Probabilità e incertezze di misura

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Piano dei due incontri

1. Rassegna critica e introduzione all'inferenza probabilistica
   - Quanto sono sensate e ben fondate le regolette per la valutazione dei cosiddetti "errori di misura"?
   - Per imparare dall'esperienza in modo quantitativo, facendo uso della logica dell'incerto, dobbiamo
     - rivedere il concetto di probabilità;
     - imparare ad . . . imparare dall'esperienza.
2. Stima delle incertezze in misure dirette e indirette
   - Sorgenti delle incertezze di misura (*decalogo ISO*).
   - Applicazione dell'inferenza probabilistica alle misure sperimentali (semplice caso di errori gaussiani):
     - singola osservazione
     - campione di osservazioni
     - stima dei parametri di un andamento lineare
   - Propagazione delle incertezze

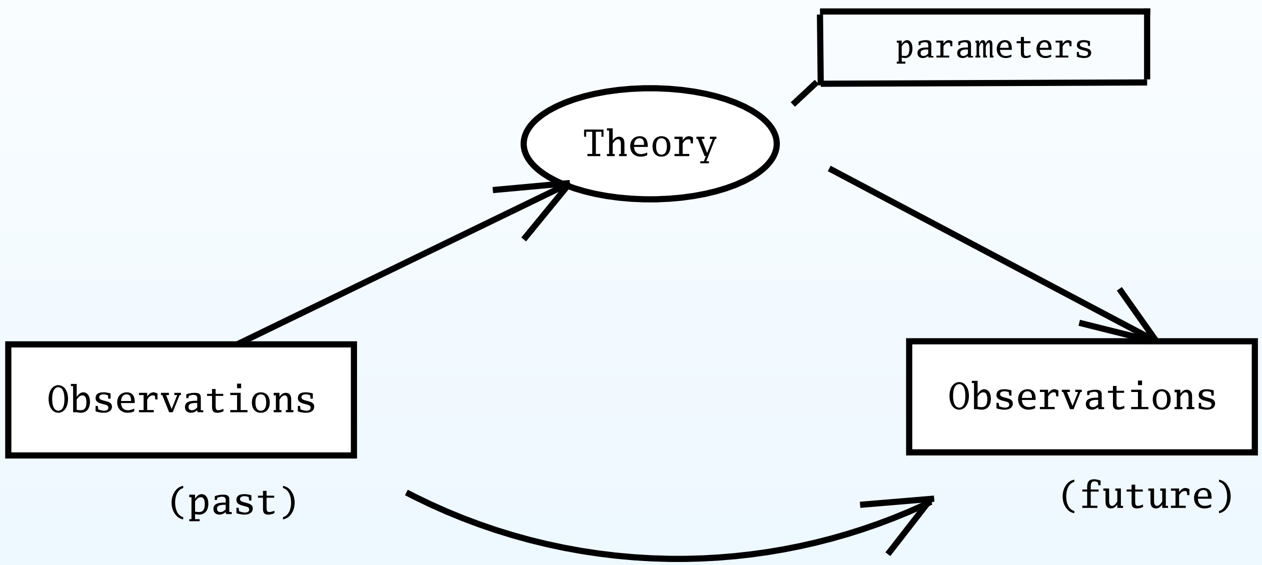
Scaletta del primo incontro

- Metodo scientifico: osservazioni e ipotesi
- Incertezza
- Cause $\longleftrightarrow$ Effetti

  "Il problema essenziale del metodo sperimentale" (Poincaré).
- L'esempio guida: il problema delle sei scatole.

  "La probabilità à riferita a casi reali o non ha alcun senso" (de Finetti).
- *Fisichettume*: una rassegna critica.
- Falsificazionismo e *variazioni statistiche* ('test').
- Approccio probabilistico.
- *Cosè la probabilità?* Regole di base della probabilità.
- Aggiornamento della probabilità alla luce delle osservazioni (formula di Bayes) $\Rightarrow$inferenza probabilistica (bayesiana)
- Conclusioni.

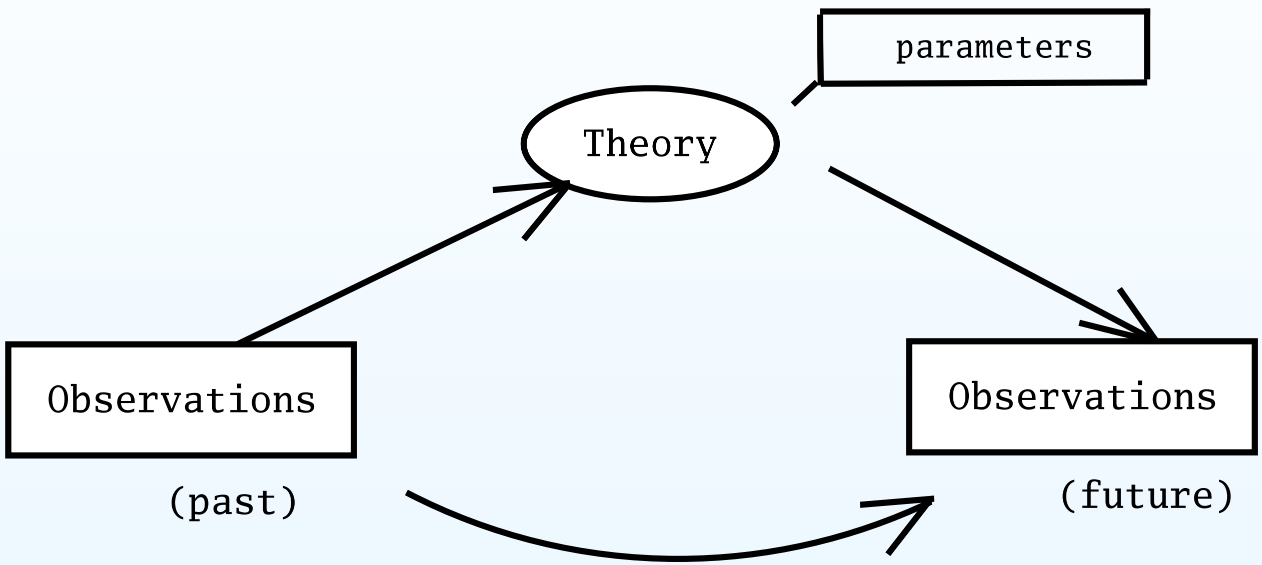
From past to future



Task of the ‘physicist’ (scientist, decision maker):

- Describe/understand the physical world
  $\Rightarrow$ inference of laws and their parameters
- Predict observations
  $\Rightarrow$ forecasting

From past to future



Process

- neither automatic
- nor purely contemplative

  $\longrightarrow$ ‘scientific method’

  $\longrightarrow$ planned experiments (‘actions’) $\Rightarrow$ decision.

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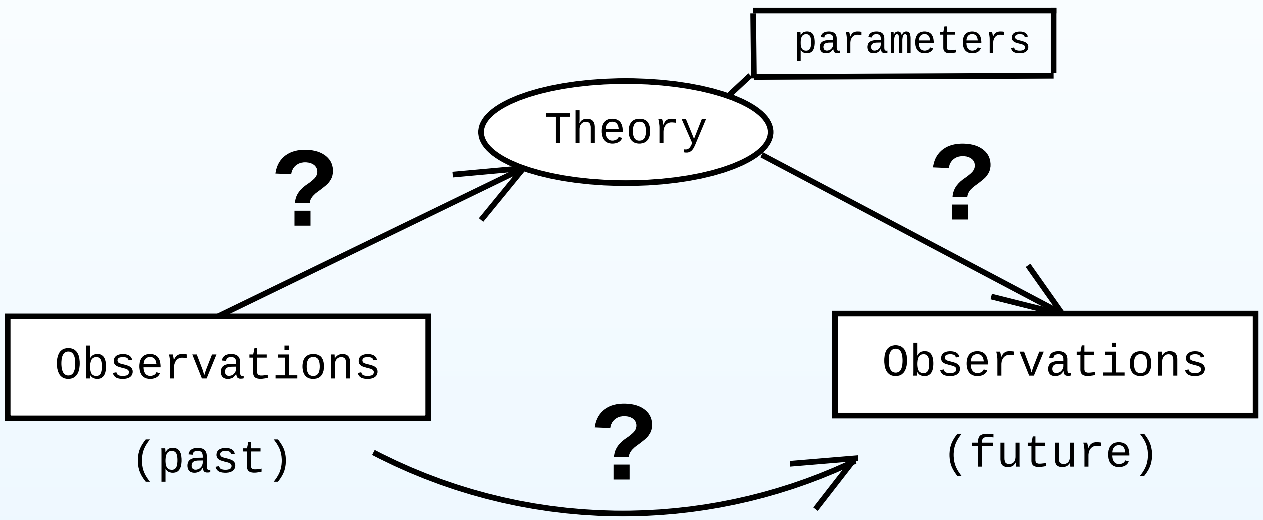
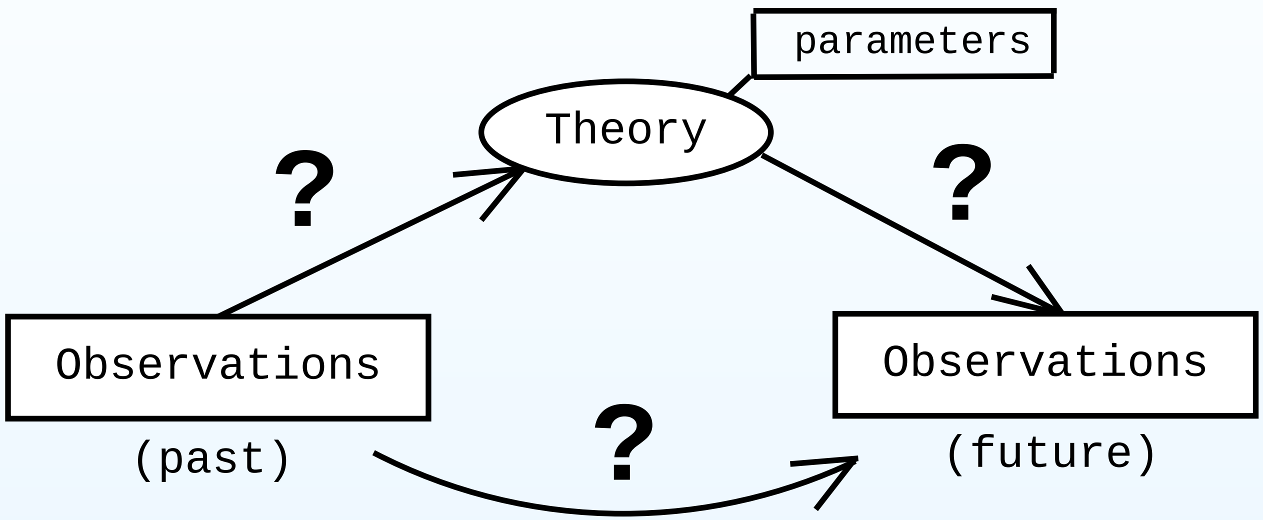


⇒ Uncertainty:

1. Given the past observations, in general we are not sure about the theory parameter (and/or the theory itself)
2. Even if we were sure about theory and parameters, there could be internal (e.g. Q.M.) or external effects (initial/boundary conditions, ‘errors’, etc) that make the forecasting uncertain.

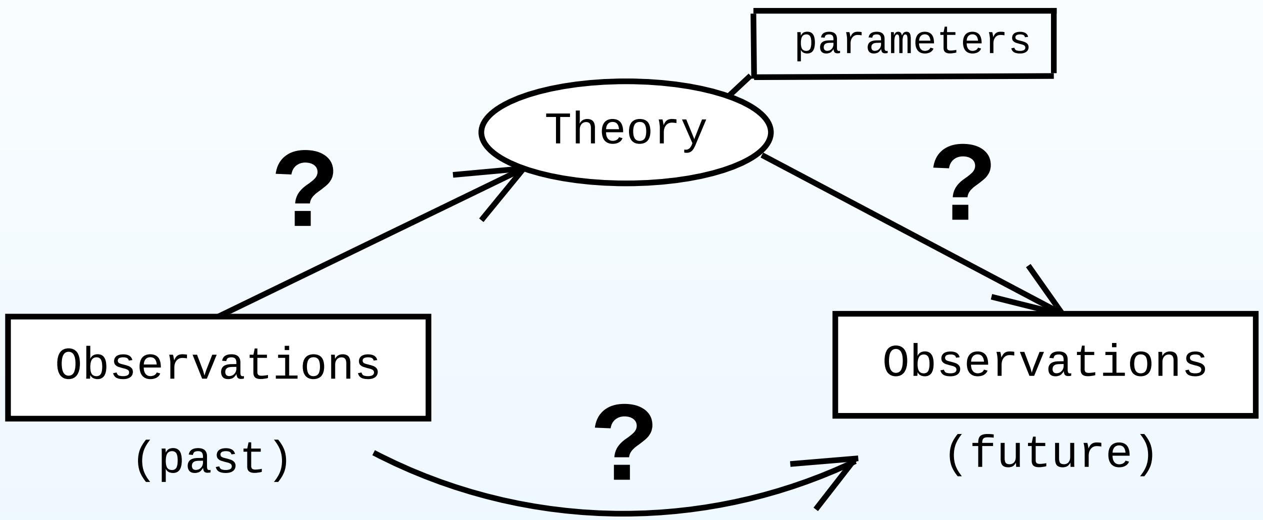
From past to future



⇒ Decision

- What is be best action ('experiment') to take in order 'to be confident' that what we would like will occur?
(Decision issues always assume uncertainty about future outcomes.)
- Before tackling problems of decision we need to learn to reason about uncertainty, possibly in a quantitative way.

From past to future



Deep reason of uncertainty

Theory	— ? ⟶	Future observations
Past observations	— ? ⟶	Theory
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About predictions

Remember:

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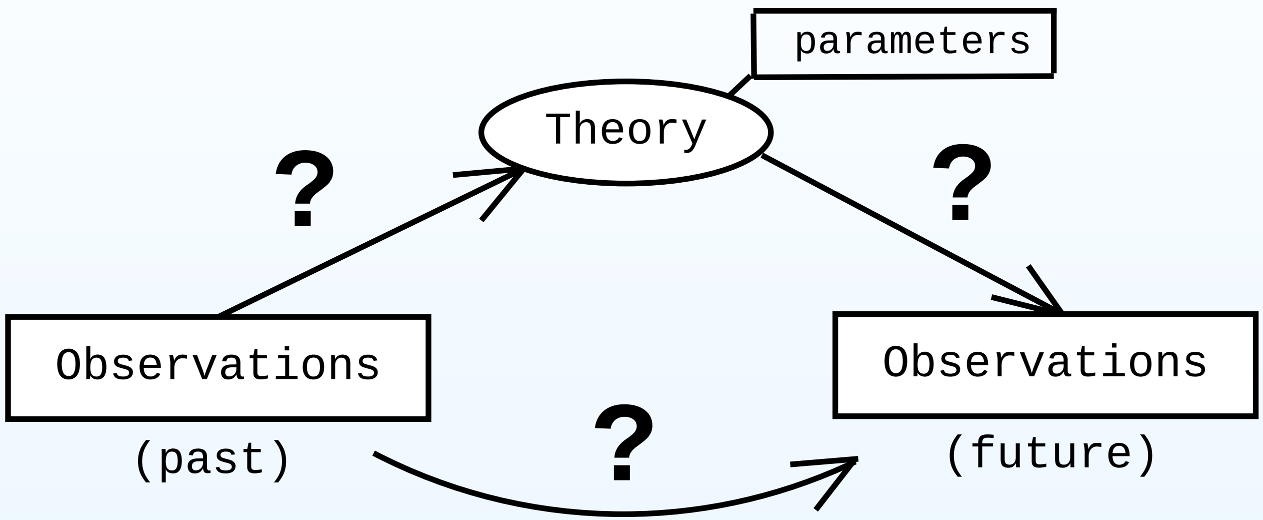
But, anyway:

"It is far better to foresee even without certainty than not to foresee at all" (Poincaré)

Deep source of uncertainty



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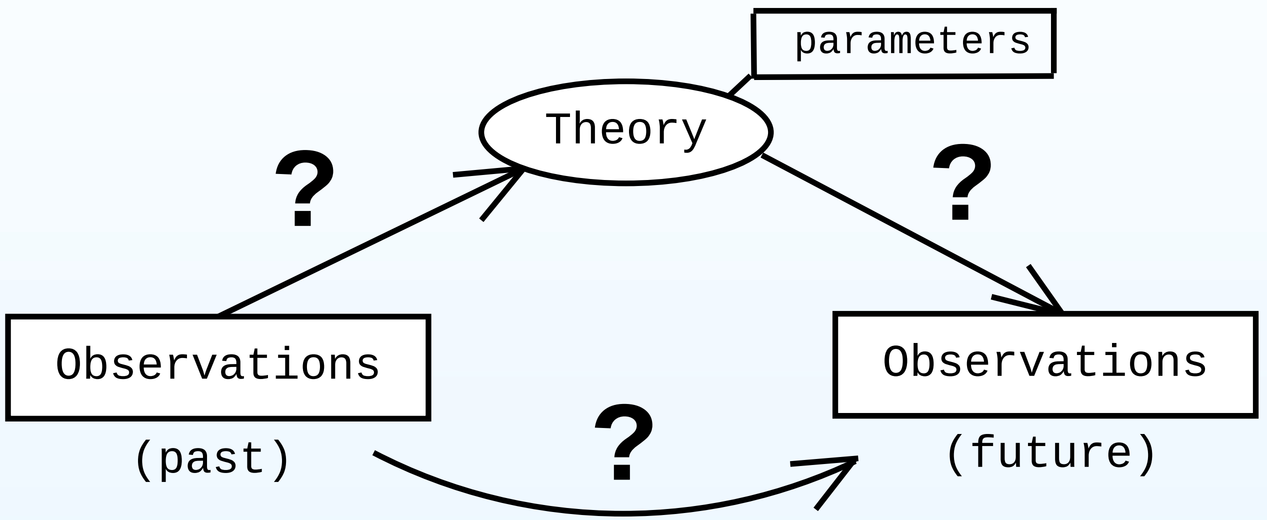

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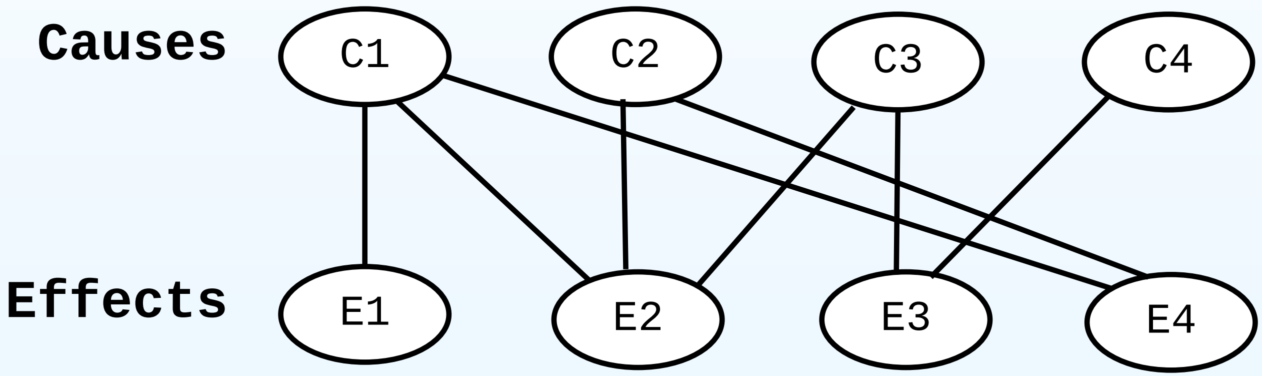
$\Longrightarrow$ **Uncertainty about causal connections**

CAUSE $\Longleftrightarrow$ EFFECT

Causes $\longrightarrow$ effects

The same *apparent* cause might produce several,different effects



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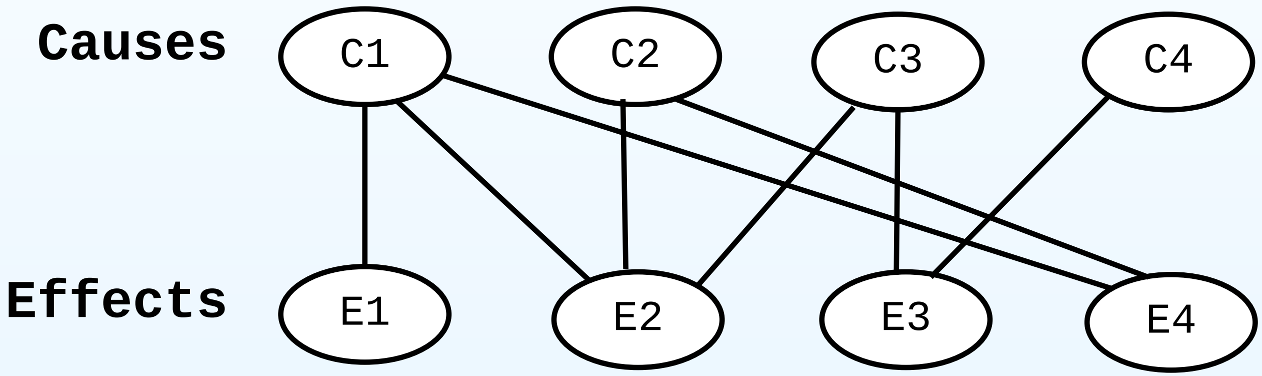

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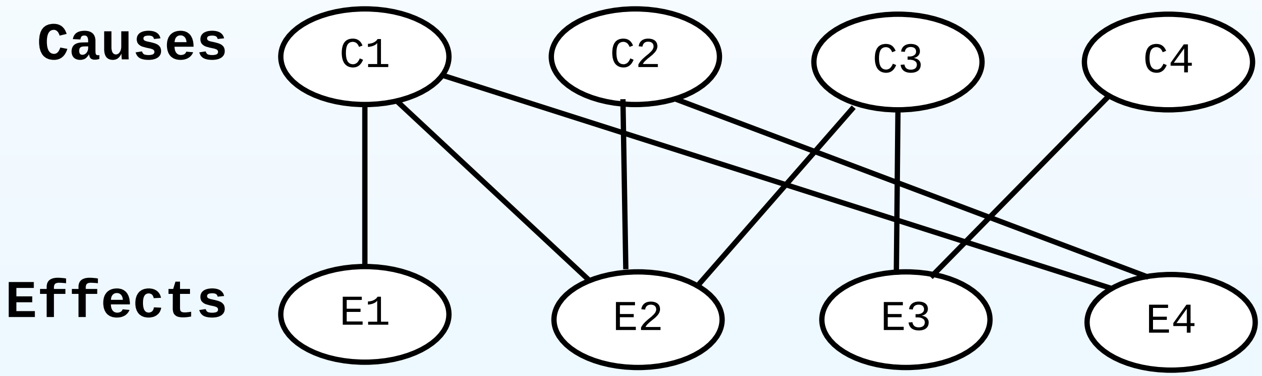

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$$\mathbf{E_2} \Rightarrow \{C_1,\, C_2,\, C_3\}?$$

The essential problem of the experimental method

“Now, these problems are classified as *probability of causes*, and are most interesting of all their scientific applications. I play at *écarté* with a gentleman whom I know to be perfectly honest. What is the chance that he turns up the king? It is 1/8. This is a problem of the *probability of effects*.

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I play with a gentleman whom I do not know. He has dealt ten times, and he has turned the king up six times. What is the chance that he is a sharper? This is a problem in the probability of causes. It may be said that it is the essential problem of the experimental method.”

(H. Poincaré – *Science and Hypothesis*)

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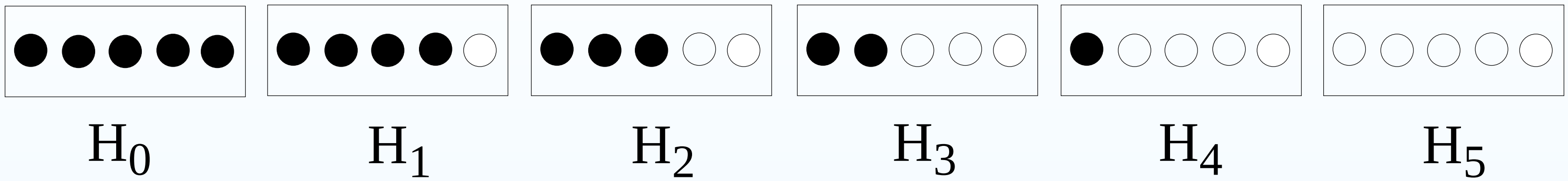
We can use similar expressions, all referring to the intuitive idea of probability.

The six box problem



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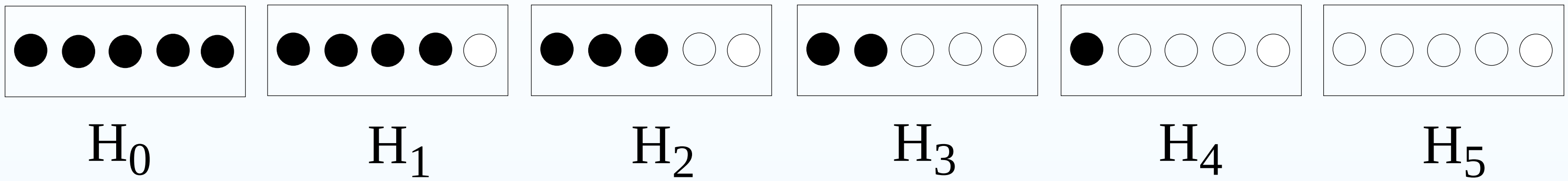

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We are in a state of uncertainty concerning several *events*, the most important of which correspond to the following questions:

(a) Which box have we chosen, $H_0, H_1, \ldots, H_5$?

(b) If we extract randomly a ball from the chosen box, will we observe a white ($E_W \equiv E_1$) or black ($E_B \equiv E_2$) ball?

Our certainty:

$$\begin{aligned} \cup_{j=0}^{5} H_j &= \Omega \\ \cup_{i=1}^{2} E_i &= \Omega\,. \end{aligned}$$

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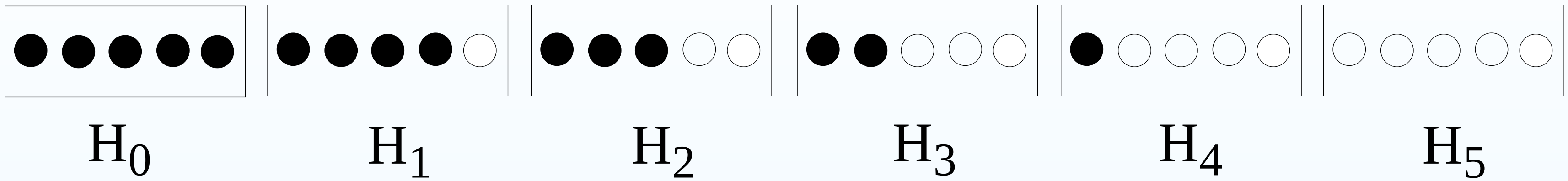

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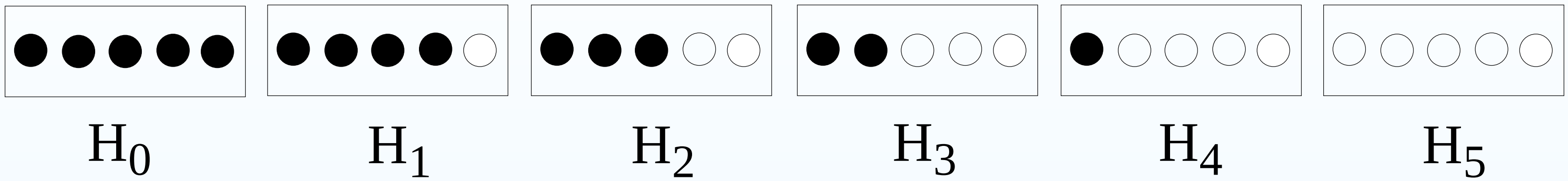

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  - Can we do it quantitatively, in an objective way?
- And after a sequence of extractions?

The toy inferential experiment

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This toy experiment is conceptually very close to what we do in Physics

- try to guess what we cannot see (the electron mass, a branching ratio, etc)

  ... from what we can see (somehow) with our senses.

The rule of the game is that we are not allowed to watch inside the box! (As we cannot open and electron and read its properties, like we read the MAC address of a PC interface)

Doing Science in conditions of uncertainty

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Indeed

“It is scientific only to say what is more likely and what is less likely” (Feynman)

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- **...Fisichettume**

  [Le varie formulette di “calcolo e propagazione degli errori”]

  ⇒ **Segue su lucidi**: vedi pp. 13-26 Ref. [2]

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Let start realizing that the method is analogous with method of the proof by contradiction of classical, deductive logic.

- Assume that a hypothesis is true
- Derive 'all' logical consequence
- If (at least) one of the consequences is known to be false, then the hypothesis is declared false.

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- What to do of all hypotheses that are not falsified? (Limbus? Get stuck?)
- What to do is nothing of what can be observed is incompatible with the hypothesis (or with many hypotheses)?

  E.g. H_i being a Gaussian $f(x \mid \mu_i, \sigma_i)$

  $\Rightarrow$ Given any pair or parameters $\{\mu_i, \sigma_i\}$, all values of x between $-\infty$ and $+\infty$ are possible.

  $\Rightarrow$ Having observed any value of x, none of H_i can be, strictly speaking, falsified.

Falsificationism and statistics

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in which the impossible is replaced by the improbable!

But from the impossible to the improbable there is not just a question of quantity, but a question of quality.

This mechanism, logically flawed, is particularly perverse, because deeply rooted in most people, due to education, but is not supported by logic.

⇒ Basically responsible of all fake claims of discoveries in the past decades.

[I am particularly worried about claims concerning our health, or the status of the planet, of which I have no control of the experimental data.]

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Example 1

Playing lotto

H: “I play honestly at lotto, betting on a rare combination”

E: “I win”

$$H \xrightarrow[\text{“practically impossible”}]{} E$$

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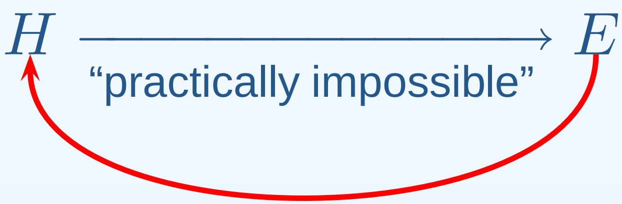

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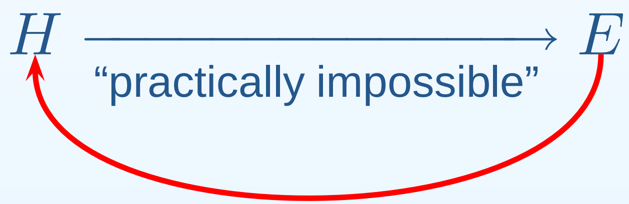

“practically to exclude”

$\Longrightarrow$ almost certainly I have cheated. . .

(or it is false that I won. . .)

Example 2

An Italian citizen is selected at random to undergo an AIDS test. Performance of clinical trial is not perfect, as customary.

Toy model:

$$\begin{aligned} P(\text{Pos} \,|\, \text{HIV}) &= 100\% \\ P(\text{Pos} \,|\, \overline{\text{HIV}}) &= 0.2\% \\ P(\text{Neg} \,|\, \overline{\text{HIV}}) &= 99.8\% \end{aligned}$$

H_1='HIV' (Infected) — E_1 = Positive

H_2='$\overline{\text{HIV}}$' (Healthy) — E_2 = Negative

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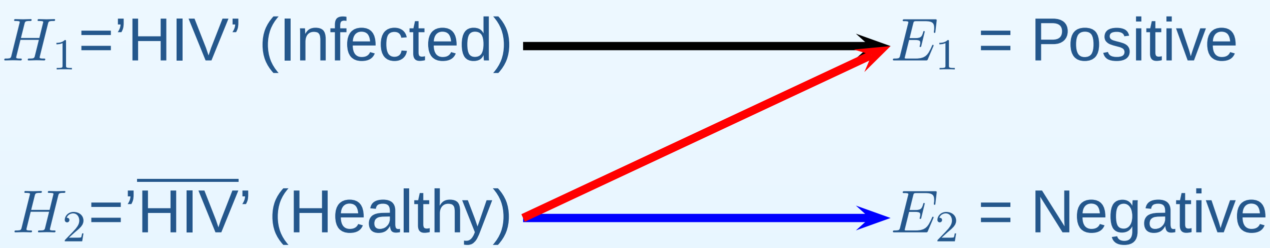

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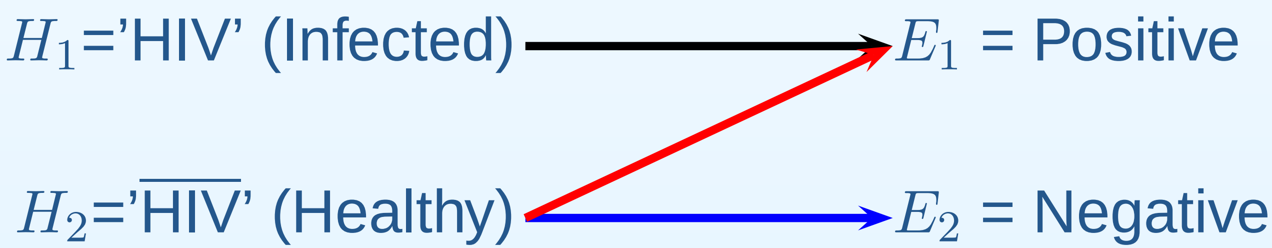

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(We will see in the sequel how to evaluate it correctly)

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  - as far as logic is concerned, the situation is worsened (. . . although p-values ‘often, by chance work’).
- Mistrust statistical tests, unless you know the details of what it has been done.

  $\longrightarrow$ You might take bad decisions!

Conflict: natural thinking $\Leftrightarrow$ cultural superstructure

Why? ‘Who’ is responsible?

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- In this, still dominant approach (frequentism) it is forbidden to speak about probability of hypotheses, probability of causes, probability of values of physical quantities, etc.

Conflict: natural thinking $\Leftrightarrow$ cultural superstructure

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What to do?

$\Longrightarrow$ Back to the past

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But benefitting of

- Theoretical progresses in probability theory
- Advance in computation (both symbolic and numeric)

$\longrightarrow$ many frequentistic ideas had their *raison d'être* in the computational barrier (and many simplified – often simplistic – methods were ingeniously worked out)

$\longrightarrow$ no longer an excuse!

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- "It's easy if you try"
- But first you have to recover the intuitive idea of probability.

What is probability?

Standard textbook definitions

$$p = \frac{\#\,\text{favorable cases}}{\#\,\text{possible equiprobable cases}}$$

$$p = \frac{\#\,\text{times the event has occurred}}{\#\,\text{independent trials under same conditions}}$$

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It is easy to check that 'scientific' definitions suffer of circularity

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It is easy to check that ‘scientific’ definitions suffer of circularity

$$p = \frac{\#\text{ favorable cases}}{\#\text{ possible equally possible cases}}$$

$$p = \frac{\#\text{ times the event has occurred}}{\#\text{ independent trials under same conditions}}$$

Laplace: *“lorsque rien ne porte à croire que l’un de ces cas doit arriver plutot que les autres”*

Pretending that replacing ‘equi-probable’ by ‘equi-possible’ is just cheating students (as I did in my first lecture on the subject. . .).

Standard textbook definitions

It is easy to check that 'scientific' definitions suffer of circularity, plus other problems

$$p = \frac{\#\ \text{favorable cases}}{\#\ \text{possible equiprobable cases}}$$

$$p = \lim_{n\rightarrow\infty} \frac{\#\ \text{times the event has occurred}}{\#\ \text{independent trials under same condition}}$$

Future ⇔ Past (believed so)

$n \rightarrow \infty$: $\longrightarrow$ *"usque tandem?"*

$\longrightarrow$ *"in the long run we are all dead"*

$\longrightarrow$ It limits the range of applications

Definitions $\longrightarrow$ evaluation rules

Very useful evaluation rules

$$A) \quad p = \frac{\#\,\text{favorable cases}}{\#\,\text{possible equiprobable cases}}$$

$$B) \quad p = \frac{\#\,\text{times the event has occurred}}{\#\text{independent trials under same condition}}$$

If the implicit beliefs are well suited for each case of application.

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BUT they cannot define the concept of probability!

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In the probabilistic approach we are going to see

- Rule A will be recovered immediately (under the assumption of equiprobability, when it applies).
- Rule B will result from a theorem (under well defined assumptions).

What is probability?

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Probability

What is probability?

It is what everybody knows what it is before going at school

→ how much we are confident that something is true

→ how much we believe something

→ "A measure of the degree of belief that an event *will* occur"

[Remark: 'will' does not imply future, but only uncertainty.]

Or perhaps you prefer this way…

"Given the state of our knowledge about everything that could possible have any bearing on the coming true[1]…,

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“Given the state of our knowledge about everything that could possible have any bearing on the coming true[1]*. . . , the numerical probability* p *of this event is to be a real number by the indication of which we try* in some cases *to setup a quantitative measure of the strength of our conjecture or anticipation, founded on the said knowledge, that the event comes true”*

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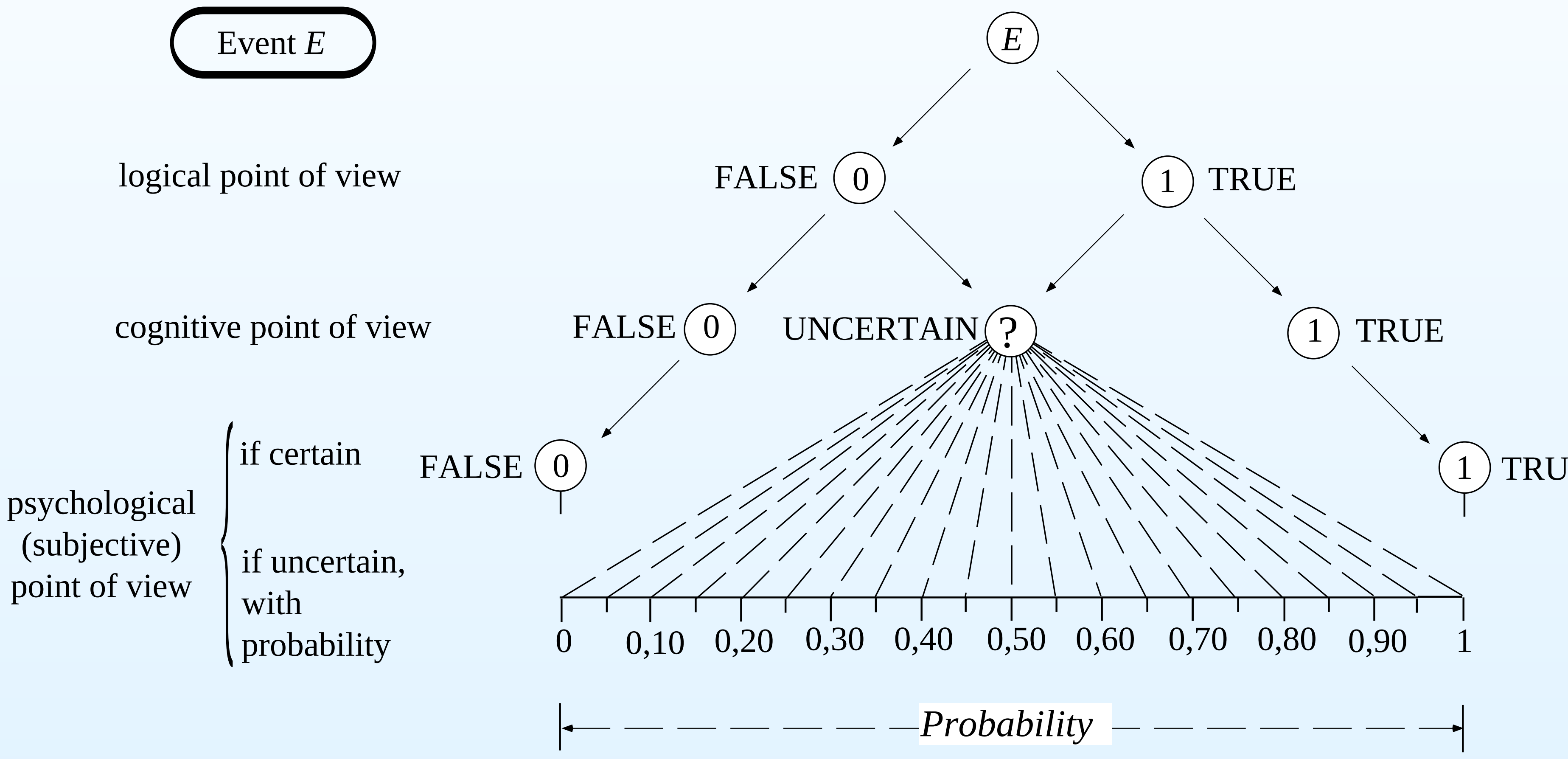
(E. Schrödinger, *The foundation of the theory of probability - I*, Proc. R. Irish Acad. 51A (1947) 51)

[1]*While in ordinary speech "to come true" usually refers to an event that is envisaged before it has happened, we use it here in the general sense, that the verbal description turns out to agree with actual facts.*

False, True and probable



Uncertainty $\longrightarrow$ probability

Probability is related to uncertainty and not (only) to the results of repeated experiments

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"If we were not ignorant there would be no probability, there could only be certainty. But our ignorance cannot be absolute, for then there would be no longer any probability at all. Thus the problems of probability may be classed according to the greater or less depth of our ignorance." (Poincaré)

Uncertainty $\longrightarrow$ probability

Probability is related to uncertainty and not (only) to the results of repeated experiments

The state of information can be different from subject to subject

$\Rightarrow$ intrinsic subjective nature.

- No negative meaning: only an acknowledgment that several persons might have different information and, therefore, necessarily different opinions.

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The state of information can be different from subject to subject

$\Rightarrow$ intrinsic subjective nature.

- No negative meaning: only an acknowledgment that several persons might have different information and, therefore, necessarily different opinions.
- *"Since the knowledge may be different with different persons or with the same person at different times, they may anticipate the same event with more or less confidence, and thus different numerical probabilities may be attached to the same event*" (Schrödinger)

Uncertainty $\longrightarrow$ probability

Probability is related to uncertainty and not (only) to the results of repeated experiments

Probability is always conditional probability

$$\text{`}P(E)\text{'} \;\longrightarrow\; P(E\,|\,I) \;\longrightarrow\; P(E\,|\,I(t))$$

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$$`P(E)' \;\longrightarrow\; P(E\,|\,I) \;\longrightarrow\; P(E\,|\,I(t))$$

- *“Thus whenever we speak loosely of ‘the probability of an event,’ it is always to be understood: probability with regard to a certain given state of knowledge*” (Schrödinger)

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- Some examples:
  - tossing a die;
  - 'three box problems';
  - two envelops' paradox.

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- Probability statements all have the same meaning no matter to what they refer and how the number has been evaluated.
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  They all convey unambiguously the same confidence on something.
- You might agree or disagree, but at least You know what this person has in his mind. (NOT TRUE with "C.L.'s"!)
- If a person has these beliefs and he/she has the chance to win a rich prize bound to one of these events, he/she has no rational reason to chose an event instead than the others.

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- Probability not bound to a single evaluation rule.
- In particular, combinatorial and frequency based 'definitions' are easily recovered as evaluation rules under well defined hypotheses.
- Keep separate concept from evaluation rule.

From the concept of probability to the probability theory

Ok, it looks nice, . . . but “how do we deal with ‘numbers’?”

From the concept of probability to the probability theory

- Formal structure: we need a mathematical structure in order to ‘propagate’ probability values to other, logically connected events:
  - basic rules
  - logic (mathematics)

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Coherent bet (de Finetti, Ramsey - ’Dutch book argument’)

It is well understood that bet odds can express confidence†

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Coherent bet $\longrightarrow$ A bet acceptable in both directions:

- You state your confidence fixing the bet odds
- . . . but somebody else chooses the direction of the bet
- best way to honestly assess beliefs.

$\longrightarrow$ see later for details, examples, objections, etc

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$\longrightarrow$ analogy to measures (we need to measure ’befiefs’)

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$\longrightarrow$ analogy to measures (we need to measure 'befiefs')

$\Rightarrow$ reference probabilities provided by simple cases in which equiprobability applies (coins, dice, turning wheels,. . .).

- Example: You are offered to options to receive a price: a) if E happens, b) if a coin will show head. Etc. . . .

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  Lindley's 'calibration' against 'standards'

$\longrightarrow$ Rational under everedays expressions like "there are 90 possibilities in 100" to state beliefs in situations in which the real possibilities are indeed only 2 (e.g. dead or alive)

- Example: a question to a student that has to pass an exam: a) normal test; b) pass it is a uniform random x will be ≤ 0.8.

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  Lindley's 'calibration' against 'standards'
- Also based on coherence, but it avoids the 'repulsion' of several person when they are asked to think directly in terms of bet (it is proved that many persons have reluctance to bet money).

Basic rules of probability

They all lead to

1. $0 \leq P(A) \leq 1$
2. $P(\Omega) = 1$
3. $P(A \cup B) = P(A) + P(B)$ $\quad$ [if $P(A \cap B) = \emptyset$]

4. $P(A \cap B) = P(A \,|\, B) \cdot P(B) = P(B \,|\, A) \cdot P(A)\,,$

where

- Ω stands for 'tautology' (a proposition that is certainly true $\rightarrow$ referring to an event that is certainly true) and $\emptyset = \overline{\Omega}$.
- $A \cap B$ is true only when both A and B are true (logical AND) (shorthands 'A, B' or $A\,B$ often used $\rightarrow$ logical product)
- $A \cup B$ is true when at least one of the two propositions is true (logical OR)

Basic rules of probability

Remember that probability is always conditional probability!

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3. $P(A \cup B \,|\, I) = P(A \,|\, I) + P(B \,|\, I)$ $\quad$ [if $P(A \cap B \,|\, I) = \emptyset$]
4. $P(A \cap B \,|\, I) = P(A \,|\, B, I) \cdot P(B \,|\, I) = P(B \,|\, A, I) \cdot P(A \,|\, I)$

I is the background condition (related to information I)

$\longrightarrow$ usually implicit (we only care on 're-conditioning')

Subjective $\neq$ arbitrary

Crucial role of the coherent bet

- You claim that <u>this</u> coin has 70% to give head?
  No problem with me: you place 70€ on head, I 30€ on tail and who wins take 100€.
  $\Rightarrow$ If OK with you, let's start.

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  $\Rightarrow$ Just reverse the bet

  (Like sharing goods, e.g. a cake with a child)

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  (Like sharing goods, e.g. a cake with a child)

$\Rightarrow$ Take into account all available information *in the most 'objective way'*
(Even that someone has a different opinion!)

$\Rightarrow$ It might seem paradoxically, but the 'subjectivist' is much more 'objective' than those who blindly use so-called objective methods.

Summary on probabilistic approach

- Probability means how much we believe something
- Probability values obey the following basic rules

1. $0 \leq P(A) \leq 1$
2. $P(\Omega) = 1$
3. $P(A \cup B) = P(A) + P(B) \quad [\text{if } P(A \cap B) = \emptyset\,]$
4. $P(A \cap B) = P(A \,|\, B) \cdot P(B) = P(B \,|\, A) \cdot P(A)\,,$

- All the rest by logic

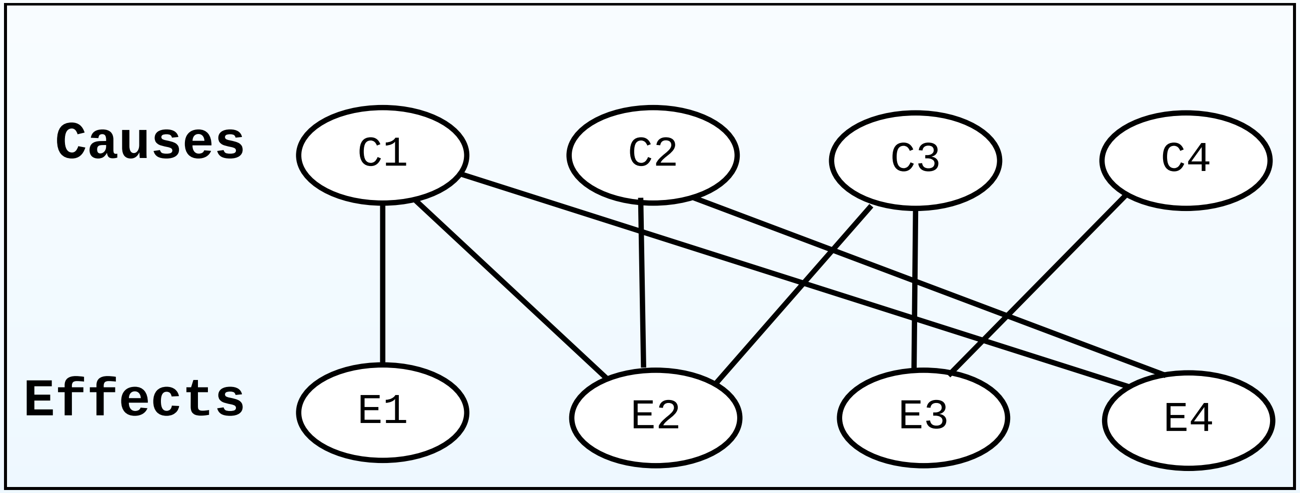
$\longrightarrow$ And, please, be coherent!

Inference

$\Longrightarrow$ How do we learn from data in a probabilistic framework?

From causes to effects and back

Our original problem:



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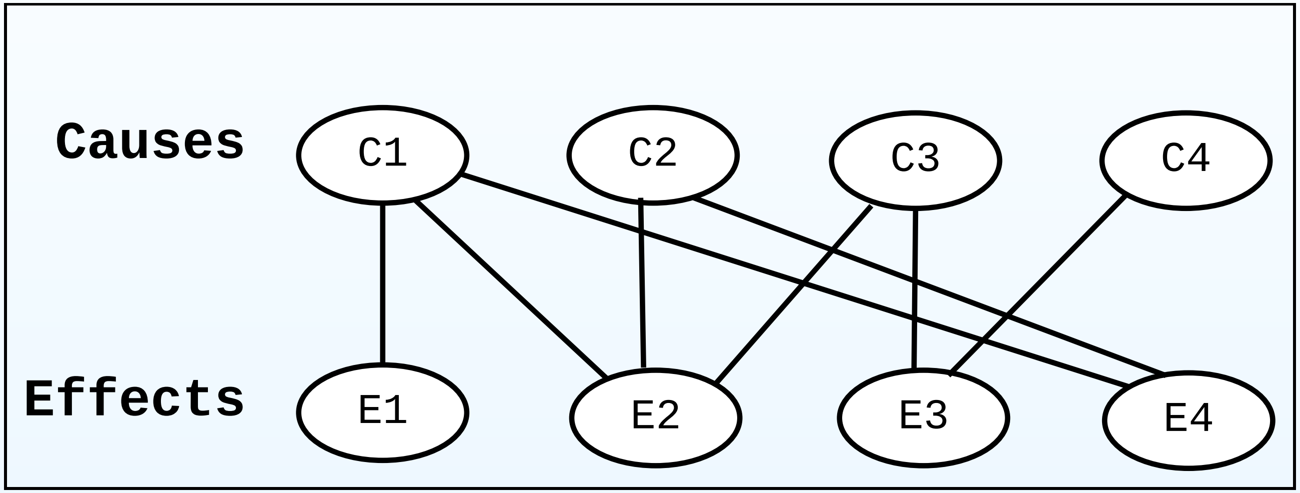

Our conditional view of probabilistic causation

$$\boxed{P(E_i \,|\, C_j)}$$

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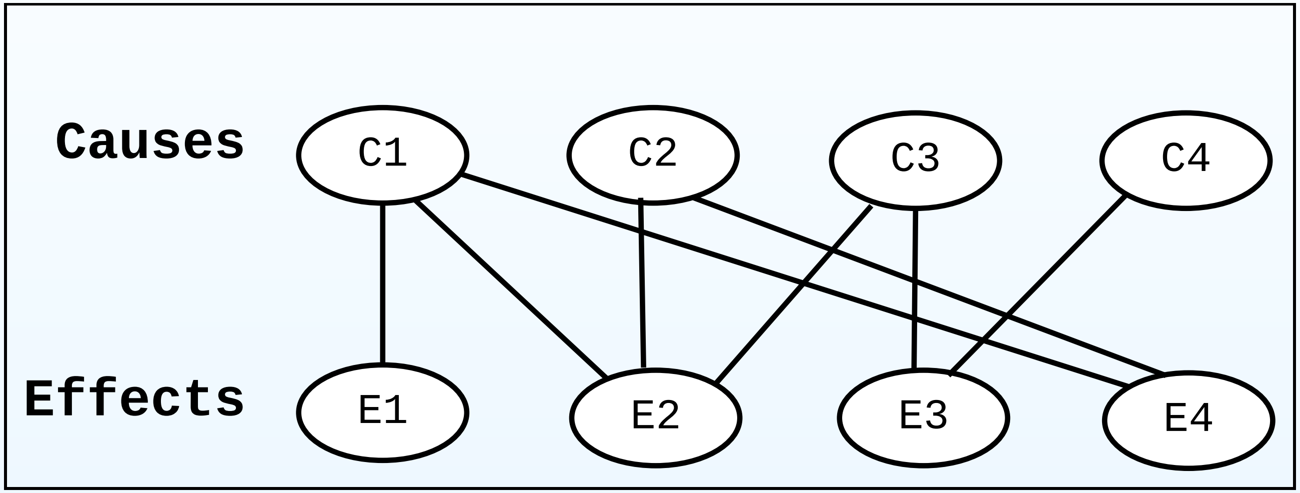

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Our conditional view of probabilistic inference

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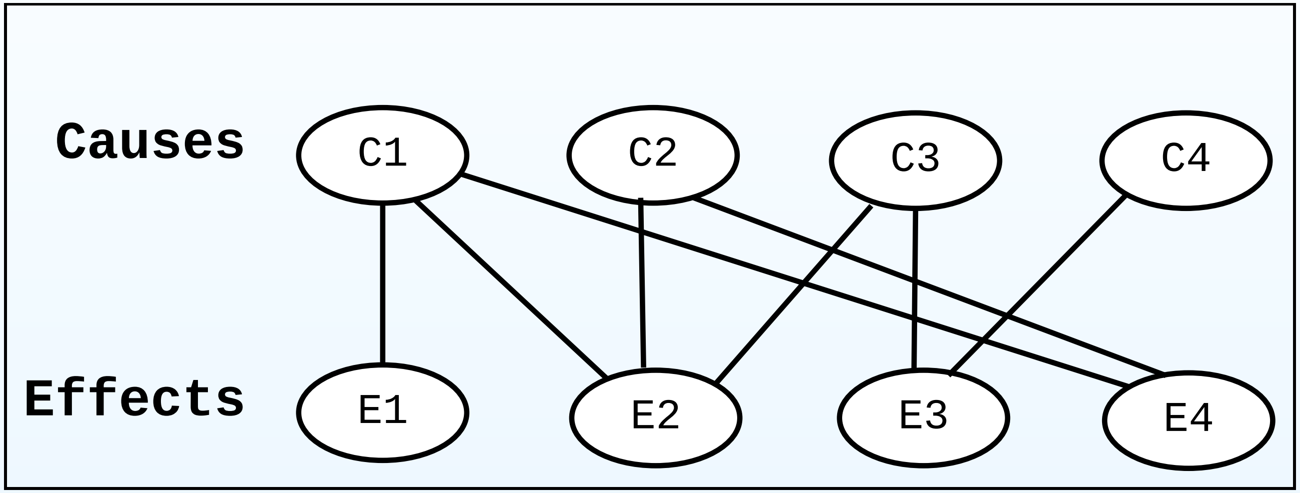

Our conditional view of probabilistic causation

$$P(E_i \,|\, C_j)$$

Our conditional view of probabilistic inference

$$P(C_j \,|\, E_i)$$

The fourth basic rule of probability:

$$P(C_j, E_i) = P(E_i \,|\, C_j)\, P(C_j) = P(C_j \,|\, E_i)\, P(E_i)$$

Symmetric conditioning

Let us take basic rule 4, written in terms of hypotheses H_j and effects E_i, and rewrite it this way:

$$\frac{P(H_j \,|\, E_i)}{P(H_j)} = \frac{P(E_i \,|\, H_j)}{P(E_i)}$$

"The condition on E_i changes in percentage the probability of H_j as the probability of E_i is changed in percentage by the condition H_j."

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Got 'after' Calculated 'before'

(where 'before' and 'after' refer to the knowledge that E_i is true.)

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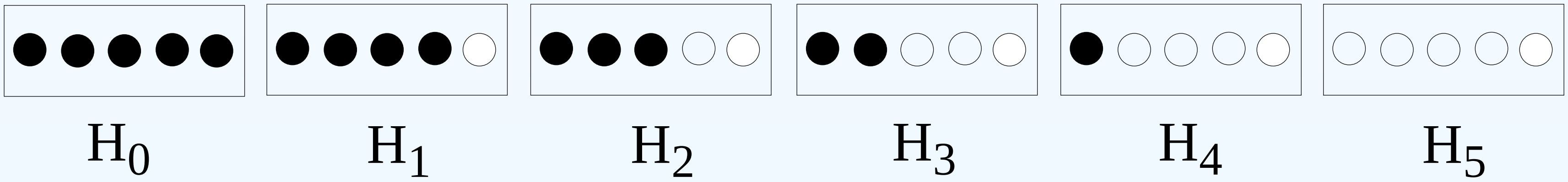
"post illa observationes" *"ante illa observationes"*

(Gauss)

Application to the six box problem



Remind:

- E_1 = White
- E_2 = Black

Collecting the pieces of information we need

Our tool:

$$P(H_j \,|\, E_i, I) = \frac{P(E_i \,|\, H_j, I)}{P(E_i \,|\, I)}\, P(H_j \,|\, I)$$

Collecting the pieces of information we need

Our tool:

$$P(H_j \,|\, E_i, I) = \frac{P(E_i \,|\, H_j, I)}{P(E_i \,|\, I)} \, P(H_j \,|\, I)$$

- $P(H_j \,|\, I) = 1/6$

Collecting the pieces of information we need

Our tool:

$$P(H_j \mid E_i, I) = \frac{P(E_i \mid H_j, I)}{P(E_i \mid I)} P(H_j \mid I)$$

- $P(H_j \mid I) = 1/6$
- $P(E_i \mid I) = 1/2$

Collecting the pieces of information we need

Our tool:

$$P(H_j \,|\, E_i, I) = \frac{P(E_i \,|\, H_j, I)}{P(E_i \,|\, I)} \, P(H_j \,|\, I)$$

- $P(H_j \,|\, I) = 1/6$
- $P(E_i \,|\, I) = 1/2$
- $P(E_i \,|\, H_j, I)$:

$$\begin{aligned} P(E_1 \,|\, H_j, I) &= j/5 \\ P(E_2 \,|\, H_j, I) &= (5-j)/5 \end{aligned}$$

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Our prior belief about H_j

Collecting the pieces of information we need

Our tool:

$$P(H_j \mid E_i, I) = \frac{P(E_i \mid H_j, I)}{P(E_i \mid I)} P(H_j \mid I)$$

- $P(H_j \mid I) = 1/6$
- $P(E_i \mid I) = 1/2$
- $P(E_i \mid H_j, I)$:

$$\begin{aligned} P(E_1 \mid H_j, I) &= j/5 \\ P(E_2 \mid H_j, I) &= (5-j)/5 \end{aligned}$$

Probability of E_i under a well defined hypothesis H_j
It corresponds to the 'response of the apparatus in measurements.
$\rightarrow$ likelihood (traditional, rather confusing name!)

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Probability of E_i taking account all possible H_j

$\rightarrow$ How much we are confident that E_i will occur.

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Probability of E_i taking account all possible H_j
$\rightarrow$ How much we are confident that E_i will occur.
Easy in this case, because of the symmetry of the problem.
But already after the first extraction of a ball our opinion about the box content will change, and symmetry will break.

Collecting the pieces of information we need

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But it easy to prove that $P(E_i \,|\, I)$ is related to the other ingredients, usually easier to 'measure' or to assess somehow, though vaguely

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But it easy to prove that $P(E_i \,|\, I)$ is related to the other ingredients, usually easier to 'measure' or to assess somehow, though vaguely

'decomposition law': $P(E_i \,|\, I) = \sum_j P(E_i \,|\, H_j, I) \cdot P(H_j \,|\, I)$

($\rightarrow$ Easy to check that it gives $P(E_i \,|\, I) = 1/2$ in our case).

Collecting the pieces of information we need

Our tool:

$$P(H_j \,|\, E_i, I) = \frac{P(E_i \,|\, H_j, I) \cdot P(H_j \,|\, I)}{\sum_j P(E_i \,|\, H_j, I) \cdot P(H_j \,|\, I)}$$

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We are ready!

$\longrightarrow$ R program

First extraction

After first extraction (and reintroduction) of the ball:

- $P(H_j)$ changes
- $P(E_j)$ for next extraction changes

Note: The box is exactly in the same status as before

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Note: The box is exactly in the same status as before

Where is probability?

$\longrightarrow$ Certainly not in the box!

Bayes theorem

The formulae used to *infer* H_i and
to *predict* $E_j^{(2)}$ are related to the name of Bayes

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Neglecting the background state of information I:

$$\frac{P(H_j \,|\, E_i)}{P(H_j)} = \frac{P(E_i \,|\, H_j)}{P(E_i)}$$

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Different ways to write the

Bayes' Theorem

Updating the knowledge by new observations

Let us repeat the experiment:

Sequential use of Bayes theorem

Old posterior becomes new prior, and so on

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$$P(H_j \,|\, E^{(1)}, E^{(2)}) \;\;\propto\;\; P(E^{(2)} \,|\, H_j, E^{(1)}) \cdot P(H_j \,|\, E^{(1)})$$

Updating the knowledge by new observations

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$$
\begin{aligned}
P(H_j \,|\, E^{(1)}, E^{(2)}) &\propto P(E^{(2)} \,|\, H_j, E^{(1)}) \cdot P(H_j \,|\, E^{(1)}) \\
&\propto P(E^{(2)} \,|\, H_j) \cdot P(H_j \,|\, E^{(1)})
\end{aligned}
$$

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&\propto P(E^{(2)} \,|\, H_j) \cdot P(E^{(1)} \,|\, H_j) \cdot P_0(H_j)
\end{aligned}
$$

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&\propto P(E^{(2)}\,|\,H_j)\cdot P(E^{(1)}\,|\,H_j)\cdot P_0(H_j) \\
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P(H_j \,|\, \text{data}) &\propto P(\text{data} \,|\, H_j) \cdot P_0(H_j)
\end{aligned}
$$

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P(H_j \,|\, \text{data}) &\propto P(\text{data} \,|\, H_j) \cdot P_0(H_j)
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$$

Bayesian inference

Updating the knowledge by new observations

Let us repeat the experiment:

Sequential use of Bayes theorem

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\begin{aligned}
P(H_j \,|\, E^{(1)}, E^{(2)}) &\propto P(E^{(2)} \,|\, H_j, E^{(1)}) \cdot P(H_j \,|\, E^{(1)}) \\
&\propto P(E^{(2)} \,|\, H_j) \cdot P(H_j \,|\, E^{(1)}) \\
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P(H_j \,|\, \text{data}) &\propto P(\text{data} \,|\, H_j) \cdot P_0(H_j)
\end{aligned}
$$

Learning from data using probability theory

Solution of the AIDS test problem

$$
\begin{aligned}
P(\text{Pos}\,|\,\text{HIV}) &= 100\% \\
P(\text{Pos}\,|\,\overline{\text{HIV}}) &= 0.2\% \\
P(\text{Neg}\,|\,\overline{\text{HIV}}) &= 99.8\%
\end{aligned}
$$

We miss something: $P_\circ(\text{HIV})$ and $P_\circ(\overline{\text{HIV}})$: Yes! We need some input from our best knowledge of the problem. Let us take $P_\circ(\text{HIV}) = 1/600$ and $P_\circ(\overline{\text{HIV}}) \approx 1$ (the result is rather stable against *reasonable* variations of the inputs!)

$$
\begin{aligned}
\frac{P(\text{HIV}\,|\,\text{Pos})}{P(\overline{\text{HIV}}\,|\,\text{Pos})} &= \frac{P(\text{Pos}\,|\,\text{HIV})}{P(\text{Pos}\,|\,\overline{\text{HIV}})} \cdot \frac{P_\circ(\text{HIV})}{P_\circ(\overline{\text{HIV}})} \\
&= \frac{\approx 1}{0.002} \times \frac{0.1/60}{\approx 1} = 500 \times \frac{1}{600} = \frac{1}{1.2}
\end{aligned}
$$

Odd ratios and Bayes factor

$$
\begin{aligned}
\frac{P(\mathsf{HIV}\,|\,\mathsf{Pos})}{P(\overline{\mathsf{HIV}}\,|\,\mathsf{Pos})} &= \frac{P(\mathsf{Pos}\,|\,\mathsf{HIV})}{P(\mathsf{Pos}\,|\,\overline{\mathsf{HIV}})} \cdot \frac{P_\circ(\mathsf{HIV})}{P(\overline{\mathsf{HIV}})} \\
&= \frac{\approx 1}{0.002} \times \frac{0.1/60}{\approx 1} = 500 \times \frac{1}{600} = \frac{1}{1.2} \\
\Rightarrow P(\mathsf{HIV}\,|\,\mathsf{Pos}) &= 45.5\%\,.
\end{aligned}
$$

Odd ratios and Bayes factor

$$\frac{P(\text{HIV}\,|\,\text{Pos})}{P(\overline{\text{HIV}}\,|\,\text{Pos})} = \frac{P(\text{Pos}\,|\,\text{HIV})}{P(\text{Pos}\,|\,\overline{\text{HIV}})} \cdot \frac{P_\circ(\text{HIV})}{P(\overline{\text{HIV}})}$$

$$= \frac{\approx 1}{0.002} \times \frac{0.1/60}{\approx 1} = 500 \times \frac{1}{600} = \frac{1}{1.2}$$

$$\Rightarrow P(\text{HIV}\,|\,\text{Pos}) = 45.5\%\,.$$

There are some advantages in expressing Bayes theorem in terms of odd ratios:

- There is no need to consider all possible hypotheses (how can we be sure?)
  We just make a comparison of any couple of hypotheses!

Odd ratios and Bayes factor

$$\frac{P(\mathsf{HIV}\,|\,\mathsf{Pos})}{P(\overline{\mathsf{HIV}}\,|\,\mathsf{Pos})} = \frac{P(\mathsf{Pos}\,|\,\mathsf{HIV})}{P(\mathsf{Pos}\,|\,\overline{\mathsf{HIV}})} \cdot \frac{P_\circ(\mathsf{HIV})}{P(\overline{\mathsf{HIV}})}$$

$$= \frac{\approx 1}{0.002} \times \frac{0.1/60}{\approx 1} = 500 \times \frac{1}{600} = \frac{1}{1.2}$$

$$\Rightarrow P(\mathsf{HIV}\,|\,\mathsf{Pos}) = 45.5\%\,.$$

There are some advantages in expressing Bayes theorem in terms of odd ratios:

- There is no need to consider all possible hypotheses (how can we be sure?)
  We just make a comparison of any couple of hypotheses!
- Bayes factor is usually much more inter-subjective, and it is often considered an 'objective' way to report how much the data favor each hypothesis.

Conclusioni

- Attenti alle formulette che girano su libri e appunti:
  $\Rightarrow$ vanno passate al vaglio della ragione
- La logica del certo inadatta alla trattazione delle incertezze: risultati assurdi o troppo conservativi
- Lo strumento concettuale corretto per trattare l'incertezza è quello di probabilità
- . . . a patto di usare il concetto intuitivo e non artefatti matematici
- $\Rightarrow$ probabilità soggettiva.
  Niente di negativo nel termine, solo accettare il fatto che la probabilità dipende dallo stato di conoscenza e che questo varia dalle persone e dal tempo.
- Lo strumento per riaggiornare la probabilità alla luce delle nuove osservazioni è il Teorema di Bayes

Prossimamente

- La prossima volta vedremo come estendere l'inferenza bayesiana alle incertezze di misura,
- . . . ma, concettualmente, abbiamo già detto tutto.

Documentazione:

$\Rightarrow$ Sito docente ($\longrightarrow$Google $\longrightarrow$Teaching)