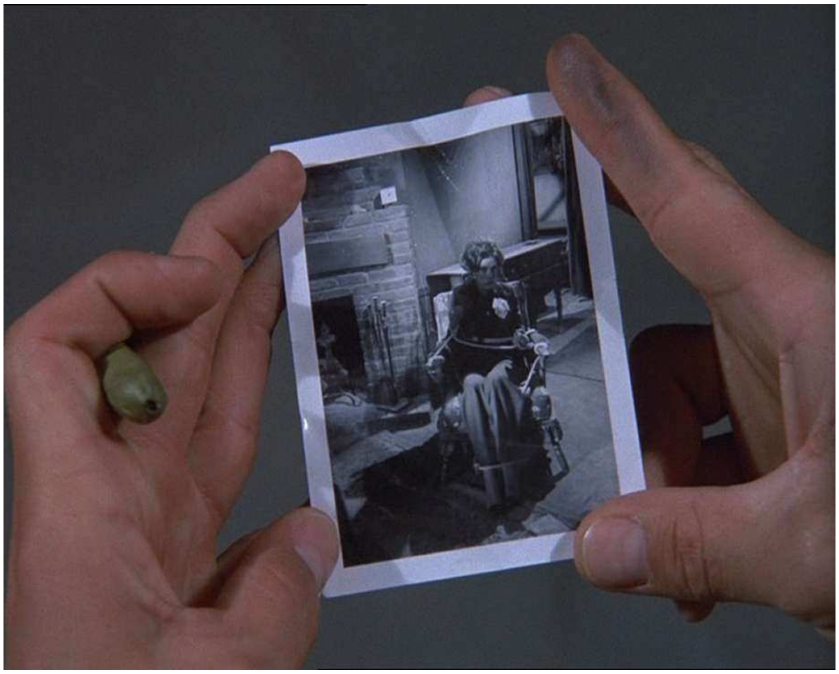

Figure 5: The discarded photograph in Columbo's hands.

> no hesitation and no sign of doubt, he picks up the one he used, that was visible, but cleverly placed in the back of others. Then, he opens that kind of old Polaroid-like camera and shows the negative inside it as the prove the picture was reversed and his alibi still valid.

But according to Columbo and his three colleagues, as well as to any TV-watcher, the *full action* incriminates him. [Finally, although the confession is irrelevant here, he realizes his mistake and his loss – murderers in the Columbo series have usually some dignity.]

5.2 How likely would have an innocent person behave that way?

As it easy to grasp, it is not just a question of picking up a camera out of 13. It is the entire sequence that is incompatible with an innocent person. Nobody, asked *directly* to pick up the camera used in a crime, would have done it on purpose, as a clear evidence of guilt.

example, the dresser had to be right of fireplace and not on its left. He could simply require to check it.

(b) Finally, his wife wore a white rosette on her left. This detail would allow him to claim with certainty that the picture has been reversed (he knew how his wife was dressed, something that could be easily verified by the police, and, moreover, rosettes hang regularly left).

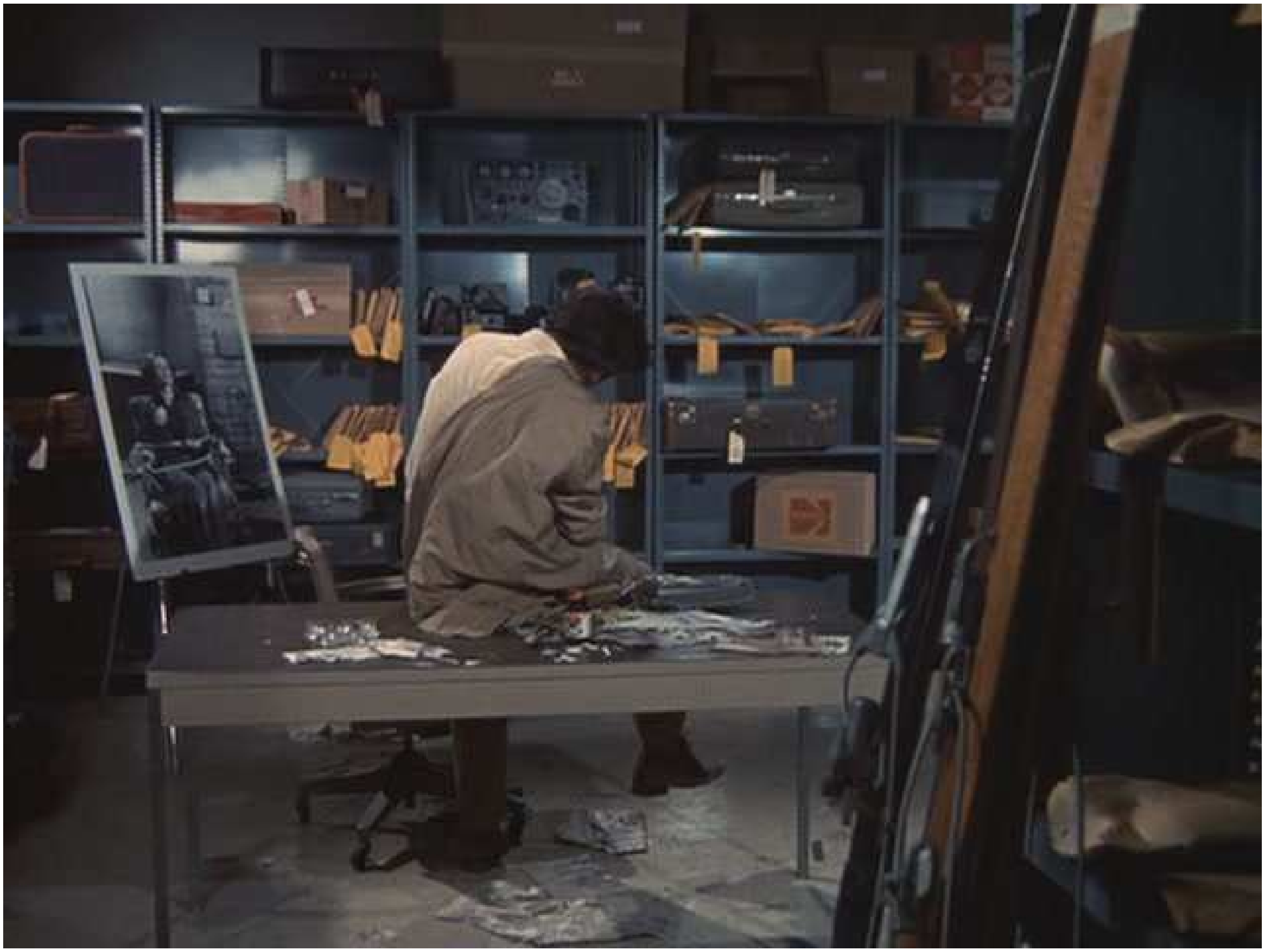

Figure 6: Final photogram of *Negative Reaction*. The incriminating camera is on the desk. The remaining twelve are in the shelf just behind Columbo's head. Desk and floor are full of the bits and pieces of the newspaper from which the lieutenant tried to reproduce the kidnapper's note.

Certainly there was an *indirect* request, implicit in the Columbo stratagem: "find the negative". But even an expert photographer would have not reacted that way if he had been innocent.

Let us assume it is reasonable he could overlook, in that particular, dramatic moment, he had other ways out (see footnote 32) and only thought at the negative of the destroyed picture. In this case he could have asked the policemen to take the camera and to look inside it. Or he would have indicated the cameras behind Columbo's shoulder, suggesting that the negative *could* be in one of those cameras.[33]

An innocent person, even put under dead stress and thinking that only the negative of the destroyed picture could save him, would perhaps jump towards the shelf, take the first camera or the first few cameras he could reach and even desperately shout "look inside them!". But he could have never resolutely displaced other cameras, taken the correct one on the back row and opened it, sure of finding the negative inside it.

But not even a cool murderer would have reacted that way, as Galesco realized a bit too late. The clever trick of Columbo was not only to ask indirectly the killer to grasp the camera he used and that only he could recognize, but, before that, to put him under stress in order to make him loose self control.

5.3 The verdict

In summary, these are our reasonable beliefs that a person would have behaved that way producing that sequence of actions ($\mathcal{A}$), depending on whether he was a killer (K) or not ($\overline{K}$), maintaining or not self-control ($SC/\overline{SC}$):

$P(\mathcal{A}\,|\,\overline{K}, I) = 0$: this is the main point, that makes the hypothesis innocent definitely impossible.

$P(\mathcal{A}\,|\,K \cap SC, I) = 0$: a cool murderer would have never reacted that way.

$P(\mathcal{A}\,|\,K \cap \overline{SC}, I) > 0$: this is the only hypotheses that can explain the action. Here '> 0' stands for 'not impossible', although not necessarily 'very probable'[34]. Let us say that, if Columbo had planned his stratagem, based on a bluff, he knew there were some chances Galesco could reacted that way, but he could not be sure about it.

Given this scenario, the 'probabilistic inversion' is rather easy, as only one hypothesis remains possible: that of a killer, who even had lost self-control.

[33] Nobody mentioned the camera was in those shelfs or even in that room! (And TV watchers didn't get the information that Galesco knew that the camera was found by the police – but this could just be a minor detail.) Moreover, only the killer and few policemen knew that the negative was left inside it by the murderer, a particular that is no obvious at all. As it was very improbable the killer used such an old-fashioned of camera. Note in fact that the camera was considered a quite old one already at the time the episode was set and it was bought in a second hand shop. In fact I remember being wondering about that writer's choice, until the very end: it was done on the purpose, so that nobody but the killer could think it was used to snap Mrs Galesco. Clever!

[34] Note that it is not required that one of the hypotheses should give with probability one, as it occurred instead of the toy example of section 2. (See also Appendix G.)

6 Comments and conclusions

Well, this was meant to be a short note. Obviously it is not just a comment to the New Scientist article, that could have been contained in a couple of sentences. In fact, discussing with several people, I felt that yet another introduction to Bayesian reasoning, not focused on physics issues, might be useful. So, at the end of the work, Columbo's cameras were just an excuse.

Let us now summarize what we have learned and make further comments on some important issues.

First, we have to be aware that often we do not see 'a fact' (e.g. Galesco killing his wife), but we infer it from other facts, assuming a causal connection among them.[35] But sometimes the observed effect can be attributed to several causes and, therefore, having observed an effect we cannot be sure about its cause. Fortunately, since our beliefs that each possible cause could produce that effect are not equal, the observation modifies our beliefs on the different causes. That is the essence of Bayesian reasoning. Since 'Bayesian' has several flavors[36] in the literature, I summarize the points of view expressed here:

- Probability simply states, in a quantitative way, how much we believe something. (If you like, you can reason the other way around, thinking that something is highly improbable if you would be highly surprised if it occurs.[37])

[35] A quote by David Hume is in order (the subdivision in paragraphs is mine):

> *All reasonings concerning matter of fact seem to be founded on the relation of Cause and Effect. By means of that relation alone we can go beyond the evidence of our memory and senses.*
>
> *If you were to ask a man, why he believes any matter of fact, which is absent; for instance, that his friend is in the country, or in France; he would give you a reason; and this reason would be some other fact; as a letter received from him, or the knowledge of his former resolutions and promises.*
>
> *A man finding a watch or any other machine in a desert island, would conclude that there had once been men in that island. All our reasonings concerning fact are of the same nature. And here it is constantly supposed that there is a connexion between the present fact and that which is inferred from it. Were there nothing to bind them together, the inference would be entirely precarious.*
>
> *The hearing of an articulate voice and rational discourse in the dark assures us of the presence of some person: Why? because these are the effects of the human make and fabric, and closely connected with it.*
>
> *If we anatomize all the other reasonings of this nature, we shall find that they are founded on the relation of cause and effect, and that this relation is either near or remote, direct or collateral."* [17]

I would like to observe that too often we tend to take for granted 'a fact', forgetting that we didn't really observed it, but we are *relying on a chain of testimonies and assumptions that lead to it.* But some of them might fail (see footnote 27 and Appendix I).

[36] Already in 1950 I.J. Good listed in Ref. [7] 9 'theories of probability', some of which could be called 'Bayesian' and among which de Finetti's approach, just to make an example, does not appear.

[37] It is very interesting to observe how people are differently surprised, in the sense of their emotional

- "Since the knowledge may be different with different persons or with the same person at different times, they may anticipate the same event with more or less confidence, and thus different numerical probabilities may be attached to the same event." [11] This is the subjective nature of probability.

- Initial probabilities can be elicited, with all the vagueness of the case,[38] on a pure subjective base (see Appendix C). *Virtual bets* or comparisons with reference events can be useful 'tools' to force ourselves or experts to provide quantitative statements of our/their beliefs. (See also Appendix C.)

- Probabilities can (but need not) be evaluated by past frequencies and can even be expressed in terms of expected frequencies of 'successes' in hypothetical trials. (See Appendix B.)

- Probabilities of causes are not generated, but only updated by new pieces of evidence.

- Evidence is not only the 'bare fact', but also all available information about it (see Appendix D). This point is often overlooked, as in the criticisms to Columbo's episode raised by New Scientist [1].

- The update depends on how differently we believe that the various causes might produce the same effect (see also Appendix G).

reaction, depending on the occurrence of events that they considered more or less probable. Therefore, contrary to I.J. Good – I have been a quite surprised about this – according to whom "to say that one degree of belief is more intense than another one is not intended to mean that there is more emotion attached to it" [7], I am definitively closer to the position of Hume:

> *Nothing is more free than the imagination of man; and though it cannot exceed that original stock of ideas furnished by the internal and external senses, it has unlimited power of mixing, compounding, separating, and dividing these ideas, in all the varieties of fiction and vision. It can feign a train of events, with all the appearance of reality, ascribe to them a particular time and place, conceive them as existent, and paint them out to itself with every circumstance, that belongs to any historical fact, which it believes with the greatest certainty. Wherein, therefore, consists the difference between such a fiction and belief? It lies not merely in any peculiar idea, which is annexed to such a conception as commands our assent, and which is wanting to every known fiction. For as the mind has authority over all its ideas, it could voluntarily annex this particular idea to any fiction, and consequently be able to believe whatever it pleases; contrary to what we find by daily experience. We can, in our conception, join the head of a man to the body of a horse; but it is not in our power to believe that such an animal has ever really existed.*
>
> *It follows, therefore, that the difference between fiction and* belief *lies in some sentiment or feeling, which is annexed to the latter, not to the former.* [17]

[38] To state it in an explicit way, I admit, contrary to others, that probability values can be themselves uncertain, as discussed in footnote 22. I understand that probabilistic statements about probability values might seem strange concepts (and this is the reason why I tried to avoid them in footnote 22), but I see nothing unnatural in statements of the kind "I am 50% confidence that the expert will provide a value of probability in the range between 0.4 and 0.6", as I would be ready to place a 1:1 bet on the event that the quoted probability value will be in that interval or outside it.

- The probability of a single hypothesis cannot be updated, if there isn't at least a second hypothesis to compare with, unless the hypothesis is absolutely incompatible with the effect [$P(E \,|\, H, I) = 0$, and not 'as little', for example, 10^{-9} or 10^{-23}]. Only in this special case an hypothesis is definitely falsified. (See Appendix G.)

- In particular, if there is only one hypothesis in the game, the final probability of this hypothesis will be one, no matter if it could produce the effect with very small probability (but not zero). [39]

- Initial probabilities depend on the information stored somehow in our brain; being, fortunately, each brain different from all others, it is quite natural to admit that, in lack of 'experimental data',"*quot capita, tot sententiae*". (See Appendix C.)

- In the probabilistic inference (i.e. that stems from probability theory) the updating rule is univocally defined by Bayes' theorem (hence the adjective 'Bayesian' related to these methods).

- This objective updating rule makes final beliefs virtually independent from the initial ones, if rational people all share the same 'solid' experimental information and are ready to change their opinion (the latter disposition has been named *Cromwell's rule* by Dennis Lindley [18]).

- In the simple case that two hypotheses are involved, the most convenient way to express the Bayes' rule is

  $$\text{final odds} = \text{Bayes factor} \times \text{initial odds},$$

  where the Bayes factor can be seen as the odds due to a single piece of evidence, if the two hypotheses were considered otherwise equally likely. (See also examples in Appendices F and G, as well as Appendix H, for comments on statistical methods based on likelihood.)

- In some cases – almost always in scientific applications – Bayes factors can be calculated exactly, or almost exactly, in the sense that all experts will agree. In many other real life cases their interpretation as 'virtual' odds (in the sense stated above) allows to elicit them with the bet mechanism as any subjective probability. (See Appendix C.)

- Bayes factors due to several independent pieces of evidence factorize.

- The multiplicative updating rule can be turned into an additive one using logarithms of the factors. (See Appendix E.)

[39] I have just learned from Ref. [7] of the following Sherlock Holmes' principle: "*If a hypothesis is initially very improbable but is the only one that explains the facts,then it must be accepted*". However, a few lines after, Good warns us that "if the only hypothesis that seems to explains the facts has very small initial odds, then this is itself evidence that some alternative hypotheses has been overlooked"...

- The base 10 logarithms has been preferred here because they are easily related to the orders of magnitudes of the odds and the name ‘judgement leanings’ (JL) has been chosen to have no conflict with other terms already engaged in probability and statistics.

- Each logarithmic addend has the meaning of weight of evidence, if the initial odds are taken as 0-th evidence.

- Individual contribution to the judgement might be small in module and even somehow uncertain, but, nevertheless, their combination might result into strong convincingness. (See Appendix G.)

- In most real life cases there are not just two alternative causes and two possible effects. Moreover, causes can be effects of other causes and effects can be themselves causes of other effects. All hypotheses in the game make up a complex ‘belief network’. Experts can certainly provide kinds of educated guesses to state how likely a cause can generate several effects, but the analysis of the full network goes well beyond human capabilities, as discussed more extensively in Appendix C and J.

- A next to simply case is when the evidence is mediated by a testimony. The formal treatment in Appendix I shows that, although experts can easily assess the required ingredients, the conclusions are really not so obvious.

- The question of the critical value of the judgement leaning, above which a suspected can be condemned, goes beyond the purpose of this notes, focused on belief. That is a delicate decision problem that inherits all issues of assessing beliefs, to which the evaluations of benefits and losses need to be added.

And Galesco? Come on, there is little to argue.
[Nevertheless, the reading of the instructive New Scientist article is warmly recommended!]

It is a pleasure to thank Pia and Maddalena, who introduced me Columbo, and Dino Esposito, Paolo Agnoli and Stefania Scaglia for having taken part to the post dinner jury that absolved him. The text has benefitted of the careful reading by Dino, Paolo and Enrico Franco (see in particular his interesting remark in footnote 44).

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A The rules of probability

Let us summarize here the rules that degrees of belief have to satisfy.

A.1 Basic rules

Given any hypothesis H (or H_i if we have many of them), also concerning the occurrence of an event, and a given state of information I, probability assessments have to satisfy the following relations:

1. $\boxed{0 \leq P(H \,|\, I) \leq 1}$

2. $\boxed{P(H \cup \overline{H} \,|\, I) = 1}$

3. $\boxed{P(H_i \cup H_j \,|\, I) = P(H_i \,|\, I) + P(H_j \,|\, I) \quad \text{if } H_i \text{ and } H_j \text{ cannot be true together}}$

4. $\boxed{P(H_i \cap H_j \,|\, I) = P(H_i \,|\, H_j, I) \cdot P(H_j \,|\, I) = P(H_j \,|\, H_i, I) \cdot P(H_i \,|\, I)}$

The first basic rule represents basically a conventional scale of probability, also indicated between 0 and 100%.

Basic rule 2 states that probability 1 is assigned to a *logical truth*, because either is true H or its opposite ("*tertium non datur*"). Indeed $H \cup \overline{H}$ represent a logical, tautological certainty (a *tautology*, usually indicated with Ω), while $H \cap \overline{H}$ is a *contradiction*, that is something impossible, indicated by $\emptyset$.

The first three basic rules are also known the 'axioms' of probability,[40] while the inverses of the fourth one, e.g. $P(H_i \,|\, H_j, I) = P(H_i \cap H_j \,|\, I)/P(H_j \,|\, I)$, are called in most literature "definition of conditional probability". In the approach followed here such a statement has no sense, because probability is always conditional probability (note the ubiquitous 'I' in all our formulae – for further comments see section 10.3 of Ref. [3]). Note that when the condition H_i does not change the probability of H_j, i.e. $P(H_i \,|\, H_j, I) = P(H_i \,|\, I)$, then H_i and H_j are said to be *independent in probability.* In this case the *joint probability* $P(H_i \cap H_j \,|\, I)$ is given by the so-called *product rule*, i.e. $P(H_i \cap H_j \,|\, I) = P(H_i \,|\, I) \cdot P(H_j \,|\, I)$.

These rules are automatically satisfied if probabilities are evaluated from favorable over possible, equally probably cases. Also relative frequencies of occurrences in the past respect

[40] Sometimes one hears of *axiomatic approach* (or even *axiomatic interpretation* – an expression that in my opinion has very little sense) of probability, also known as *axiomatic Kolmogorov approach.* In this approach 'probabilities' are just real 'numbers' in the range $[0, 1]$ that satisfy the axioms, with no interest on their *meaning*, i.e. *how they are perceived by the human mind.* This kind of approach might be perfect for a pure mathematician, only interested to develop all mathematical consequences of the axioms. However it is not suited for applications, because, before we can use the 'numbers' resulting from such a probability theory, we have to understand what they mean. For this reason one might also hear that "probabilities are real numbers which obey the axioms and that we need to 'interpret' them", an expression I deeply dislike. I like much more the other way around: *probability is probability* (how much we believe something) and probability values *can be proved to obey* the four basic rules listed above, which can *then* considered by a pure mathematician the 'axioms' from which a theory of probability can be built.

these rules, with the little difference that the probabilistic interpretation of past relative frequencies is not really straightforward, as briefly discussed in the following appendix. That beliefs satisfy, in general, the same basic rules can be proved in several ways. If we calibrate our degrees of beliefs against 'standards', as illustrated in section 3, this is quite easy to understand. Otherwise it can be proved by the normative principle of the *coherent bet* [10].

A.2 Other important rules

Important relations that follow from the basic rules are (A is also a generic hypothesis):

$$\begin{aligned}
P(\overline{H}\,|\,I) &= 1 - P(H\,|\,I) && (29)\\
P(H\cap\overline{H}\,|\,I) &= 0 && (30)\\
P(H_i\cup H_j\,|\,I) &= P(H_i\,|\,I) + P(H_j\,|\,I) - P(H_i\cap H_j\,|\,I) && (31)\\
P(A\,|\,I) &= P(A\cap H\,|\,I) + P(A\cap\overline{H}\,|\,I) && (32)\\
&= P(A\,|\,H,I)\cdot P(H\,|\,I) + P(A\,|\,\overline{H},I)\cdot P(\overline{H}\,|\,I) && (33)\\
P(A\,|\,I) &= \sum_i P(A\cap H_i\,|\,I) \quad \text{(if H_i form a \textit{complete class})} && (34)\\
&= \sum_i P(A\,|\,H_i,I)\cdot P(H_i\,|\,I) \quad \text{(idem)}\,. && (35)
\end{aligned}$$

The first two rules are quite obvious. Eq. (31) is an extension of the third basic rule in the case two hypotheses are not mutually exclusive. In fact, if this is not case, the probability of $H_i\cap H_j$ is double counted and needs to be subtracted. Eq. (32) is also very intuitive, because either A is true together with H or with its opposite.

Formally, Eq. (33) follows from Eq. (32) and basic rule 4. Its interpretation is that the probability of any hypothesis can be seen as 'weighted average' of conditional probabilities, with weights given by the probabilities of the conditionands [remember that $P(H\,|\,I) + P(\overline{H}\,|\,I) = 1$ and therefore Eq. (33) can be rewritten as

$$P(A\,|\,I) = \frac{P(A\,|\,H,I)\cdot P(H\,|\,I) + P(A\,|\,\overline{H},I)\cdot P(\overline{H}\,|\,I)}{P(H\,|\,I) + P(\overline{H}\,|\,I)}\,,$$

that makes self evident its weighted average interpretation].

Eq. (34) and (35) are simple extensions of Eq. (32) and (33) to a generic 'complete class', defined as a set of mutually exclusive hypotheses [$H_i\cap H_j = \emptyset$, i.e. $P(H_i\cap H_j\,|\,I) = 0$], of which at least one must be true [$\cup_i H_i = \Omega$, i.e. $\sum_i P(H_i\,|\,I) = 1$]. It follows then that Eq. (35) can be rewritten as the ('more explicit') weighted average

$$P(A\,|\,I) = \frac{\sum_i P(A\,|\,H_i,I)\cdot P(H_i\,|\,I)}{\sum_i P(H_i\,|\,I)}\,.$$

[Note that any hypothesis H and its opposite $\overline{H}$ form a complete class, because $P(H\cap\overline{H}\,|\,I) = 0$ and $P(H\cup\overline{H}\,|\,I) = 1$.]

B Belief versus frequency

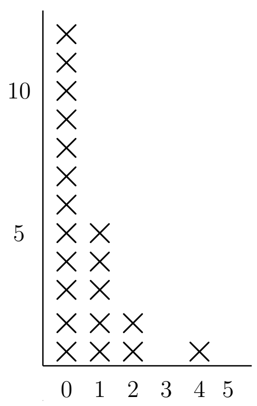
B.1 Beliefs from past frequencies

There is no doubt that

> *"where different effects have been found to follow from causes, which are to appearance exactly similar, all these various effects must occur to the mind in transferring the past to the future, and enter into our consideration, when we determine the probability of the event. Though we give the preference to that which has been found most usual, and believe that this effect will exist, we must not overlook the other effects, but must assign to each of them a particular weight and authority, in proportion as we have found it to be more or less frequent."* [17]

However, some comments about how our minds perform these operations are in order.

Before they are turned into beliefs, observed frequencies are somehow smoothed, either intuitively or by mathematical algorithms. In both cases, consciously or unconsciously, some models of regularities are somehow 'assumed' (a word that in this context means exactly 'believed'). Think, for example, at an experiment in which the number of counts are recorded in a defined interval of time, under conditions apparently identical. Imagine that the numbers of counts in 20 independent measurements are: 0, 0, 1, 0, 0, 0, 1, 2, 0, 0, 1, 1, 0, 4, 2, 0, 0, 0, 0, 1. The results are reported in the histogram. The question is "what do we expect in the 21-st observation, provided the experimental conditions remain unchanged?". It is rather out of discussion that, if a prize is offered on the occurrence of a count, everyone will bet on 0, because it happened most frequently. But can we state that our belief is exactly 60% (12/20)? Moreover, I am also pretty sure that, if you were asked to place your bet on 3 or 4, you would prefer 3, although this number of count has not occurred in the first 20 observations. In an analogous way you might not believe that 5 is impossible. That is because we tend to see regularities in nature.[41] Therefore going from past frequencies to probabilities can be quite a sophisticated process, that requires a lot of assumptions (again *priors*!).



[41] I find that the following old joke conveys well the message. A philosopher, a physicist and a mathematician travel by train through Scotland. The train is going slowly and they see a cow walking along a country road parallel to the railway. The philosopher look at the others, then very seriously states "In Scotland cows are black". The physicist replies that we cannot make such a generalization from a single individual. We are only authorized to state, he maintains, that "In Scotland there is at least one black cow". The mathematician looks well at cow, thinks a while, and then, he said, "I am afraid you are both incorrect. The most we can say is that in Scotland at least one cow has a black side".

B.2 Relative frequencies from beliefs

The question of how relative frequencies of occurrence follow from beliefs is much easier. It is a simple consequence of probability theory and can be easily understood by anyone familiar with the binomial distribution, taught in any elementary course on probability. If we think at n independent trials, for each of which we believe that the 'success' will occur with probability p, the *expected number* of successes is np, with a *standard uncertainty* $\sqrt{np(1-p)}$. We expect then a relative frequency p [that is $(np)/n$] with an uncertainty $\sqrt{p(1-p)/n}$ [that is $\sqrt{np(1-p)}/n$]. When n is very large, the uncertainty goes to zero and we become 'practically sure' to observe a relative frequency very close to p. This asymptotic feature goes under the name of *Bernoulli theorem.* It is important to remark that this reasoning can be *purely hypothetical* and has nothing to do with the so called frequentistic definition of probability.

To conclude this section, probabilities can be evaluated from (past) frequencies and (future, or hypothetical) frequencies can be evaluated from probabilities, but *probability is not frequency.*[42]

C Intuitions versus formal, possibly computer aided, reasoning

Contrary to 'robotized Bayesians'[43] I think it is quite natural that different persons might have initially different opinions, that will necessarily influence the beliefs updated by ex-

[42] The following de Finetti's quote is in order. *"For those who seek to connect the notion of probability with that of frequency, results which relate probability and frequency in some way (and especially those results like the 'law of large numbers') play a pivotal rôle, providing support for the approach and for the identification of the concepts. Logically speaking, however, one cannot escape from the dilemma posed by the fact that the same thing cannot both be assumed first as a definition and then proved as a theorem; nor can one avoid the contradiction that arises from a definition which would assume as certain something that the theorem only states to be very probable." [10]*

[43] This expression refers the robot of E.T. Jaynes' [9] and followers, according to which probabilities should not be subjective. Nevertheless, contrary to frequentists, they allow the possibility of 'probability inversions' via Bayes' theorem, but they have difficulties with priors, that, according to them, shouldn't be subjective. Their solution is that the evaluation of priors should be then delegated to some 'principles' (e.g. *Maximum Entropy* or *Jeffrey priors*). But it is a matter of fact that unnecessary principles (that can be, anyway, used as convenient rules in particular, well understood situations) are easily misused (see e.g. comments on maximum likelihood principle in the Appendix H – several years ago, remarking this attitude by several Bayesian fellows, I wrote a note on *Jeffreys priors versus experienced physicist priors; arguments against objective Bayesian theory*, whose main contents went lately into Ref. [26]), the approach becomes dogmatic and uncritical use of some methods might easily lead to absurd conclusions. For comments on anti-subjective criticisms (mainly those expressed in chapter 12 of Ref. [9]), see section 5 of Ref. [22]. As an example of a bizarre result, although considered by many Jaynes' followers as one of the jewels of their teacher's thought, let me mention the famous die problem. "A die has been tossed a very large number N of times, and we are told that the average number of spots up per toss was not 3.5, as we might expect from an honest die, but 4.5. Translate this information into a probability assignment $P_n, n = 1, 2, \ldots, 6$, for the n-th face to come up on the next toss."[23] The celebrated Maximum Entropy solution is that the probabilities for the six faces are, in increasing order, 5.4%, 7.9%, 11.4%, 18.5%, 24.0% and 34.8%. I have several times raised my

perimental evidence, although the updating rule is well defined, because based on probability theory. But we have also seen, in a formal way, that when the combined weight of evidence in favor of either hypothesis is much larger than the prior judgement leaning, i.e. $|\Delta \mathrm{JL}_{1,2}(\boldsymbol{E}, I)| \gg |\mathrm{JL}_{1,2}(I)|$, then priors become irrelevant and we reach highly inter-subjective conclusions.

I am not in the position to try to discuss the internal processes of the human mind that lead us to react in a certain way to different *stimuli.* I only acknowledge that there are experts of different fields that can make (in most case good) decisions in an fantastically short reacting time. There is no need to think to doctors or engineer in emergency situations, football players, fighter pilots, and many other examples. It is enough to observe us in the everyday actions of driving a car or recognizing people from very partial information (and the context plays and important role! How many times has happened to you not to immediately recognize/identify a neighbor, a waiter or a clerk if you meet him/her in a place you didn't expect him/her at all?). We are brought to think that much of the way in which external information is processed is not analytical, but somehow hard-wired in the brain.

A part of the automatic reasoning of the mind is innate, as we can understand observing children, animals, or even rational adults when they are possessed by pulsions and emotions. Another part comes from the experience of the individual, where by 'experience' it is meant all inputs received, of which he/she might be conscious (like education and training) or unconscious, but all processed and organized (again consciously or not) by the *causality principle*[17], that allows us to anticipate (again consciously or not) the consequences of our and somebody else's actions. As a matter of fact, and coming to the main issue of this paper, there is no doubt that experienced policemen, lawyers and judges, thanks to their experience, have developed kinds of automatic reasonings, that we might call instinct or intuitive behavior (see footnote 2) and that certainly help them in their work.

We have seen in section 3 that priors and even individual weights of evidence can be elicited on a pure subjective way, possibly with the help of virtual bets or of comparison to reference events. The problem arrives when the situation becomes a bit more complicate than just one cause and a couple of effects, and the network of causes-effects become complex. Appendix I shows that the little complication of considering the possibility that the evidence could be somehow reported in an erroneous way, as well known to psychologists, of even fabricated by the investigators makes the problem difficult and the intuition could fail. Appendix J shows an extension of the toy model of section 2 in which several 'testimonies'

perplexities about the solution, but the reaction of Jaynes' followers was, let's say, exaggerated. Recently this result has been questioned by the somewhat quibbling Ref. [24] (one has to recognize that the original formulation of the problem had anyhow the assumption that the die was tossed a large number of times), which, however, also misses the crucial point: *numbers on a die faces are just labels*, having no intrinsic order, as instead it would be the case of the indications on a measuring device. I find absurd making this kind of inferences without even giving a look at a real die! (Any reasonable person, used to try to observe and understand nature, would first observe careful a die and try to guess how it could have been loaded to favor the faces having larger number of spots.)

need to be taken into account.

In summary, the intuition of experts is fundamental to define the priors of the problem. It can be also very important, and sometimes it is the only possibility we have, to assess the degree of belief that some causes can produce some effects, needed to evaluate the Bayes factors and, when the situation becomes complex, to set up a 'network of beliefs' (see Appendix J). A different story is to process the resulting network, on the base of the acquired evidences, in order to evaluate the probabilities of interest. Intuition can be at lost, or miserably fail.

To make clearer the point consider this very rude example. Imagine you are interested in the variable z, that you think for some reasons is related to x and y by the following relation:

$$z = \frac{y \times \sin(\pi^4 + x^2)}{\sqrt{x^3 + y^2}}\,.$$

You might have good reason to state that x is about 10, most likely not less than 9 and not more than 11, and that in this interval you have no reasons to prefer a value with respect to another one. Similarly, you might thing that the value of y you trust mostly is 20, but it could go down to 15 and up to 30 with decreasing beliefs. What do you expect for z. Which values of z should you believe, consistently with your basic assumptions? If a rich prize is give to the person that predicts the interval of width 0.02 in which z will occur, which interval would you choose? What is the value of z (let us call it z_m) such that there is 50% chance that z will occur below this value? What is the probability that z will be above 10? [The solution is in next page (figure 7).]

Anyway, if you consider this example a bit too 'technical' you might want to check the capabilities of your intuition on the much simpler one of Appendix J. (Try first to read the caption of figure 11 and to reply the questions.)

D Bare facts and complete state of information

As it has been extensively discussed in section 5, saying that a person has taken a camera out of thirteen is a piece of information, but it is not all, and it is not enough to update correctly our beliefs.

This is true in general, even in fields of research that are considered by outsider to be the realm of objectivity, where only 'facts' count. Stated with Peter Galison words [20],

> *"Experiments begin and end in a matrix of beliefs. . . . beliefs in instrument type, in programs of experiment enquiry, in the trained, individual judgments about every local behavior of pieces of apparatus."*

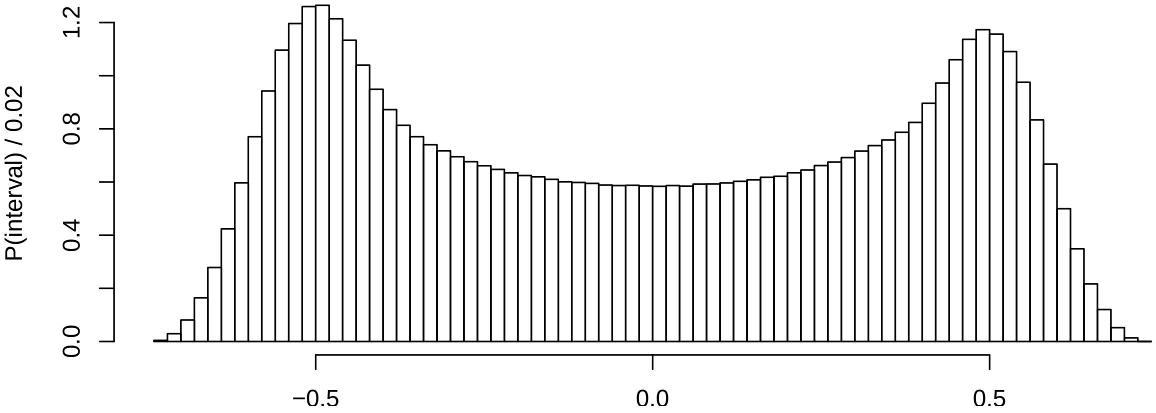


Figure 7: This histogram shows in a graphical way **'a' solution to the question at the end of Appendix C** (details depend, obviously, on how the initial assumptions have been modelled, but the gross features do not change if different reasonable models, consistent with the assumptions, are used – here x has been taken uniform between 9 and 11; y has been modelled with an asymmetric triangular distribution ranging between 15 and 30, with maximum belief in 20).

The values of z we have to believe mostly are those around -0.5, but also all others in the range -0.7 to 0.7 cannot be really neglected. In particularly, values around 0.5 are almost as likely as those around -0.5. As we can see, there is about 50% that z occurs below 0 (-0.02, to be precise) and 50% above. Note that, although the center of the distribution is around 0 (-0.14, to be precise), the most believable values are far from it. In other words, even if the *expected value* is -0.14 and the *standard uncertainty* (quantified by the *standard deviation*) is 0.40, if a prize is assigned to whom predicts the interval of width 0.02 in which the *uncertain number* z will occur, we should place that interval at -0.5. Apart from the technical complications, the message of this example is that one thing is to state the basic assumptions and subjective beliefs in some of the variables of the game, a much more complicate issue is to evaluate all *logical consequences of the premises*. In other words, if you agree on the premises of this problem, but not on the conclusions, you run into contradiction. Now, it is a matter of fact that *contradictions of this kind are rather frequent* because the evaluation of the consequences is not commonly done using formal logic and probability theory. The extension to complex belief networks is straightforward, although, as we shall see in Appendix J, also a very simple network is enough to challenge our ability to provide intuitive answers.

> [Then, taking as an example the discovery of the positron:]
> *"Taken out of time there is no sense to the judgment that Anderson's track 75 is a positive electron; its textbook reproduction has been denuded of the prior experience that made Anderson confident in the cloud chamber, the magnet, the optics, and the photography."*

My preferred toy examples to convey this important messages are *the three box problem(s)'* and *the two envelopes 'paradox'* (see section 3.13 of Ref. [3] – I remind briefly here only the box ones). The box problems are a series of recreational/educational problems, the basic one being rather famous as 'Monthy Hall problem'. The great majority of people (my usual target are physics PhD students) get mad with them because they have not been educated to take into account all available information. Therefore they have quite some difficulties to understand that if a contestant has taken one box (yet unopened) and there is still another un-opened box to choose, the probability that this box contains the prize (only one of the three boxes does) depends on whether the opened (and empty) box was got by chance or was chosen with the intention to take a box without prize.[44] (And there is often somebody in the audience that when he/she listens the formulation of the problem in which the box was opened by chance, he/she smiles at the others, and than gives the solution…of the version in which the conductor opens on purpose and empty box.)

I found that the issue of considering into account all available information is shown in a particular convincing way in the 'three prisoner paradox' (isomorph[45] to Monthy Hall, but more a headache than this, perhaps because it involves humans) and in the 'thousand prisoner problem' of Ref. [21]: not only bare facts enter the evaluation of probability, but also all contextual knowledge about them, including the question asked to acquire their knowledge.

[44] Reading the draft of this paper, my colleague Enrico Franco has remarked that in the way the box problems (or the Monthy Hall) are presented there are additional pieces of information which are usually neglected, as I also did in Ref. [3] ('then' was not underlined in the original):

> (1) In the first case, imagine two contestants, each of whom chooses one box at random. Contestant B opens his chosen box and finds it does not contain the prize. Then the presenter offers player A the opportunity to exchange his box, still un-opened, with the third box. …
> (2) In the second case there is only one contestant, A. After he has chosen one box the presenter tells him that, although the boxes are identical, he knows which one contains the prize. Then he says that, out of the two remaining boxes, he will open one that does not contain the prize.… [3]

It makes quite some difference if the conductor announces he will propose the exchange before the boxe(s) is/are initially taken by the contestant(s) that, or if he does it later, as I usually formulate the problems. In the latter case, in fact, contestant A can have a legitimate doubt concerning the malicious intention of the conductor, who might want to induce him to lose. Mathematics oriented guys would argue then that the problem does have a solution. But the question is that in real life one has to act, and one has to finally make his decision, based on the best knowledge of the game and of the conductor, in a finite amount of time.

[45] This is true only neglecting the complication taken into account in the previous footnote. Indeed, in one case the 'exchange game' is initiated by the conductor, while in the second by the prisoner, therefore Enrico Franco's comment does not apply to the three prisoner problem.

E Some remarks on the use of logarithmic updating of the odds

The idea of using (natural) logarithms of the odds is quite old, going back, as far as I know, to Charles Sanders Peirce [6]. He related them to what he called *feeling of belief* (or *intensity of belief*), that, according to him, “should be as the logarithm of the chance, this latter being the expression of the state of facts which produces the belief” [6], where by ‘chance’ he meant exactly probability ratios, i.e. the odds.

Peirce proposed his ”thermometer for the proper intensity of belief” [6] for several reasons.

- First because of considerations that when the odds go to zero or to infinity, then the intensity of belief on either hypothesis goes to infinity;[46] when “an even chance is reached [the feeling of believing] should completely vanish and not incline either toward or away from the proposition.” [6] The logarithmic function is the simplest one to achieve the desired feature. (Another interesting feature of the odds is described in footnote 16.)

- Then because (expressing the question in our terms), if we started from a state of indifference (initial odds equal to 1), each piece of evidence should produce odds equal to its Bayes factor [our $\tilde{O}_{i,j}(E_i)$]. The combined odds will be the product of the individual odds [Eq. 19]. But, mixing now Pierce’s and our terminology, when we combine several arguments (pieces of evidence), they “ought to produce a belief equal to the sum of the intensities of belief which either would produce separately”. [6] Then “because we have seen that the chances of independent concurrent arguments are to be multiplied together to get the chance of their combination, and therefore the quantities which best express the intensities of belief should be such that they are to be *added* when the *chances* are multiplied…Now, the logarithm of the chance is the only quantity which fulfills this condition”. [6]

- Finally, Peirce justifies his choice by the fact that human perceptions go often as the logarithm of the stimulus (think at subjective feeling of sound and light – even ‘utility’, meant as the ‘value of money’ is supposed to grow logarithmically with the amount of money): “There is a general law of sensibility, called Fechner’s psychophysical law. It is that the intensity of any sensation is proportional to the logarithm of the external force which produces it.”[6] (Table 1 provides a comparisons between the different quantities involved, to show that the human sensitivity on probabilistic judgement is indeed logarithmic, with a resolution about the first decimal digit of the base 10 logarithms.)

[46] In this respect, belief becomes similar to other human sentiments, for which in normal speech we use a scale that goes to infinity – think at expressions like ‘infinite love’, ‘infinite hate’, and so on (see also footnote 37).

As far as the logarithms in question, I have done a short research on their use, which, actually, lead me to discover Peirce's *Probability of Intuition* [6] and Good's *Probability and the weighing of Evidence* [7].[47] As far as I understand, without pretension of completeness or historical exactness:

- Peirce' 'chances' are introduced as if they were our odds, but are used if they were Bayes factors ("the chances of independent concurrent arguments are to be multiplied together to get the chance of their combination" [6]). Then he takes the *natural* logarithm of these 'chances', to which he also associates an idea of *weight of evidence* ("our belief ought to be proportional to the weight of evidence, in the sense, that two arguments which are entirely independent, neither weakening nor strengthening each other, ought, when they concur, to produce a belief equal to the sum of the intensities of belief which either would produce separately" [6]).

- According to Ref. [8] the modern use of the logarithms of the odds seem to go back to I.J. Good, who used to call *log-odds* the *natural* logarithm of the odds.[48]

- However, reading later Ref. [8] it is clear that Good, following a suggestion of A.M. Turing, proposes a decibel-like (db) notation[49], giving proper names both to the logarithm of the odds and to the logarithm of the Bayes factor:

  - "($10\,\log_{10} f$) db ... may be also described as the *weight of evidence* or amount of information for H given E" [7];
  - "($10\,\log_{10} o$) db may be called the *plausibility* corresponding to odds o" [7].

  It follows then that

$$\text{"Plausibility gained} \;=\; \text{weight of evidence".[7]} \qquad (36)$$

[47] Peirce article is a mix of interesting intuitions and confused arguments, as in the "bag of beans" example of pages 709-710 (he does not understand the difference between the observation of 20 black beans and that of 1010 black and 990 white for the evaluation of the probability that another bean extracted from the same bag is white or black, arriving thus to a kind of paradox – from Bayes' rule it is clear that weights of evidence sum up to form the intensity of belief on two bag compositions, not on the outcomes from the boxes [27]). Of a different class is Good's book, one of the best on probabilistic reasoning I have met so far, perhaps because I feel myself often in tune with *Good thinking* (including the passion for footnotes and internal cross references shown in Ref. [7]).

[48] But Goods mentions that "In 1936 Jeffreys had already appreciated the importance of the logarithm of the [Bayes] factor and had suggested for it the name 'support'." [7]

[49] "In acoustic and electrical engineering the bel is the logarithm to base 10 of the ratio of two intensities of sound. Similarly, if f is the [Bayes] factor in favor of a hypothesis has gained $\log_{10} f$ bels, or ($10\,\log_{10} f$) db." [7] [Good uses the name 'factor' for what we call Bayes factor, "the factor by which the initial odds of H must be multiplied in order to obtain the final odds. Dr. A.M. Turing suggested in a conversation in 1940 that the word 'factor' should be regarded as the technical term in this connexion, and that it could be more fully described as *the factor in favor of the hypothesis H in virtue of the result of the experiment.*" [7]]

- Decibel-like logarithms of the odds are used since at least forty years with under the name *evidence.* [23] .

Personally, I think that the decibel-like definition is not very essential (decibels themselves tend already to confuse normal people, also because for some applications the factor 10 is replaced by a factor 20). Instead, as far as names are concerned:

- 'plausibility' is difficult to defend, because it is too similar to probability in everyday use, and, as far as I understand, has decayed;
- 'weight of evidence' seems to be a good choice, for the reasons already well clear to Peirce.
- 'evidence' in the sense of Ref. [23] seems, instead, quite bad for a couple of reasons:
  - First, because 'evidence' has already too many meanings, including, in the Bayesian literature, the denominator of the r.h.s. of Eq. (3).
  - Second, because this name is given to the logs of the odds (including the initial ones), but not to those of the Bayes factors to which no name is given. Therefore, the name 'evidence', as used in Ref. [23] in this context, is not related to the evidence.

I have taken the liberty to use the expression 'judgment leaning' first because it evokes the famous *balance of Justice*, then because all other expressions I thought about have already a specific meaning, and some of them even several meanings.[50] It is clear, especially comparing Eq. (36) with Eq. (24), that, besides the factor ten multiplying the base ten logarithms and the notation, I am quite in tune with Good. I have also to admit I like Peirce' 'intensity of belief' to name the JL's, although it is too similar to 'degree of belief', already widely used to mean something else.

So, in summary, these are the symbols and names used here:

$\mathbf{JL}_{i,j}(\cdot)$ is the *judgement leaning* in favor of hypothesis i and against j, with the conditions in parenthesis. If we only consider an hypothesis (H) and its opposite $\overline{H}$, that could

[50] Many controversies in probability and statistics arise because there is no agreement on the meaning of the words (including 'probability' and 'statistics'), or because some refuse to accept this fact. For example, I am perfectly aware that many people, especially my friends physicists, tend to to assign to the word 'probability' the meaning of a kind of *propension* 'nature' has to behave more in a particular way than in other way, although in many other cases – and more often! – they also mean by the same word how much they believe something (see e.g. chapters 1 and 10 of Ref. [3]). For example, one might like to think that kind B_1 boxes of section 2 have a 100% *propensity* to produce white balls and 0 to produce black balls, while type B_2 have 7.7% propension to produce white and 92.3% to produce black. Therefore, if one knows the box composition and is only interested to the outcome of the extraction, then probability and propensity coincide in value. But if the composition is unknown this is no longer true, as we shall see in Appendix J. [By the way, all interesting questions we shall see in Appendix J have no meaning (and no clean answers) for ideologizied guy who refuse to accept that probability primarily means how much we believe something. (See also comments in Appendix H.)]

be possibly related to the occurrence of the event E or its opposite $\overline{E}$, also the notation $\mathrm{JL}_H(\cdot)$, or $\mathrm{JL}_E(\cdot)$, will be used (as in table 2 of Appendix I).
(Sometimes I have also tempted to call a JL 'intensity of belief' if it is clear from the contest that the expression does not refer to a probability.)

$\Delta\mathbf{JL}_{i,j}(\cdot)$, with the same meaning of the subscript and of the argument, is the variation of judgement leaning produced by a piece of evidence and it is called here *weight of evidence*, although it differs by a factor from the analogous names used by Peirce and Good[51].

F AIDS test

Let us make an example of general interest, that exhibits some of the issues that also arise in forensics.

Imagine an Italian citizen is chosen *at random* to undergo an AIDS test. Let us assume the analysis used to test for HIV infection is not perfect. In particular, infected people (HIV) are declared 'positive' (Pos) with 99.9% probability and 'negative' (Neg) with 0.1%; there is, instead, a 0.2% chance a healthy person ($\overline{\mathrm{HIV}}$) is told positive (and 99.8% negative).

The other information we need is the prevalence of the virus in Italy, from which we evaluate our initial belief that the randomly chosen person is infect. We take 1/400 or 0.25% (roughly 150 thousands in a population of 60 millions).

To summarize, these are the pieces of information relevant to work the exercise:[52]

$$\begin{aligned}
P(\mathrm{Pos}\,|\,\mathrm{HIV}, I) &= 99.9\%,\\
P(\mathrm{Neg}\,|\,\mathrm{HIV}, I) &= 0.1\%,\\
P(\mathrm{Pos}\,|\,\overline{\mathrm{HIV}}, I) &= 0.2\%\\
P(\mathrm{Neg}\,|\,\overline{\mathrm{HIV}}, I) &= 99.8\%\\
P(\mathrm{HIV}\,|\,I) &= 0.25\%\\
P(\overline{\mathrm{HIV}}\,|\,I) &= 99.75\%,
\end{aligned}$$

from which we can calculate initial odds, Bayes factors and JL's [we use here the notation $O_{\mathrm{HIV}}(I)$, instead of our usual $O_{1,2}(I)$ to indicate odds in favor of the hypothesis HIV and against the opposite hypothesis ($\overline{\mathrm{HIV}}$); similarly for JL_{HIV} and $\Delta\mathrm{JL}_{HIV}$]:

$$\begin{aligned}
O_{\mathrm{HIV}}(I) &= 1/399 = 0.0025 && \Rightarrow\ \mathrm{JL}_{\mathrm{HIV}}(I) = -2.6\\
\tilde{O}_{\mathrm{HIV}}(\mathrm{Pos}, I) &= 99.9/0.2 = 499.5 && \Rightarrow\ \Delta\mathrm{JL}_{\mathrm{HIV}}(\mathrm{Pos}, I) = +2.7\\
\tilde{O}_{\mathrm{HIV}}(\mathrm{Neg}, I) &= 0.1/99.8 = 1/998 = 0.001002 && \Rightarrow\ \Delta\mathrm{JL}_{\mathrm{HIV}}(\mathrm{Neg}, I) = -3.0\,.
\end{aligned}$$

[51] $\log_{10} x = \ln x/\ln 10 = (10\,\log_{10} x)/10$.

[52] The performance of the test are of pure fantasy, while the prevalence is somehow realistic, although not pretended to be the real one. But it will be clear that the result is rather insensitive on the precise figures.

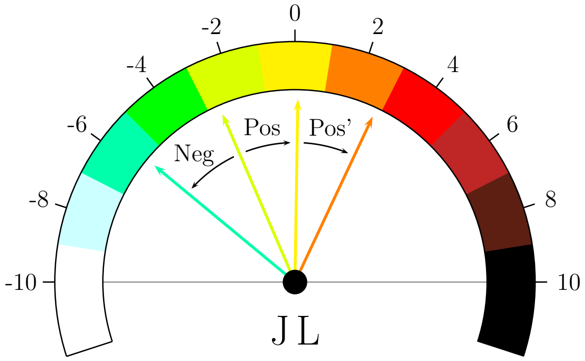


Figure 8: AIDS test illustrated with judgement leanings.

A positive result adds a weight of evidence of 2.7 to -2.6, yielding the negligible leaning of $+0.1$. Instead a negative result has the negative weight of -3.0, shifting the leaning to -5.6, definitely on the safe side (see fig. 8).

The figure shows also the effect of a second, *independent*[53] analysis, having the same performances of the first one and in which the person results again positive. As it clear from the figure, the same conclusion would be reached if only one test was done on a subject for which a doctor could be in serious doubt if he/she had AIDS or not (JL ≈ 0).

From this little example we learn that if we want to have a good discrimination power of a test, it should have a ΔJL very large in module. Absolute discrimination can only be achieved if the weight of evidence is infinite, i.e. if either hypothesis is impossible given the observation.

G Which generator?

Imagine two (pseudo-) random number generators: H_1, Gaussian with mean 0 and standard deviation 1, and H_2, also Gaussian, but with mean 0.4 and standard deviation 2 (see figure 9).

A program chooses at random, with equal probability, H_1 or H_2; then the generator produces a number, that, rounded to the 7-th decimal digit, is $x_E = 0.3986964$. The question is, from which random generator does x_E come from?

At this point, the problem is rather easy to solve, if we know the probability of each

[53] Note that 'independent' does not mean the analysis has simply been done by somebody else, possibly in a different laboratory, but also that the *principle of measurement* is independent.

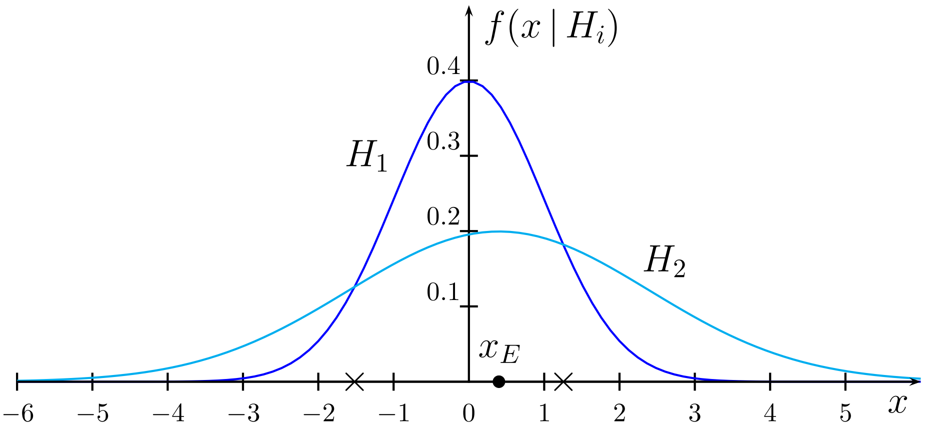


Figure 9: Which random number generator has produced x_E? Which hypothesis favors the points indicated by '×'?

generator to give x_E. They are[54]

$$\begin{aligned} P(x_E\,|\,H_1,I) &= 3.68\times 10^{-8} \quad (1 \text{ in } \approx 27 \text{ millions}) \\ P(x_E\,|\,H_2,I) &= 1.99\times 10^{-8} \quad (1 \text{ in } \approx 50 \text{ millions})\,, \end{aligned}$$

from which we can calculate Bayes factor and weight of evidence:

$$\tilde{O}_{1,2}(x_E,I) \;=\; 1.85 \qquad \Rightarrow \quad \Delta\mathrm{JL}_{1,2}(x_E,I) = +0.27\,.$$

Therefore, the observation of x_E provides a slight evidence in favor of H_1, *no matter if this generator has very little probability to give* x_E, as it has very little probability to give *any* particular number.

What matters when comparing hypotheses is never, stated in general terms, the absolute probability $P(E\,|\,H_i,I)$. In particular, it doesn't make sense saying "$P(H_i\,|\,E,I)$ is small because $P(E\,|\,H_i,I)$ is small".[55] As a consequence, from a consistent probabilistic point of view, *it makes no sense to test a single, isolated hypothesis*, using 'funny arguments',

[54] The curves $f(x\,|\,H_i)$ in figure 9 represent *probability density functions* ('pdf'), i.e. they give the probability per unit x, i.e. $P([x-\Delta x/2, x+\Delta x/2])/\Delta x$, for small Δx (remember that 'densities' are always local). Rounding to the 7-th digit means that the number before rounding was in the interval of $\Delta x = 10^{-7}$ centered x_E. It follows that the probability a generator would produce that number can be calculated as $f(x_E\,|\,H_i)\times\Delta x$. Indeed, we can see that in the calculation of Bayes factors the width Δx simplifies and what really matter is the ratio of the two pdf's, i.e.

$$\tilde{O}_{1,2}(x_E,I) \;=\; \frac{P(x_E\,|\,H_1)}{P(x_E\,|\,H_2)} = \frac{f(x_E\,|\,H_1)\times\Delta x}{f(x_E\,|\,H_2)\times\Delta x} = \frac{f(x_E\,|\,H_1)}{f(x_E\,|\,H_2)}\,.$$

The Bayes factor is therefore the ratio of the ordinates of the curves in figure 9 for the same x_E. Note that $f(x_E\,|\,H_1)\times\Delta x$ can be small at will, but, nevertheless, hypothesis H_1 can receive a very high weight of evidence from x_E if $f(x_E\,|\,H_1) \gg f(x_E\,|\,H_2)$.

[55] Sometimes this might be qualitatively correct, because it easy to *imagine there could be* an alternative

like how far if x_E from the peak of $f(x\,|\,H_i)$, or how large is the area below $f(x\,|\,H_i)$ from $x = x_E$ to infinity. In particular, if two models give exactly the same probability to produce an observation, like the two points indicated by '×' in fig. 9, the evidence provided by this observation is absolutely irrelevant [$\Delta\text{JL}_{1,2}(\text{'}\times\text{'}) = 0$].

To get a bit familiar with the weight of evidence in favor of either hypothesis provided by different observations, the following table, reporting Bayes factors and JL's due to the integers between -6 and $+6$, might be useful.

x_E	$\tilde{O}_{1,2}(x_E)$	$\Delta\text{JL}_{1,2}(x_E)$
-6	5.1×10^{-6}	-5.3
-5	2.9×10^{-4}	-3.5
-4	7.5×10^{-3}	-2.1
-3	9.4×10^{-2}	-1.0
-2	0.56	-0.3
-1	1.5	0.2
0	2.0	0.3
1	1.3	0.1
2	0.37	-0.4
3	5.2×10^{-2}	-1.3
4	3.4×10^{-3}	-2.5
5	1.0×10^{-4}	-4.0
6	1.5×10^{-6}	-5.8

As we see from this table, and as we better understand from figure 9, numbers large in module are in favor of H_2, and very large ones are in its strong favor. Instead, the numbers laying in the interval defined by the two points marked in the figure by a cross provide evidence in favor of H_1. However, while individual pieces of evidence in favor of H_1 can only be weak (the maximum of ΔJL is about 0.3, reached around $x = 0$, namely -0.13, to be precise, for which ΔJL reaches 0.313), those in favor of the alternative hypothesis can be sometimes very large. It follows then that one gets easier convinced of H_2 rather than of H_1.

We can check this by a little simulation. We choose a model, extract 50 random variables and analyze the data as if we didn't know which generator produced them, although considering H_1 and H_2 equally likely. We expect that, as we go on with the extractions, the pieces of evidence accumulate until we *possibly* reach a level of practical certainty. Obviously, the individual pieces of evidence do not provide the same ΔJL, and also the sign can fluctuate, although we expect more positive contributions if the points are generated by H_1

hypothesis H_j such that:

1. $P(E\,|\,H_j, I) \gg P(E\,|\,H_i, I)$, such that the Bayes factor is strongly in favor of H_j;
2. $P(H_j\,|\,I) \approx P(H_i\,|\,I)$, that is H_j is roughly as credible as H_i.

(For details see section 10.8 of Ref.[3].)

and the other way around if they came from H_2. Therefore, as a function of the number of extractions the accumulated weight of evidence follows a kind of *asymmetric random walk* (imagine the JL indicator fluctuating as the simulated experiment goes on, but drifting 'in average' in one direction).

Figure 10 shows 200 inferential stories, half per generator. We see that, in general, we get practically sure of the model after a couple of dozens of extractions. But there are also cases in which we need to wait longer before we can feel enough sure on one hypothesis.

It is interesting to remark that the leaning in favor of each hypothesis grows, *in average*, linearly with the number of extractions. That is, a little piece of evidence, which is in average positive for H_1 and negative for H_2, is added after each extraction. However, around the average trend, there is a large varieties of individual inferential histories. They all start at $\Delta\text{JL} = 0$ for $n = 0$, but in practice there are no two identical 'trajectories'. All together they form a kind of 'fuzzy band', whose 'effective width' grows also with the number of extractions, *but not linearly.* The widths grows as the square root of n.[56] This is the reason why, as n increases, the bands tend to move away from the line JL = 0. Nevertheless, individual trajectories can exhibit very 'irregular'[57] behaviors as we can also see in figure 10.

[56] We can evaluate the *prevision* ('expected value') of the variation of leaning at each random extraction for each hypotheses, calculated as the average value of $\Delta\text{JL}_{1,2}(H_i)$. We can also evaluate the *uncertainty of prevision*, quantified by the standard deviation. We get for the two hypotheses

$$\left\{\begin{array}{rcl} \text{E}[\Delta\text{JL}_{1,2}(H_1)] &=& 0.15 \\ \sigma[\Delta\text{JL}_{1,2}(H_1)] &=& 0.24 \\ u_R[\Delta\text{JL}_{1,2}(H_1)] &=& 1.6 \end{array}\right. \qquad \left\{\begin{array}{rcl} \text{E}[\Delta\text{JL}_{1,2}(H_2)] &=& -0.38 \\ \sigma[\Delta\text{JL}_{1,2}(H_2)] &=& 0.97 \\ u_R[\Delta\text{JL}_{1,2}(H_2)] &=& 2.6 \end{array}\right.$$

where also the *relative uncertainty* u_R has been reported, defined as the uncertainty divided by the absolute value of the prevision. The fact that the uncertainties are relatively large tells clearly that we *do not expect* that a single extraction will be sufficient to convince us of either model. But this does not mean we cannot take the decision because the number of extraction *has been* too small. If a very large fluctuation provides a ΔJL of -5 (the table in this section shows that this is not very rare), we have already got a very strong evidence in favor of H_2. Repeating what has been told several time, what matters is the cumulated judgement leaning. It is irrelevant if a JL of -5 comes from ten individual pieces of evidence, only from a single one, or partially from evidence and partially from prior judgement.

When we plan to make n extractions from a generator, probability theory allows us to calculate expected value and uncertainty of $\text{JL}_{1,2}(n)$:

$$\begin{array}{rcl} \text{E}[\Delta\text{JL}_{1,2}(n, H_i)] &=& n \times \text{E}[\Delta\text{JL}_{1,2}(H_i)] \\ \sigma[\Delta\text{JL}_{1,2}(n, H_i)] &=& \sqrt{n} \times \sigma[\Delta\text{JL}_{1,2}(H_i)] \\ u_R[\Delta\text{JL}_{1,2}(n, H_i)] &=& \dfrac{1}{\sqrt{n}} \times u_R[\Delta\text{JL}_{1,2}(H_i)]\,. \end{array}$$

In particular, for $n = 50$ we get $\Delta\text{JL}_{1,2}(H_1) = 7.5 \pm 1.7$ ($u_R = 22\%$) and $\Delta\text{JL}_{1,2}(H_2) = -19 \pm 7$ ($u_R = 37\%$), that explain the gross feature of the bands in figure 10.

[57] I find the issue of 'statistical regularities' to be often misunderstood. For example, the trajectories in figure 10 that do not follow the general trend *are not exceptions*, being generated by the same rules that produces all of them.

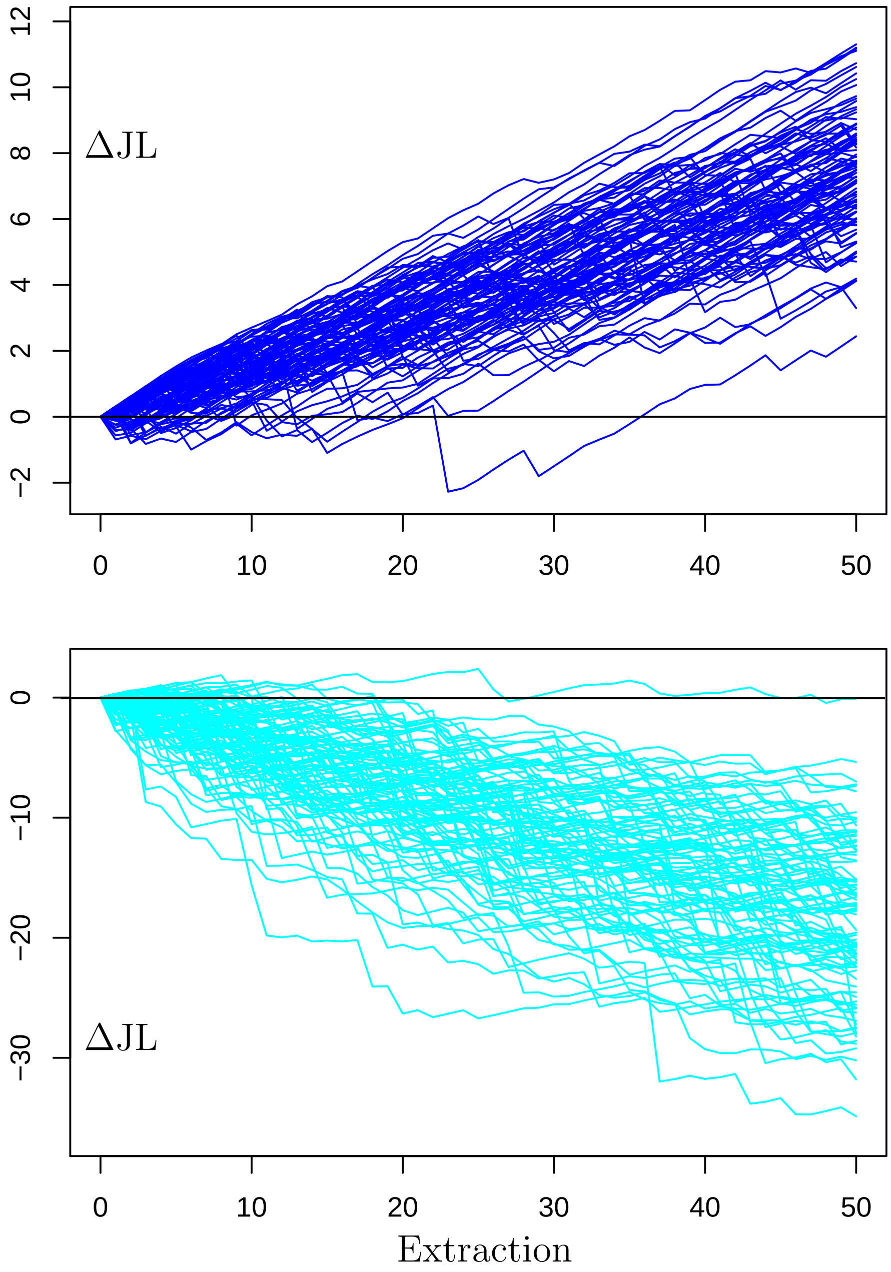


Figure 10: Combined weights of evidence in simulated experiments. The above (blue) combined JL sequences have been obtained by the generator H_1, as it can be recognized because they tend to large positive values as the number of extractions increases. The below one are generated by H_2.

H Likelihood and maximum likelihood methods

Some comments on *likelihood* are also in order, because the reader might have heard this term and might wonder if and how it fits in the scheme of reasoning expounded here.

One of the problems with this term is that it tends to have several meanings, and then to create misunderstandings. In plane English 'likelihood' is "1. the condition of being likely or probable; probability", or "2. something that is probable"[58]; but also "3. (Mathematics & Measurements / Statistics) the probability of a given sample being randomly drawn regarded as a function of the parameters of the population".

Technically, with reference to the example of the previous appendix, the likelihood is simply $P(x_E \,|\, H_i, I)$, where x_E is fixed (the observation) and H_i is the 'parameter'. Then it can take two values, $P(x_E \,|\, H_1, I) = 3.68 \times 10^{-8}$ and $P(x_E \,|\, H_2, I) = 1.99 \times 10^{-8}$.

If, instead of only two models we had a continuity of models, for example the family of all Gaussian distributions characterized by central value μ and 'effective width' (standard deviation) σ, our likelihood would be $P(x_E \,|\, \mu, \sigma, I)$, i.e.

$$\mathcal{L}(\mu, \sigma\,;\, x_E) \;\;=\;\; P(x_E \,|\, \mu, \sigma, I)\,, \tag{37}$$

written in this way to remember that: 1) a likelihood is a function of the model parameters and not of the data; 2) $\mathcal{L}(\mu, \sigma\,;\, x_E)$ *is not* a probability (or a probability density function) of μ and σ. Anyway, for the rest of the discussion we stick to the very simple likelihood based on the two Gaussians. That is, instead of a double infinity of possibilities, our space of parameters is made only of two points, $\{\mu_1 = 0, \sigma_1 = 1\}$ and $\{\mu_1 = 0.4, \sigma_2 = 2\}$. Thus the situation gets simpler, although the main conceptual issues remain substantially the same.

In principle there is nothing bad to give a special name to this function of the parameters. But, frankly, I had preferred statistics gurus named it after their dog or their lover, rather than call it 'likelihood'.[59] The problem is that it is very frequent to hear students, teachers and researcher explaining that the 'likelihood' tells "how likely the parameters are" (this is *the probability of the parameters! not the 'likelihood'*). Or they would say, with reference to our example, "it is the probability that x_E comes from H_i" (again, this expression would be the probability of H_i given x_E, and not the probability of x_E given the models!) Imagine if we have only H_1 in the game: x_E comes with certainty from H_1, although H_1 does not yield with certainty x_E.[60]

[58] See e.g. `http://www.thefreedictionary.com/likelihood`.

[59] Note added: I have just learned, while making the short research on the use of the logarithmic updating of the odds presented in Appendix E, that "the term [likelihood] was introduced by R. A. Fisher with the object of *avoiding* the use of Bayes' theorem" [7].

[60] As further example, you might look at `http://en.wikipedia.org/wiki/Likelihood_principle`, where it is stated (January 28, 2010, 15:40) that a likelihood "gives a measure of how 'likely' any particular value of θ is" (note the quote mark of 'likely', as in the example of footnote 61). But, fortunately we find in `http://en.wikipedia.org/wiki/Likelihood_function` that "This is not the same as the probability that those parameters are the right ones, given the observed sample. Attempting to interpret the likelihood of a

Several methods in 'conventional statistics' use somehow the likelihood to decide which model or which set of parameters describes at best the data. Some even use the likelihood ratio (our Bayes factor), or even the logarithm of it (something equal or proportional, depending on the base, to the weight of evidence we have indicated here by JL). The most famous method of the series is the *maximum likelihood principle*. As it is easy to guess from its name, it states that the *best estimates* of the parameters are those which maximize the likelihood.

All that *seems* reasonable and in agreement with what it has been expounded here, but it is not quite so. First, for those who support this approach, likelihoods are not just a part of the inferential tool, they are everything. Priors are completely neglected, more or less because of the objections in footnote 9. This can be acceptable, if the evidence is overwhelming, but this is not always the case. Unfortunately, as it is now easy to understand, neglecting priors is mathematically equivalent to consider the alternative hypotheses equally likely! As a consequence of this statistics miseducation (most statistics courses in the universities all around the world only teach 'conventional statistics' and never, little, or badly probabilistic inference) is that too many unsuspectable people fail in solving the AIDS problem of appendix B, or confuse the likelihood with the probability of the hypothesis, resulting in misleading scientific claims (see also footnote 60 and Ref. [3]).

The second difference is that, since "there are no priors", the result cannot have a probabilistic meaning, as it is openly recognized by the promoters of this method, who, in fact, do not admit we can talk about probabilities of causes (but most practitioners seem not to be aware of this 'little philosophical detail', also because frequentistic gurus, having difficulties to explain what is the meaning of their methods, they say they are 'probabilities', but in quote marks![61]). As a consequence, the resulting 'error analysis', that in human terms

hypothesis given observed evidence as the probability of the hypothesis is a common error, with potentially disastrous real-world consequences in medicine, engineering or jurisprudence. See prosecutor's fallacy[*] for an example of this." ([*] see `http://en.wikipedia.org/wiki/Prosecutor%27s_fallacy`.)
Now you might understand why I am particular upset with the name likelihood.

[61] For example, we read in Ref. [25] (the authors are influential supporters of the use frequentistic methods in the particle physics community):

> *When the result of a measurement of a physics quantity is published as $R = R_0 \pm \sigma_0$ without further explanation, it simply implied that R is a Gaussian-distributed measurement with mean R_0 and variance σ_0^2. This allows to calculate various confidence intervals of given "probability", i.e. the "probability" P that the true value of R is within a given interval.*

(Quote marks are original and nowhere in the paper is explained why probability is in quote marks!)
The following Good's words about frequentistic *confidence intervals* (e.g. '$R = R_0 \pm \sigma_0$' of the previous citation) and "probability" might be very enlighting (and perhaps shocking, if you always thought they meant something like 'how much one is confident in something'):

> *Now suppose that the functions $\underline{c}(E)$ and $\overline{c}(E)$ are selected so that $[\overline{c}(E), \overline{c}(E)]$ is a confidence interval with coefficient α, where α is near to 1. Let us assume that the following instructions are issued to all statisticians.*
>
> *"Carry out your experiment, calculate the confidence interval, and state that c belong to this interval. If you are asked whether you 'believe' that c belongs to the confidence interval you must refuse to answer. In the long run your assertions, if independent of each other, will be*

means to assign different beliefs to different values of the parameters, is cumbersome. In practice the results are reasonable only if the possible values of the parameters are initially equally likely and the 'likelihood function' has a 'kind shape' (for more details see chapters 1 and 12 of Ref. [3]).

I Evidences mediated by a testimony

In most cases (and practically always in courts) pieces of evidence are not acquired directly by the person who has to form his mind about the plausibility of a hypothesis. They are usually accounted by an intermediate person, or by a chain of individuals. Let us call E_T the report of the evidence E provided in a *testimony.* The inference becomes now $P(H_i\,|\,E_T, I)$, generally different from $P(H_i\,|\,E, I)$.

In order to apply Bayes' theorem in one of its form we need first to evaluate $P(E_T\,|\,H_i, I)$. Probability theory teaches us how to get it [see Eq. (33) in Appendix A]:

$$P(E_T\,|\,H_i, I) \;=\; P(E_T\,|\,E, I)\cdot P(E\,|\,H_i, I) + P(E_T\,|\,\overline{E}, I)\cdot P(\overline{E}\,|\,H_i, I) \qquad (38)$$

(E_T could be due to a true evidence or to a fake one). Three new ingredients enter the game:

- $P(E_T\,|\,E, I)$, that is the probability of the evidence to be correctly reported as such.
- But the testimony could also be incorrect the other way around (it could be incorrectly reported, simply by mistake, but also it could be a 'fabricated evidence'), and therefore also $P(E_T\,|\,\overline{E}, I)$ is needed. Note that the probabilities to lie could be in general asymmetric, i.e. $P(\overline{E}_T\,|\,E, I) \neq P(E_T\,|\,\overline{E}, I)$, as we have seen in the AIDS problem of Appendix F, in which the response of the analysis models false witness well.
- Finally, since $P(\overline{E}\,|\,H_i, I)$ enters now directly, the 'naïve' Bayes factor, only depending on $P(E\,|\,H_i, I)$, is not longer enough.

Taking our usual two hypotheses, $H_1 = H =$ 'guilty' and $H_2 = \overline{H} =$ 'innocent', we get the following Bayes factor based on the *testified evidence* E_T (hereafter, in order to simplify the notation, we use the subscript 'H' in odds and Bayes factors, instead of 'i, j', to indicate

right in approximately a proportion α of cases." (Cf. Neyman (1941), 132-3) [7]

[Neyman (1941) stands for J. Neyman's "Fiducial argument and the theory of confidence intervals", *Biometrica*, **32**, 128-150.]
(For comments about what is in my opinion a "kind of condensate of frequentistic nonsense", see Ref. [3], in particular section 10.7 on *frequentistic coverage.* You might get a feeling of what happens taking Neyman's prescriptions literally playing with the 'the ultimate confidence intervals calculator' available in `http://www.roma1.infn.it/~dagos/ci_calc.html`.)

that they are in favor of H and against $\overline{H}$, as we already did in the AIDS example of Appendix F):

$$\tilde{O}_H(E_T, I) \;=\; \frac{P(E_T \,|\, E, I)\cdot P(E \,|\, H, I) + P(E_T \,|\, \overline{E}, I)\cdot P(\overline{E} \,|\, H, I)}{P(E_T \,|\, E, I)\cdot P(E \,|\, \overline{H}, I) + P(E_T \,|\, \overline{E}, I)\cdot P(\overline{E} \,|\, \overline{H}, I)}\,. \tag{39}$$

As expected, this formula is a bit more complicate that the Bayes factor calculated taking E for granted, which is recovered if the lie probabilities vanish

$$\tilde{O}_H(E_T, I) \quad \xrightarrow[P(E_T \,|\, \overline{E}, I)\,\to\, 0]{} \quad \tilde{O}_H(E, I)\,, \tag{40}$$

i.e. only when we are absolutely sure the witness does not err or lie reporting E (but Peirce reminds us that "absolute certainty, or an infinite chance, can never be attained by mortals" [6]).

In order to single out the effects of the new ingredients, Eq. (39) can be rewritten as[62]

$$\tilde{O}_H(E_T, I) \;=\; \tilde{O}_H(E, I)\times\frac{1+\lambda(I)\cdot\left[\frac{1}{P(E \,|\, H, I)} - 1\right]}{1+\lambda(I)\cdot\left[\frac{\tilde{O}_H(E,I)}{P(E \,|\, H, I)} - 1\right]}\,, \tag{41}$$

where

$$\lambda(I) \;=\; \frac{P(E_T \,|\, \overline{E}, I)}{P(E_T \,|\, E, I)}\,, \tag{42}$$

under the *condition*[63] $P(E \,|\, H, I) > 0$ and $P(E \,|\, \overline{H}, I) > 0$, i.e. $\tilde{O}_H(E, I)$ *positive and finite.* The parameter $\lambda(I)$, ratio of the *probability of fake evidence* and the *probability that the evidence is correctly accounted*, can be interpreted as a kind of *lie factor*. Given the human

[62] Factorizing $P(E \,|\, H, I)$ and $P(E \,|\, \overline{H}, I)$ respectively in the numerator and in the denominator, Eq. (39) becomes

$$\tilde{O}_H(E_T, I) \;=\; \tilde{O}_H(E, I)\times\frac{1+\frac{P(E_T \,|\, \overline{E}, I)}{P(E_T \,|\, E, I)}\cdot\frac{P(\overline{E} \,|\, H, I)}{P(E \,|\, H, I)}}{1+\frac{P(E_T \,|\, \overline{E}, I)}{P(E_T \,|\, E, I)}\cdot\frac{P(\overline{E} \,|\, \overline{H}, I)}{P(E \,|\, \overline{H}, I)}}\,.$$

Then $P(E_T \,|\, \overline{E}, I)/P(E_T \,|\, E, I)$ can be indicated as $\lambda(I)$, $P(\overline{E} \,|\, H_i, I)$ is equal to $1 - P(E \,|\, H_i, I)$ and, finally, $P(E \,|\, \overline{H}, I)$ can be written as $P(E \,|\, H, I)/\tilde{O}_H(E, I)$.

[63] Otherwise, obviously $\tilde{O}_H(E, I)$ cannot be factorized. The effective odds $\tilde{O}_H(E_T, I)$ can however be written in the following convenient forms

$$\begin{aligned}\left.\tilde{O}_H(E_T, I)\right|_{P(E \,|\, H, I)=0} &= \frac{1}{P(\overline{E} \,|\, \overline{H}) + P(E \,|\, \overline{H})/\lambda}\\ \left.\tilde{O}_H(E_T, I)\right|_{P(E \,|\, \overline{H}, I)=0} &= \lambda\, P(E \,|\, H) + P(\overline{E} \,|\, H)\,,\end{aligned}$$

although less interesting than Eq. (41).

roughly logarithmic sensibility to probability ratios, it might be useful to define, in analogy to the JL,

$$\mathrm{J}\lambda(I) \;=\; \log_{10}[\lambda(I)]\,. \tag{43}$$

Let us make some instructive limits of Eq. (41).

$$\tilde{O}_H(E_T,I) \;\xrightarrow[\lambda(I)\to 0]{}\; \tilde{O}_H(E,I) \tag{44}$$

$$\tilde{O}_H(E_T,I) \;\xrightarrow[\lambda(I)\to 1]{}\; 1 \tag{45}$$

$$\tilde{O}_H(E_T,I) \;\xrightarrow[P(E\,|\,H,I)\to 0]{}\; 1 \tag{46}$$

$$\tilde{O}_H(E_T,I) \;\xrightarrow[\tilde{O}_H(E,I)\to\infty]{}\; \frac{P(E\,|\,H,I)}{\lambda(I)}+1-P(E\,|\,H,I) \tag{47}$$

As we have seen, the ideal case is recovered if the lie factor vanishes. Instead, if it is equal to 1, i.e. $\mathrm{J}\lambda(I)=0$, the reported evidence becomes useless. The same happens if $P(E\,|\,H,I)$ vanishes [this implies that $P(E\,|\,\overline{H},I)$ vanishes too, being $P(\overline{H},I)=P(E\,|\,\overline{H},I)/\tilde{O}_H(E,I)$].

However, the most remarkable limit is the last one. It states that, even if $\tilde{O}_H(E,I)$ is very high, the effective Bayes factor cannot exceed the inverse of the lie factor:

$$\tilde{O}_H(E_T,I) \;\le\; \frac{P(E\,|\,H,I)}{\lambda(I)} \le \frac{1}{\lambda} \quad [\text{if } \tilde{O}_H(E,I)\to\infty]\,, \tag{48}$$

or, using logarithmic quantities

$$\Delta\mathrm{JL}(E_T,I) \;\le\; -\mathrm{J}\lambda+\log_{10}P(E\,|\,H,I)\le -\mathrm{J}\lambda \quad [\text{if } \Delta\mathrm{JL}(E,I)\to\infty]\,. \tag{49}$$

At this point some numerical examples are in order (and those who claim they can form their mind on pure intuition get all my admiration... *if* they really can). Let us imagine that E would ideally provide a weight of evidence of 6 [i.e. $\Delta\mathrm{JL}_H(E,I)=6$]. We can study, with the help of table 2, how the *weight of the reported evidence* $\Delta\mathrm{JL}_H(E_T,I)$ depends on the other beliefs [in this table logarithmic quantities have been used throughout, therefore $\mathrm{JL}_E(H,I)$ is the base ten logarithm of the odds in favor of E given the hypothesis H; the table provides, for comparisons, also $\Delta\mathrm{JL}_H(E_T,I)$ from $\Delta\mathrm{JL}_H(E,I)$ equal to 3 and 1].

The table exhibits the limit behaviors we have seen analytically. In particular, if we fully trust the report, i.e. $\mathrm{J}\lambda(I)=-\infty$, then $\Delta\mathrm{JL}_H(E_T,I)$ is exactly equal to $\Delta\mathrm{JL}_H(E,I)$, as we already know. But as soon as the absolute value of the lie factor is close to $\mathrm{JL}_H(E,I)$, there is a sizeable effect. The upper bound can be the be rewritten as

$$\tilde{O}_H(E_T,I) \;\le\; \min\,[\tilde{O}_H(E,I),\,\frac{1}{\lambda}]\,, \tag{50}$$

or

$$\Delta\mathrm{JL}_H(E_T,I) \;\le\; \min\,[\Delta\mathrm{JL}_H(E,I),\,-\mathrm{J}\lambda(I)]\,, \tag{51}$$

$\Delta\mathrm{JL}_H(E,I)=6$									
$\mathrm{J}\lambda(I)$		$\mathrm{JL}_H(E_T,I)$							
	$\mathrm{JL}_E(H,I)$:	10	3	2	1	0	-1	-3	-10
$\to -\infty$		6.00	6.00	6.00	6.00	6.00	6.00	6.00	6.00
-8		6.00	6.00	6.00	6.00	5.99	5.95	4.96	4×10^{-3}
-7		5.96	5.96	5.96	5.95	5.92	5.68	4.00	4×10^{-4}
-6		5.70	5.70	5.70	5.68	5.52	4.92	3.00	4×10^{-5}
-5		4.96	4.96	4.95	4.92	4.68	3.95	2.00	4×10^{-6}
-4		4.00	4.00	3.99	3.95	3.70	2.96	1.04	4×10^{-7}
-3		3.00	3.00	3.00	2.96	2.70	1.96	0.30	4×10^{-8}
-2		2.00	2.00	2.00	1.95	1.70	1.00	0.04	4×10^{-9}
-1		1.00	1.00	1.00	0.96	0.74	0.26	0.004	4×10^{-10}
0		0	0	0	0	0	0	0	0

$\Delta\mathrm{JL}_H(E,I)=3$									
$\mathrm{J}\lambda(I)$		$\mathrm{JL}_H(E_T,I)$							
	$\mathrm{JL}_E(H,I)$:	10	3	2	1	0	-1	-3	-10
$\to -\infty$		3.00	3.00	3.00	3.00	3.00	3.00	3.00	3.00
-5		3.00	3.00	3.00	3.00	2.99	2.95	1.96	4×10^{-6}
-4		2.96	2.96	2.96	2.95	2.92	2.68	1.04	4×10^{-7}
-3		2.70	2.70	2.70	2.68	2.52	1.93	0.30	4×10^{-8}
-2		1.96	1.96	1.96	1.92	1.68	1.00	0.04	4×10^{-9}
-1		1.00	1.00	0.99	0.96	0.74	0.26	0.004	4×10^{-10}
0		0	0	0	0	0	0	0	0

$\Delta\mathrm{JL}_H(E,I)=1$									
$\mathrm{J}\lambda(I)$		$\mathrm{JL}_H(E_T,I)$							
	$\mathrm{JL}_E(H,I)$:	10	3	2	1	0	-1	-3	-10
$\to -\infty$		1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
-3		1.00	1.00	1.00	1.00	0.99	0.96	0.26	4×10^{-8}
-2		0.96	0.96	0.96	0.96	0.93	0.72	0.04	4×10^{-9}
-1		0.72	0.72	0.72	0.70	0.58	0.23	0.003	4×10^{-10}
-0.5		0.41	0.41	0.41	0.39	0.27	0.07	8×10^{-4}	4×10^{-10}
0		0	0	0	0	0	0	0	0

Table 2: Dependence of the judgement leaning due to a reported evidence [$\Delta\mathrm{JL}_H(E_T,I)$] for $\Delta\mathrm{JL}_H(E,I)=6$, 3 and 1 as a function the other ingredients of the inference (see text). Note the upper limit of $\Delta\mathrm{JL}_H(E_T,I)$ to $-\mathrm{J}\lambda$, if this value is $\leq \Delta\mathrm{JL}_H(E,I)$.

a relation valid in the region of interest when thinking about an evidence in favor of H, i.e. $\Delta\mathrm{JL}_H(E,I) > 0$ and $\mathrm{J}\lambda(I) < 0$.

This upper bound is very interesting. Since minimum conceivable values of $\mathrm{J}\lambda(I)$ for human beings can be of the order of -6 (to perhaps ≈ -8 or ≈ -9, but in many practical applications -2 or -3 can already be very generous!), in practice the effective weights of evidence cannot exceed values of about $+6$ (I have no strong opinion on the exact value of this limit, my main point is that *you consider there might be such a practical human limit.*)

This observation has an important consequence in the combination of evidences, as anticipated at the end of section 3.5. Should we give more consideration to a single strong piece of evidence, virtually weighing $\Delta\mathrm{JL}(E) = 10$, or 10 independent weaker evidences, each having a ΔJL of 1? As it was said, in the ideal case they yield the same global leaning factor. But as soon as human fallacy (or conspiracy) is taken into account, and we remember that our belief is based on E_T and not on E, then we realize that $\Delta\mathrm{JL}(E_T) = 10$ is well above the range of JL that we can reasonably conceive. Instead the weaker pieces of evidence are little affected by this doubt and when they sum up together, they really can provide a ΔJL of about 10.

J A simple Bayesian network

Let us go back to our toy model of section 2 and let us complicate it just a little bit, adding the possibility of incorrect testimony (but we also simplify it using uniform priors, so that we can focus on the effect of the *uncertain evidence*). For example, imagine you do not see directly the color of the ball, but this is reported to you by a collaborator, who, however, might not tell you always the truth. We can model the possibility of a lie in following way: after each extraction he tosses a die and reports the true color only if the die gives a number smaller than 6. Using the formalism of Appendix I, we have

$$P(E_T \,|\, E, I) = 5/6 \qquad (52)$$

$$P(E_T \,|\, \overline{E}, I) = 1/6\,, \qquad (53)$$

i.e.

$$\lambda(I) = 1/5. \qquad (54)$$

The resulting *belief network*,[64] relative to five extractions and to the corresponding five reports is shown in figure 2, redrawn in a different way in figure 11. In this diagram the

[64] In complex situations an effects might have several (con-)causes; or an effect can be itself a cause of other effects; and so on. As it can be easily imagined, causes and effects can be represented by a graph, as that of figure 2. Since the connections between the *nodes* of the resulting *network* have usually the meaning of probabilistic links (but also deterministic relations can be included), this graph is called a *belief network*. Moreover, since Bayes' theorem is used to update the probabilities of the possible *states* of the nodes (the node 'Box', with reference to our toy model, has states B_1 and B_2; the node 'Ball' has states W and B), they are also called *Bayesian networks*. For more info, as well as tutorials and demos of powerful packages having also a friendly graphical user interface, I recommend visiting Hugin [12] and Netica [13] web sites. (My

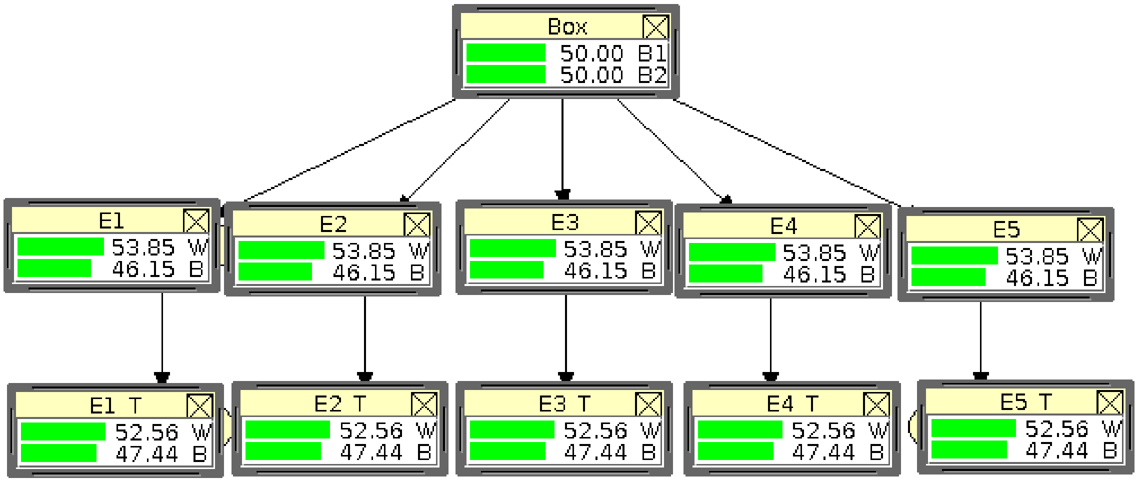


Figure 11: Same belief network of figure 2. This representation shows the 'monitors' giving the initial probabilities of all states of the variables. **If you like to test your intuition**, try the guess how *all* probabilities change when the information changes in the following order: a) witness 1 says white ($E_1T = W$); b) witness 2 also reports white ($E_2T = W$); c) then witness 3 claims, contrary to the previous two, to have observed black ($E_3T = B$); c) finally we directly observe the result of the fourth extraction, resulting black ($E_4 = B$). The **solutions** are in figures 13 to 16.

nodes are represented by 'monitors' that provide the probability of each *state* of the *variable*. The green bars mean that we are in condition of uncertainty with respect to all states of all variable. Let us describe the several nodes:

- Initial box compositions have probability 50% each, that was our assumption.
- The probability of white and black are the same for all extractions, with white a bit more probable than black (14/26 versus 12/26, that is 53.85% versus 46.15%).
- There is also higher probability that the 'witness' reports white, rather than black, but the difference is attenuated by the 'lie factors'.[65] In fact, calling W_T and B_T the reported colors we have

$$\begin{aligned}
P(W_T\,|\,I) &= P(W_T\,|\,W,I)\cdot P(W\,|\,I) + P(W_T\,|\,B,I)\cdot P(B\,|\,I) \qquad (55)\\
P(B_T\,|\,I) &= P(B_T\,|\,W,I)\cdot P(W\,|\,I) + P(B_T\,|\,B,I)\cdot P(B\,|\,I)\,. \qquad (56)
\end{aligned}$$

Let us now see what happens if we *observe white* (red bar in figure 12). All probabilities of

preference for Hugin is mainly due to the fact that it is multi-platform and runs nicely under Linux.) For a book introducing Bayesian networks in forensics, Ref. [14] is recommended. For a monumental probabilistic network on the '**case that will never end**', see Ref. [15] (if you like classic thrillers, the recent paper of the same author might be of your interest [16]).

[65] Note that there are in general two lie factors, one for E and one for $\overline{E}$. For simplicity we assume here they have the same value.

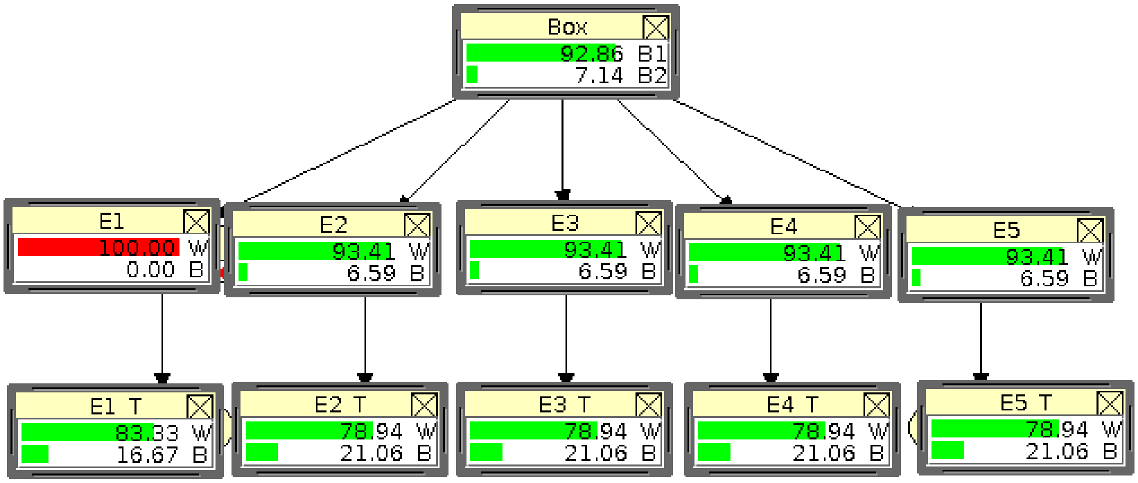


Figure 12: Status of the network after the observation of a white ball.

the network have been updated (Hugin [12] has nicely done the job for us[66]). We recognize the 93% of box B_1, that we already know. We also see that the increased belief on this box makes us more confident to observe white balls in the following extractions (after re-introduction).

More interesting is the case in which our inference is based on the reported color (figure 13). The fact that the witness could lie reduces, with respect to the previous case, our confidence on B_1 and on white balls in future extractions. As an exercise on what we have learned in appendix H, we can evaluate the 'effective' Bayes factor $\tilde{O}_{B_1}(W_T,I)$ that takes into account the testimony. Applying Eq. (41) we get

$$\begin{aligned}\tilde{O}_{B_1}(W_T,I) &= \tilde{O}_{B_1}(W,I)\times\frac{1+\lambda(I)\cdot\left[\frac{1}{P(W\,|\,B_1,I)}-1\right]}{1+\lambda(I)\cdot\left[\frac{\tilde{O}_H(W,I)}{P(W\,|\,H,I)}-1\right]} \qquad (57)\\ &= 13\times\frac{5}{17}=3.82\,, \qquad (58)\end{aligned}$$

or $\Delta\mathrm{JL}_{B_1}(W_T,I)=0.58$, about a factor of two smaller than $\Delta\mathrm{JL}_{B_1}(W,I)$, that was 1.1 (this mean we need two pieces of evidence of this kind to recover the loss of information due to the testimony).

The network gives us also the probability that the witness has really told us the truth, i.e. $P(W\,|\,W_T,I)$, that is *different* from $P(W_T\,|\,W,I)$, the reason being that white was initially a bit more probable than black.

Let us see now what happens if we get two concording testimonies (figure 14). As ex-

[66] The Hugin file can be found in `http://www.roma1.infn.it/~dagos/prob+stat.html#Columbo`.

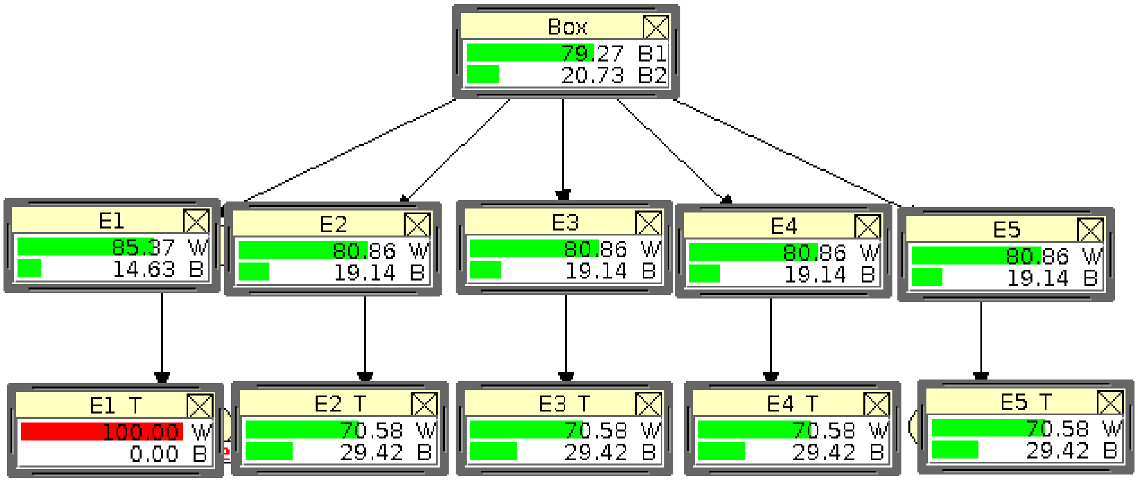


Figure 13: Status of the network after the report of a white ball (compare with figure 12).

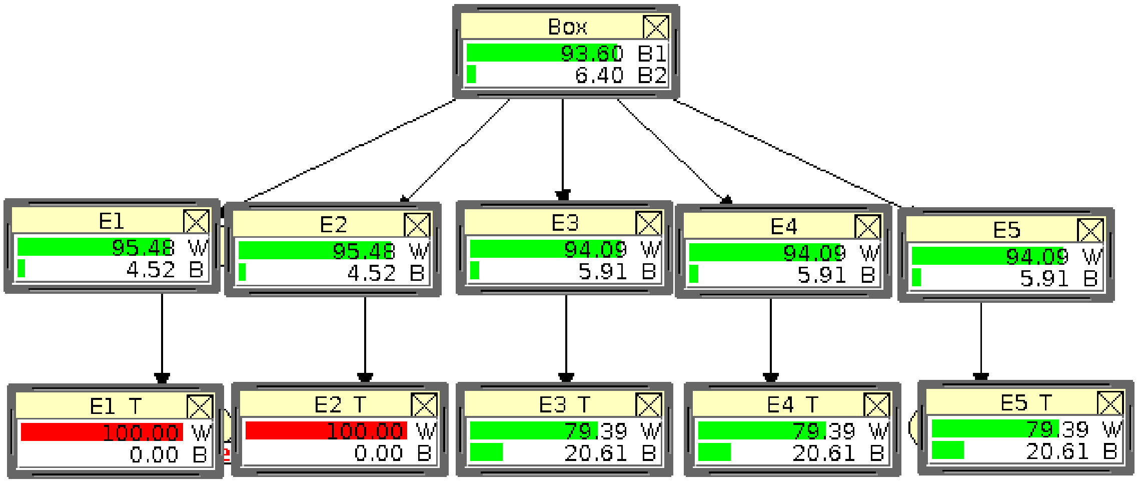


Figure 14: Network of figure 13 updated by a second testimony in favor of white.

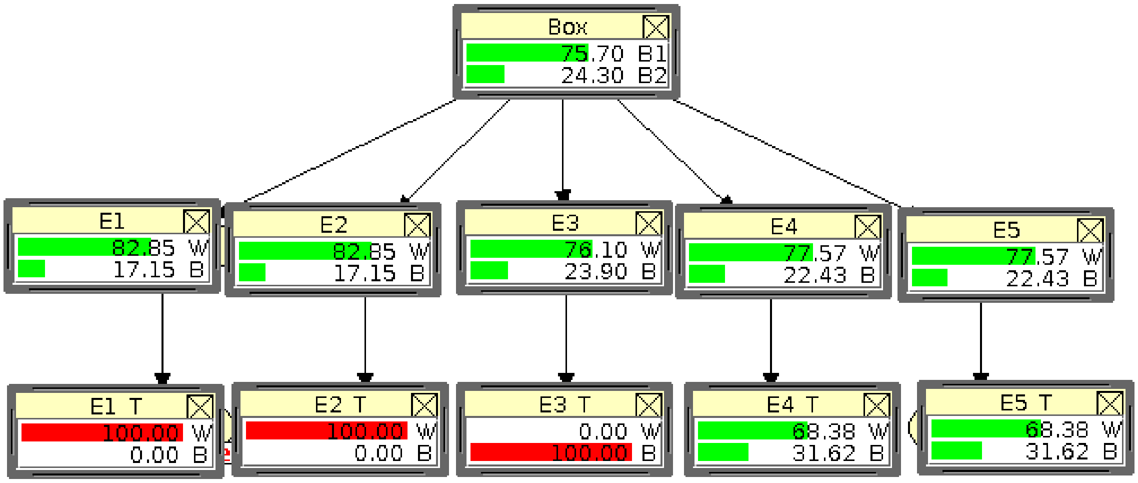


Figure 15: Network of figure 14 updated by a third testimony in favor of black.

pected, the probability of B_1 increases and becomes closer to the case of a direct observation of white. As usual, also the probabilities of future white balls increase.

The most interesting thing that comes from the result of the network is how the probabilities that the two witness lie change. First we see that they are the same, about 95%, as expected for symmetry. But the surprise is that the probability the the first witness said the truth has increased, passing from 85% to 95%. We can justify the variation because, in qualitative agreement with intuition, if we have concordant witnesses, *we tend to believe to each of them more than what we believed individually*. Once again, the result is, perhaps after an initial surprise, in qualitative agreement with intuition. The important point is that intuition is unable to get quantitative estimates. Again, the message is that, once we agree on the basic assumption and we check, whenever it is possible, that the results are reasonable, it is better to rely on automatic computation of beliefs.

Let's go on with the experiment and suppose the third witness says black (figure 15). This last information reduces the probability of B_1, but does not falsify this hypothesis, as if, instead, we had *observed* black. Obviously, it does also reduce the probability of white balls in the following extractions.

The other interesting feature concerns the probability that each witness has reported the truth. Our belief that the previous two witnesses really saw what they said is reduced to 83%. But, nevertheless we are more confident on the first two witnesses than on the third one, that we trust only at 76%, although the lie factor is the same for all of them. The result is again in agreement with intuition: if many witnesses state something and fewer say the opposite, *we tend to believe the majority*, if we initially consider all witnesses equally reliable. But a Bayesian network tells us also how much we have to believe the many more then the fewer.

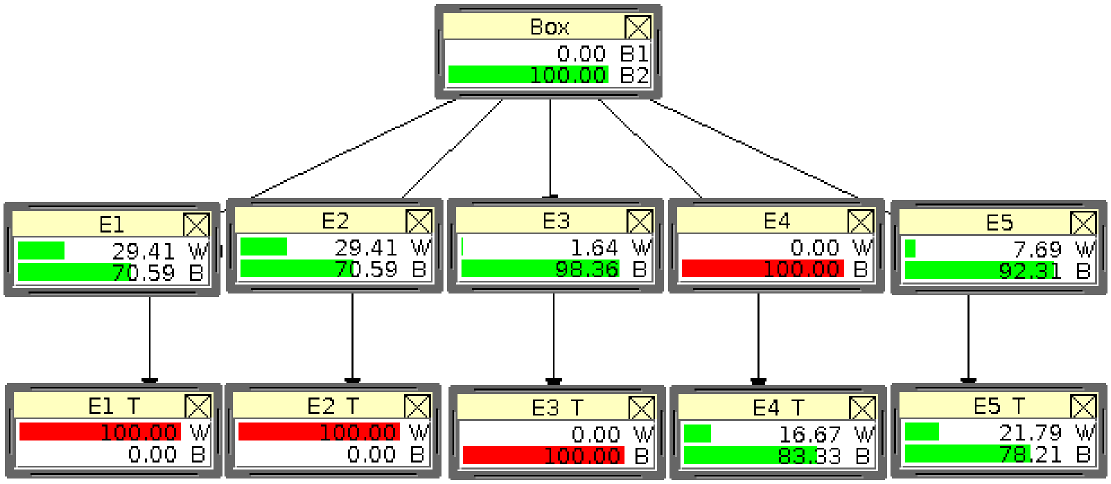


Figure 16: Network of figure 15 updated by a direct observation of a black ball.

Let us do, also in this case the exercise of calculating the effective Bayes factor, using however the first formula in footnote 63: the effective odds $\tilde{O}_H(B_T, I)$ can be written as

$$\tilde{O}_{B_1}(B_T, I) \quad = \quad \frac{1}{P(W\,|\,B_2) + P(B\,|\,B_2)/\lambda}, \tag{59}$$

i.e. $1/[1/13 + (12/13)/(1/5)] = 13/61 = 0.213$, smaller then 1 because they provide an evidence against box B_1 ($\Delta\text{JL} = -0.67$). It is also easy to check that the resulting probability of 75.7% of B_1 can be obtained summing up the three weights of evidence, two in favor of B_1 and two against it: $\Delta\text{JL}_{B_1}(W_T, W_T, B_T, I) = 0.58 + 0.58 - 0.67 = 0.49$, i.e. $\tilde{O}_{B_1}(W_T, W_T, B_T, I) = 10^{0.49} = 3.1$, that gives a probability of B_1 of 3.1/(1+3.1)=76%.

Finally, let us see what happens if we really see a black ball (E_4 in figure 16). Only in this case we become certain that the box is of the kind B_2, and the game is, to say, finished. But, nevertheless, we still remain in a state on uncertainty with respect to several things. The first one is the probability of a white ball in future extractions, that, from now becomes 1/13, i.e. 7.7%, and does not change any longer. But we also remain uncertain on whether the witnesses told us the truth, because what they said is not incompatible with the box composition. But, and again in qualitative agreement with the intuition, we trust much more whom told black (1.6% he lied) than the two who told white (70.6% they lied).

Another interesting way of analyzing the final network is to consider the probability of a black ball in the five extractions considered. The fourth is one, because we have seen it. The fifth is 92.3% (12/13) because we know the box composition. But in the first two extractions the probability is smaller than it (70.6%), while in the third is higher (98.4%). That is because in the two different cases we had an evidence respectively against and in favor of them.