Minimal solution of uncertainty propagation

The most general problem:

$$f(x_1, x_2, \ldots, x_n) \xrightarrow[Y_j = Y_j(X_1, X_2, \ldots, X_n)]{} f(y_1, y_2, \ldots, y_m)\,.$$

The 'minimal' solution: linear combinations, i.e.

$$\left\{ \begin{array}{l} \mathrm{E}(X_i) \\ \sigma(X_i) \\ \rho(X_i, X_{i'}) \end{array} \right. \xrightarrow[Y_j = c_{j0} + c_{j1}X_1 + c_{j2}X_2 + \cdots + c_{jn}X_n]{} \left\{ \begin{array}{l} \mathrm{E}(Y_j) \\ \sigma(Y_j) \\ \rho(Y_j, Y_{j'}) \end{array} \right.$$

But not forgetting the correlations!

Linear combination of independent variables

Simple, but instructive and important case:

- One (*output*) variable (Y) depending from many (*input*) quantities X_i, $i = 1, 2, \ldots, n$.

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Less general (it holds only if X_i are independent) property:

$$\begin{aligned} \mathrm{Var}[Y] &= c_1^2 \mathrm{Var}[X_1] + c_2^2 \mathrm{Var}[X_2] + \cdots + c_n^2 \mathrm{E}[X_n] \\ &= \sum_i c_i^2 \mathrm{Var}[X_i] \end{aligned}$$

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These two properties (and the extension of the second in the case of *correlated input variable*) are the main reason to prefer, as mostly representative summaries of distributions,

- expected value
- standard deviation ($= \sqrt{\mathrm{Var}}$)

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"A linear combinations of Gaussians
is still Gaussian"!

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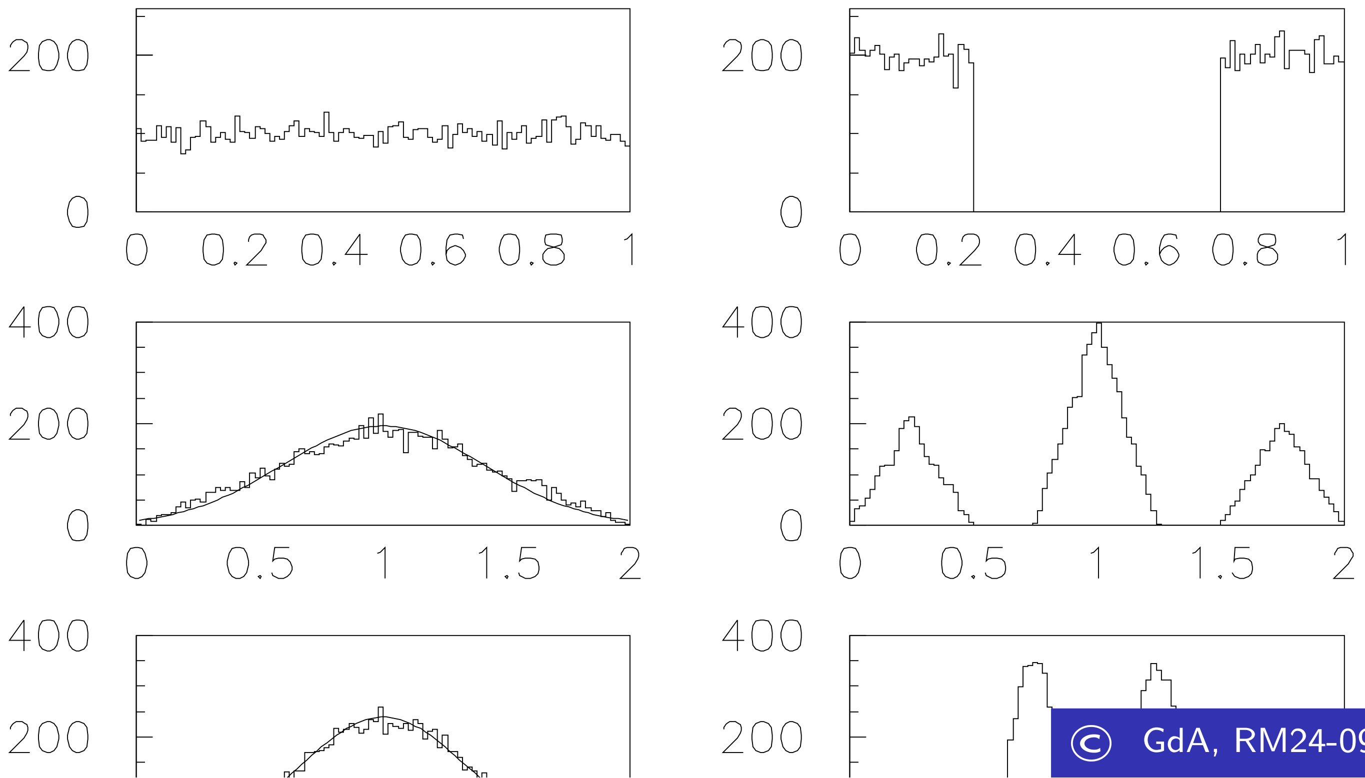
Central Limit Theorem:

$$\text{“}n \to \infty\text{''} \Longrightarrow Y \sim \mathcal{N}\left(\sum_{i=1}^{n} c_i\, \mathrm{E}(X_i), \left(\sum_{i=1}^{n} c_i^2\, \sigma_i^2\right)^{\frac{1}{2}}\right)$$

if $c_i^2\, \sigma_i^2 << \sum_{i=1}^{n} c_i^2\, \sigma_i^2$ for all X_i not described by a Gaussian!
(i.e. a single non-Gaussian variable has not to dominate the uncertainty about Y.)

Central Limit Theorem: a cartoon 'proof'



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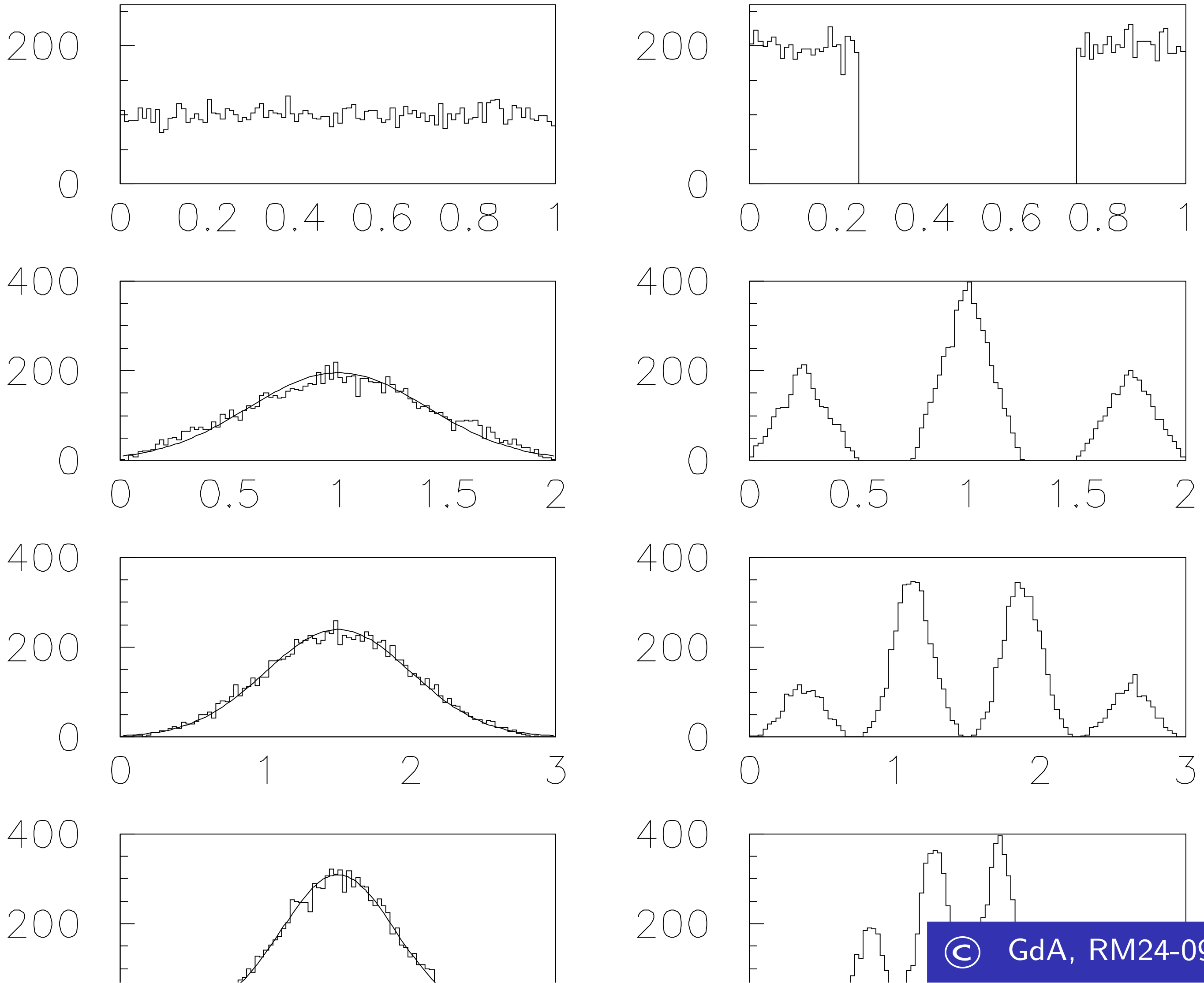

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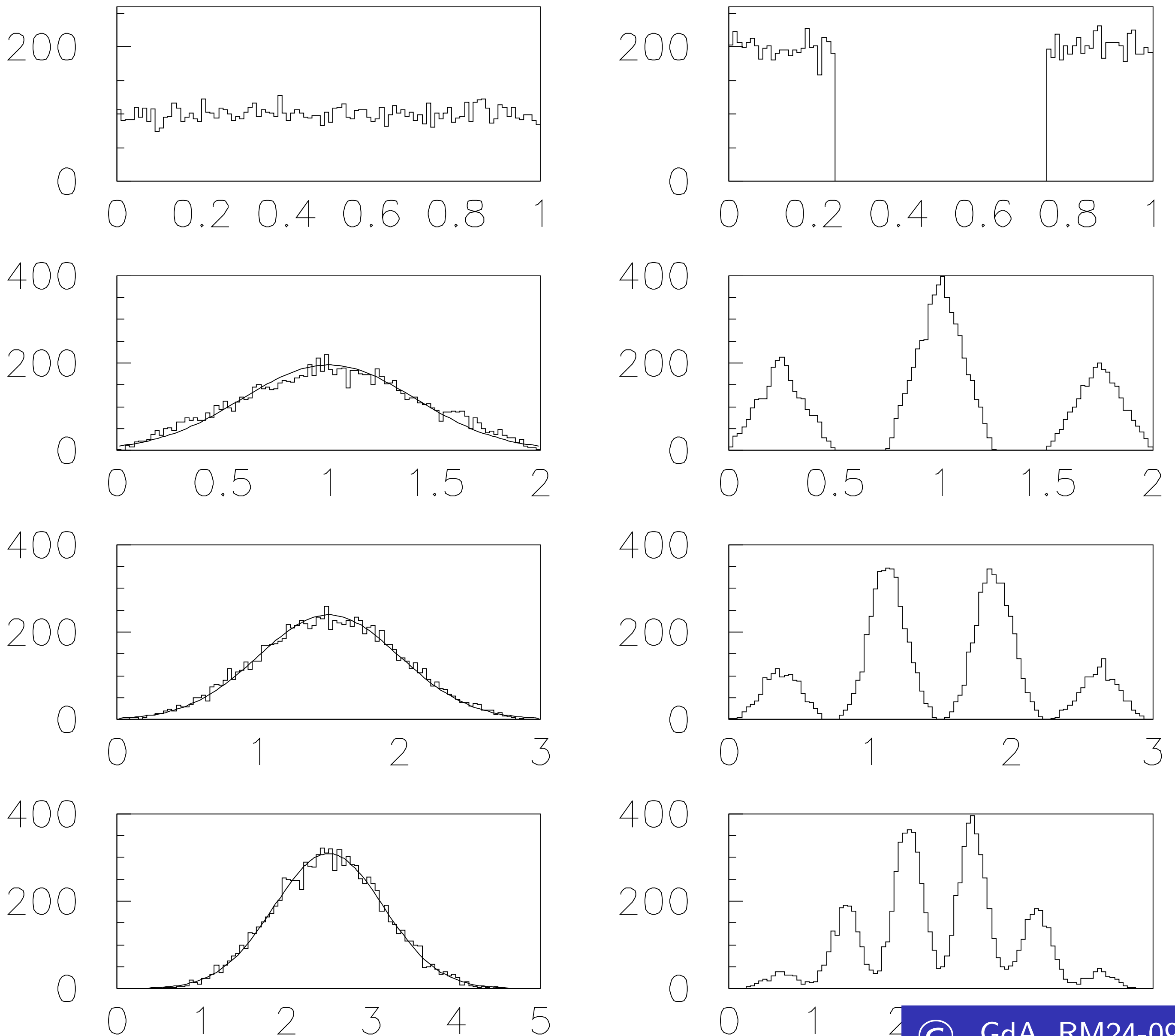

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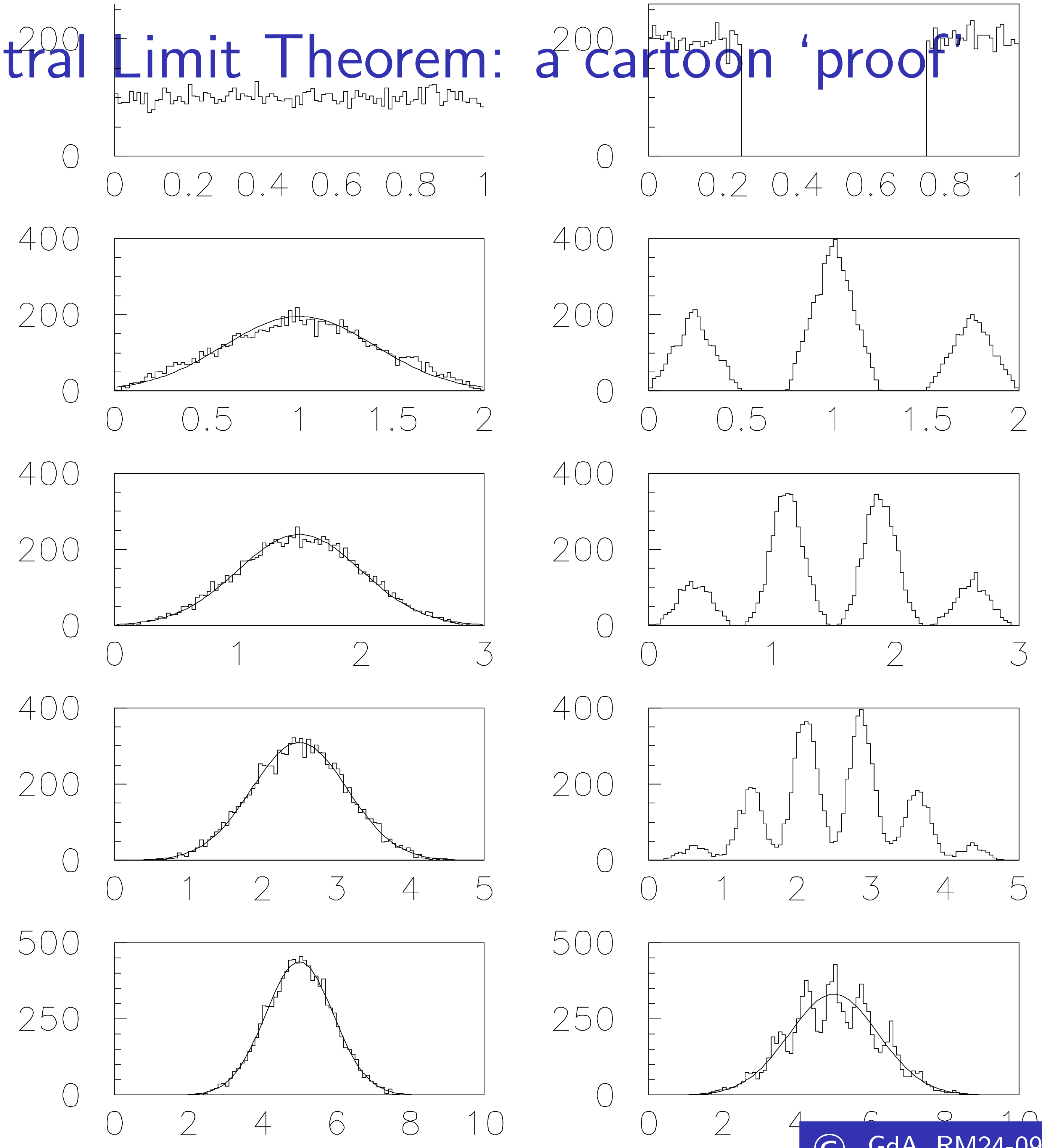

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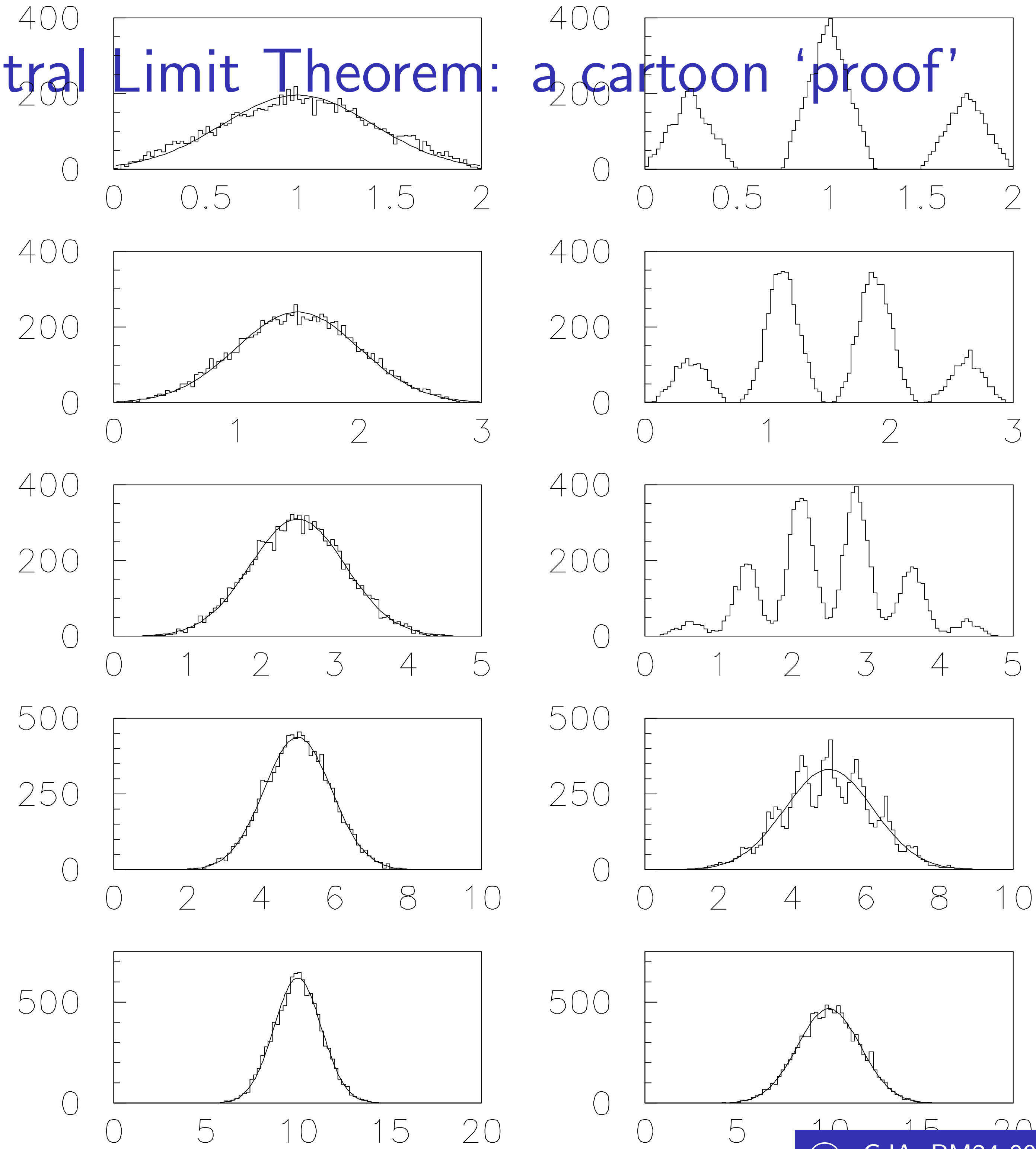

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