Inferring Poisson's λ

$$f(\lambda\,|\,x,\mathcal{P}) \;=\; \frac{\frac{\lambda^x\,e^{-\lambda}}{x!}\,f_\circ(\lambda)}{\int_0^\infty \frac{\lambda^x\,e^{-\lambda}}{x!}\,f_\circ(\lambda)\,\mathrm{d}\lambda}\,.$$

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Assuming $f_\circ(\lambda)$ constant up to a certain $\lambda_{max} \gg x$ and making the integral by parts we obtain

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$$F(\lambda\,|\,x,\mathcal{P}) \;=\; 1 - e^{-\lambda}\left(\sum_{n=0}^{x}\frac{\lambda^n}{n!}\right)$$

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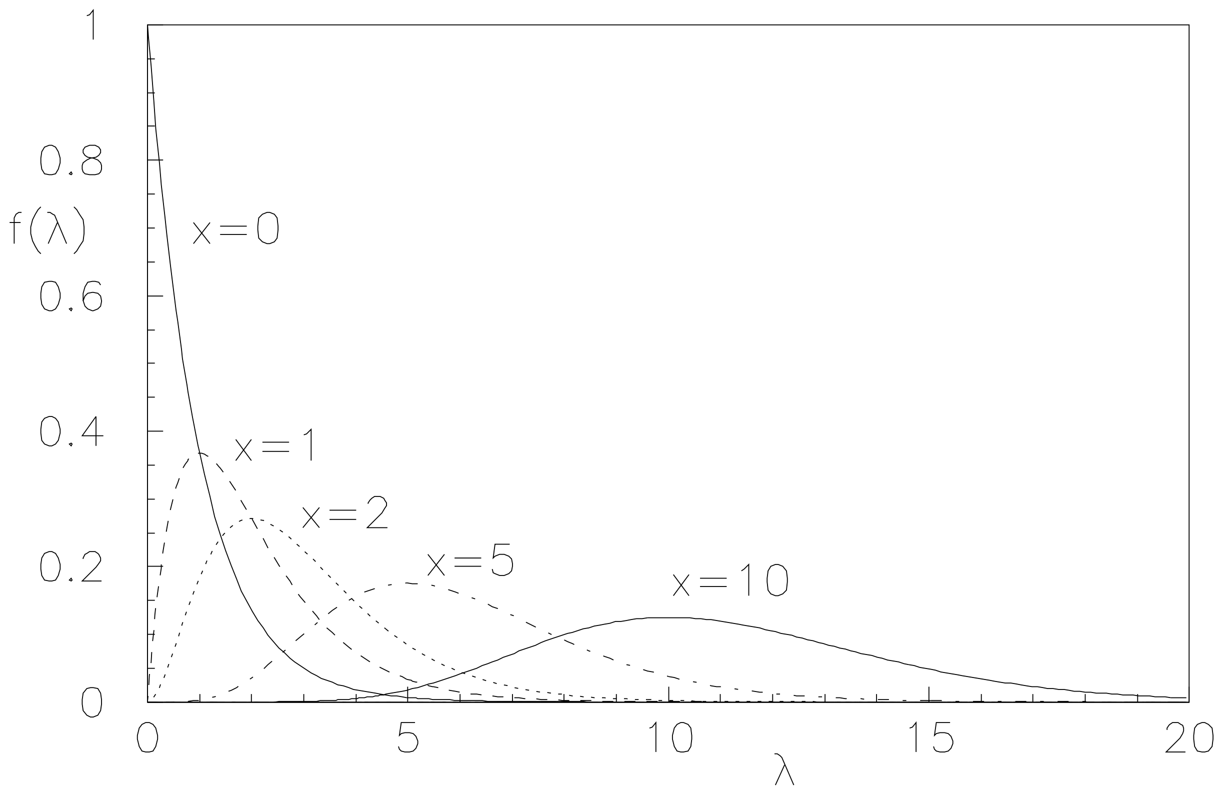
$$\begin{aligned} f(\lambda\,|\,x,\mathcal{P}) &= \frac{\lambda^x\,e^{-\lambda}}{x!} \\ F(\lambda\,|\,x,\mathcal{P}) &= 1 - e^{-\lambda}\left(\sum_{n=0}^{x}\frac{\lambda^n}{n!}\right) \end{aligned}$$

Summaries

$$\begin{aligned} \mathrm{E}(\lambda) &= x+1, \\ \mathrm{Var}(\lambda) &= x+1, \\ \lambda_m &= x \end{aligned}$$

Some examples of $f(\lambda)$



For 'large' x $f(\lambda\,|\,x)$ becomes Gaussian with expected value x and standard deviation $\sqrt{x}$.

The difference between the most probable λ and its expected value for small x is due to the asymmetry of $f(\lambda)$.

Inferring Poisson's λ

Conjugate prior

$$f(\lambda \,|\, x) \;\propto\; \lambda^x \, e^{-\lambda} \cdot f_\circ(\lambda)$$

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Does such a probability function 'exist'?

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$\Rightarrow$ Gamma distribution

Gamma distribution

$X \sim \text{Gamma}(c, r)$:

$$f(x \,|\, \text{Gamma}(c, r)) = \frac{r^c}{\Gamma(c)} x^{c-1} e^{-r\,x} \qquad \begin{cases} r,\ c > 0 \\ x \geq 0 \end{cases} ,$$

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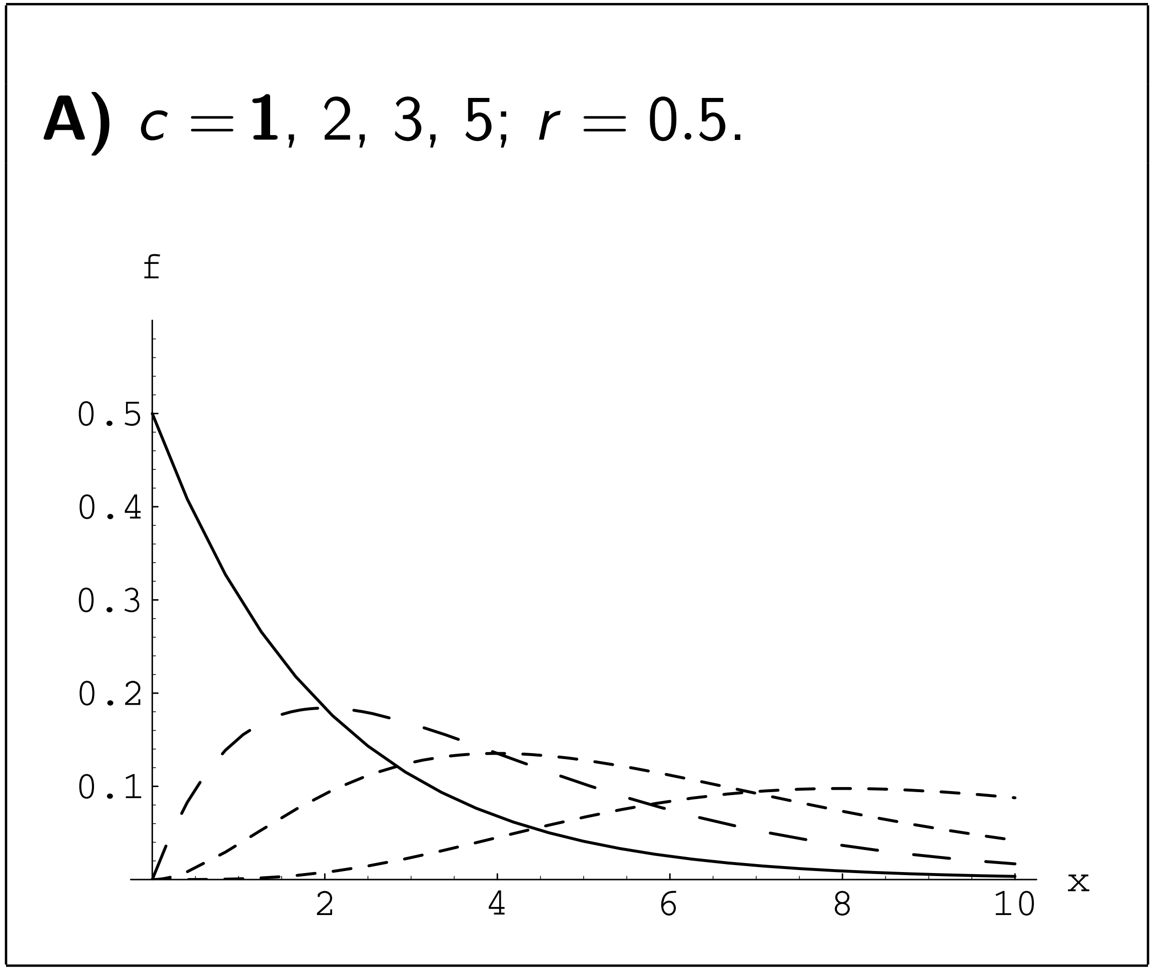
- **The Gamma is a key distribution!**
  The Erlang distribution is important to get a physical intuition of the properties of Gamma and then of the χ^2!

Gamma distribution

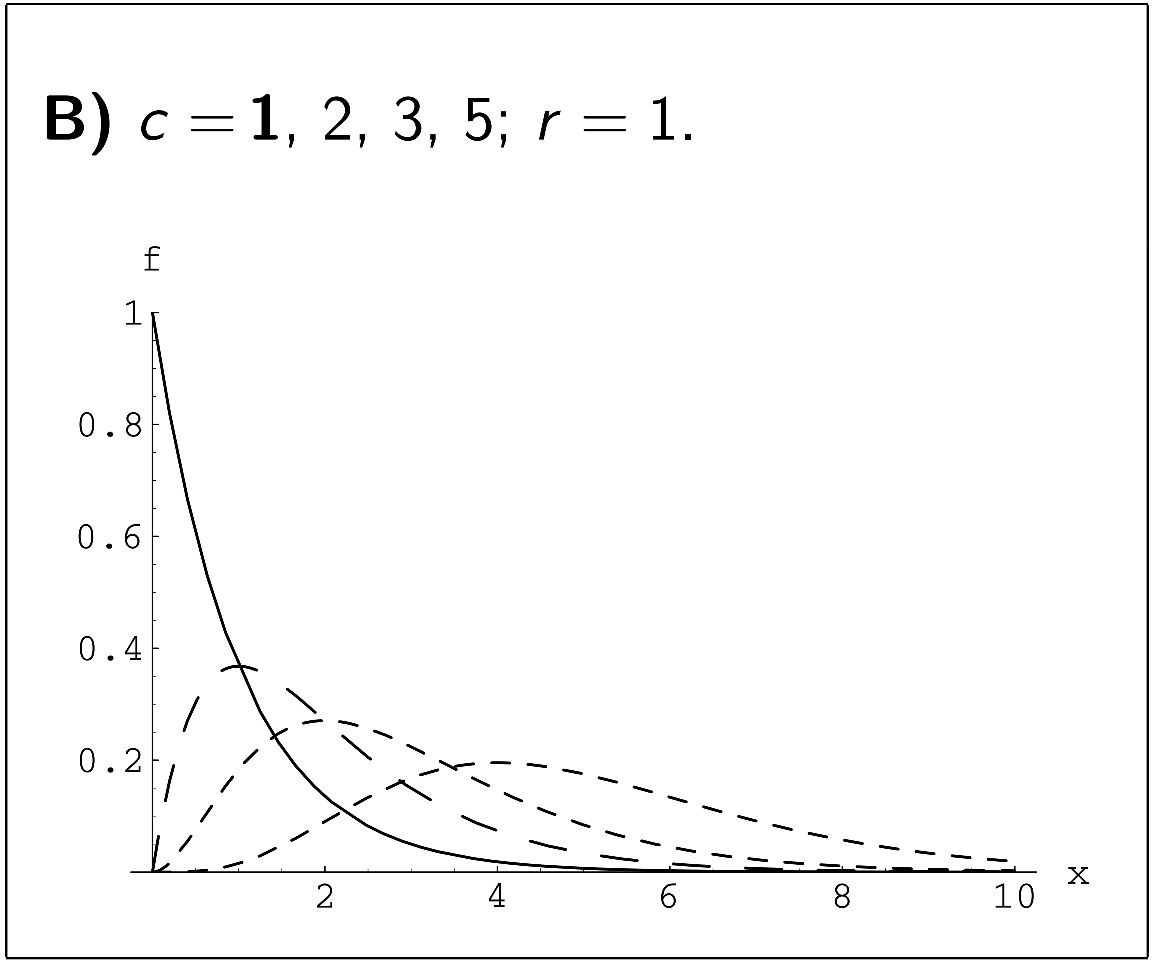
Some examples



r: rate (if the variable is a time, then r is Poisson rate).

Gamma distribution

Some examples



r: rate (rate increases $\rightarrow$ distributions squized)

Gamma distribution

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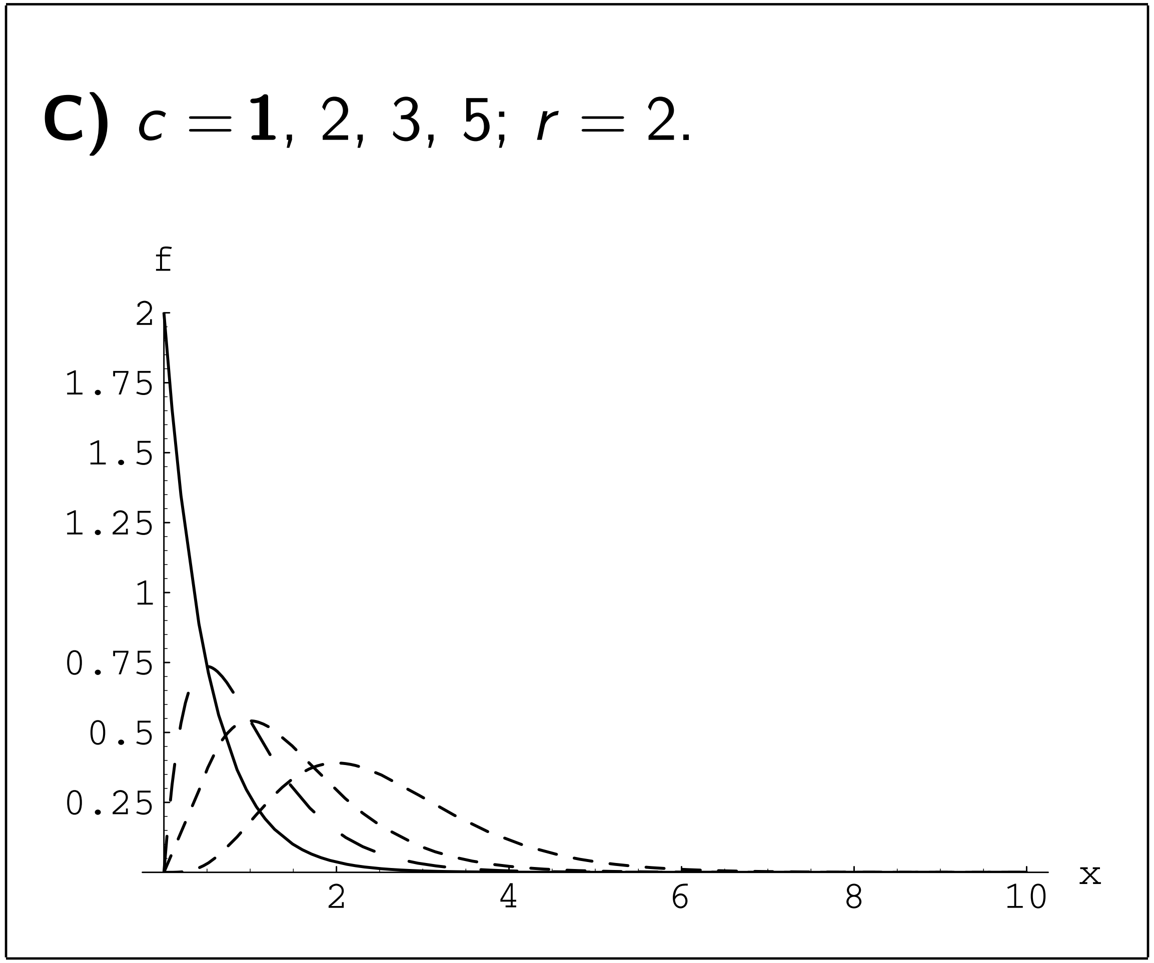

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Gamma (and χ^2) distribution

Summaries

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\mathrm{E}(X) &= \frac{c}{r} \\
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Therefore, for the χ^2 ($\rightarrow c = \nu/2,\ r = 1/2$)

$$
\begin{aligned}
E(\chi^2) &= \nu \\
\mathrm{Var}(\chi^2) &= 2\,\nu \\
\mathrm{mode}(\chi^2) &= \begin{cases} 0 & \text{if } \nu \le 2 \\ \nu - 2 & \text{if } \nu > 2 \end{cases}
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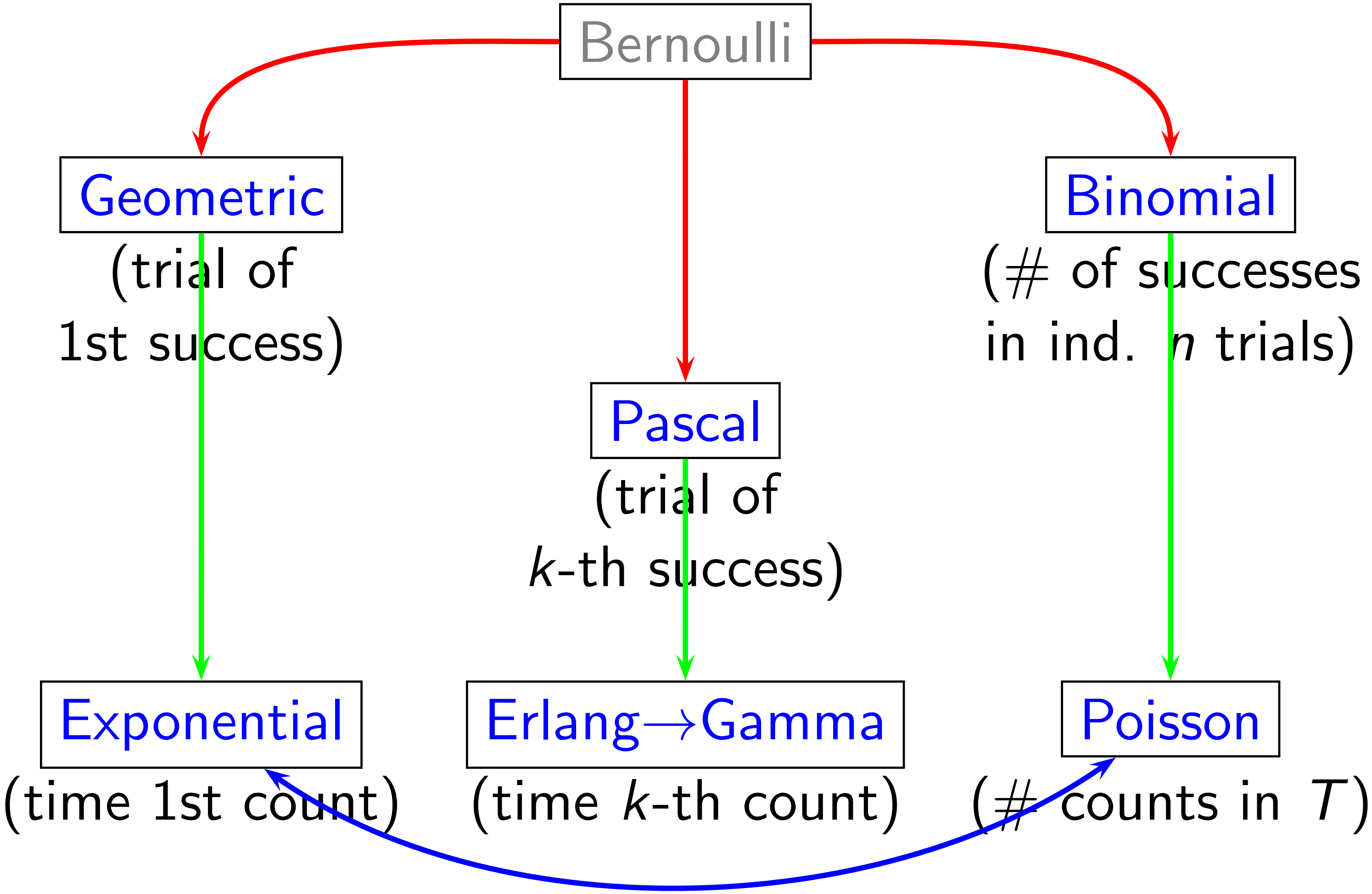
Distributions derived from the Bernoulli process



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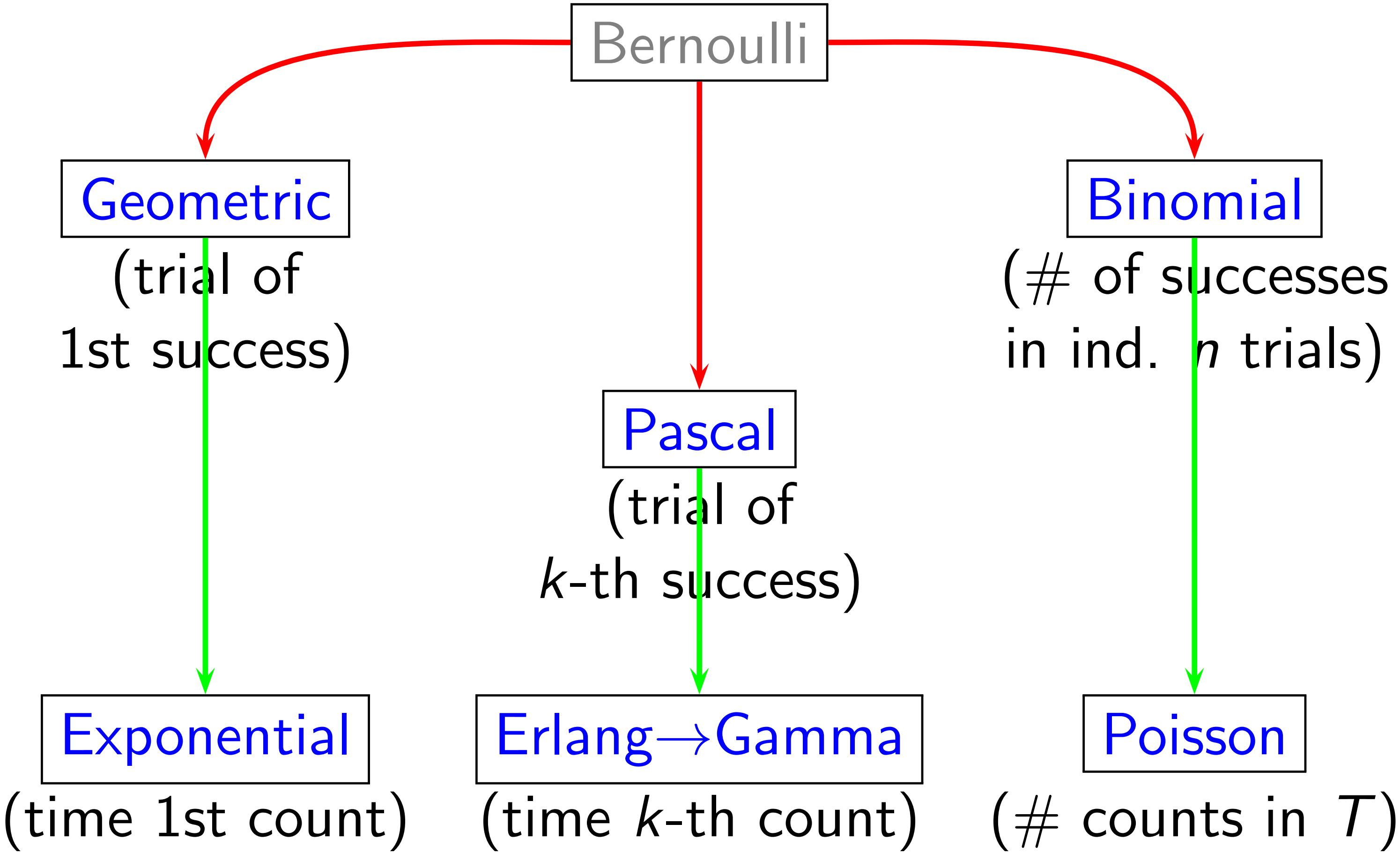

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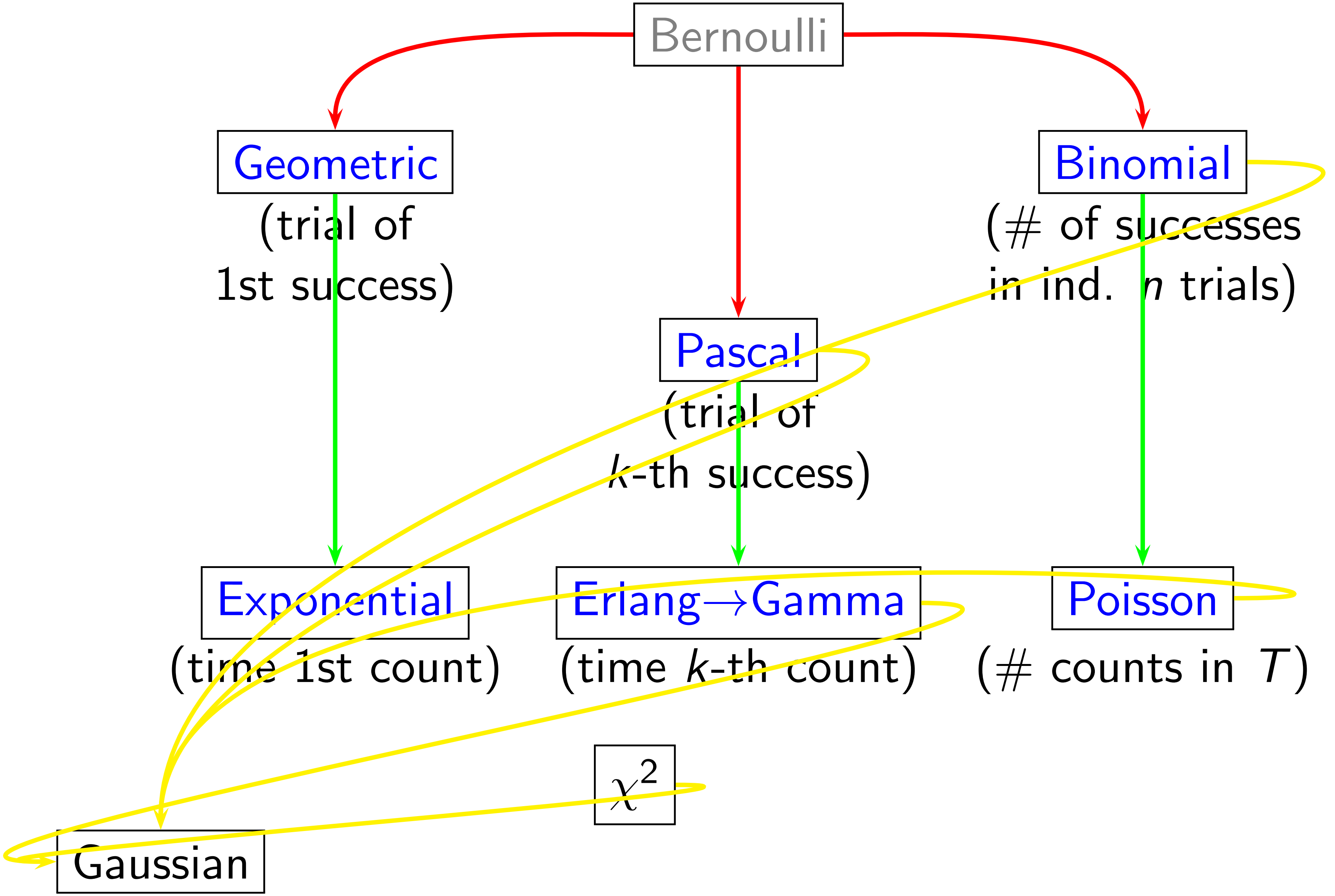

Inferring the Poisson's λ

Use of gamma conjugate prior

- $$f(\lambda \,|\, x, \text{Gamma}(c_i, r_i)) \;\;\propto\;\; \left[\lambda^x e^{-\lambda}\right] \times \left[\lambda^{c_i - 1} e^{-r_i\,\lambda}\right]$$

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$$\begin{aligned} c_f &= c_i + x \\ r_f &= r_i + 1 \end{aligned}$$

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- $c = 1$, $r \rightarrow 0$

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