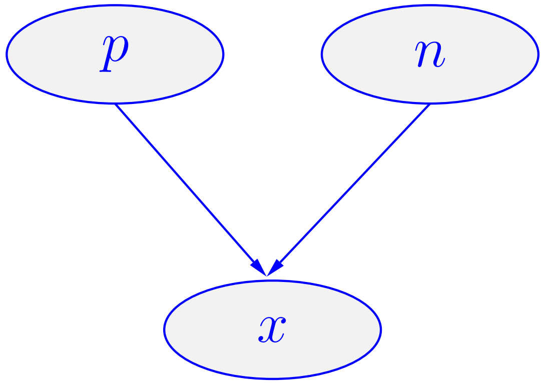
n independent Bernoulli processes

General case

n independent Bernoulli processes

General case

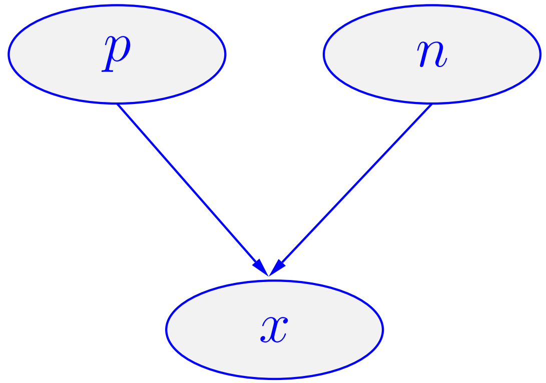
Model



n independent Bernoulli processes

General case

Model



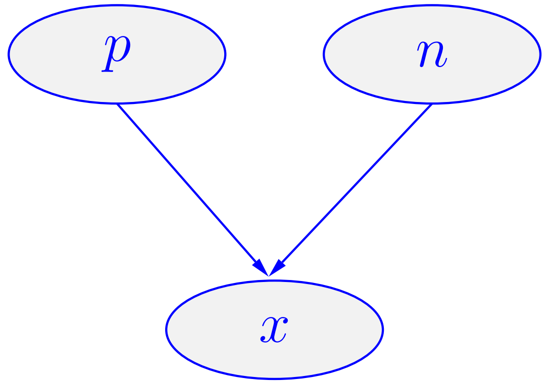
Joint pdf (omitting background condition I):

$$f(x, p, n) \;=\; f(x \,|\, p, n)$$

n independent Bernoulli processes

General case

Model



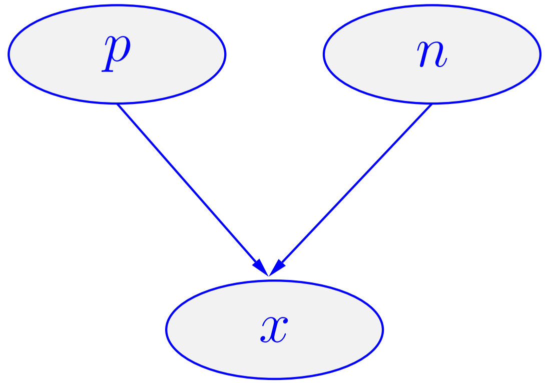
Joint pdf (omitting background condition I):

$$f(x, p, n) \;=\; f(x \,|\, p, n) \cdot f(p, n)$$

n independent Bernoulli processes

General case

Model



Joint pdf (omitting background condition I):

$$
\begin{aligned}
f(x, p, n) &= f(x \mid p, n) \cdot f(p, n) \\
&= f(x \mid p, n) \cdot f(p \mid n) \cdot f(n)
\end{aligned}
$$

n independent Bernoulli processes

General case

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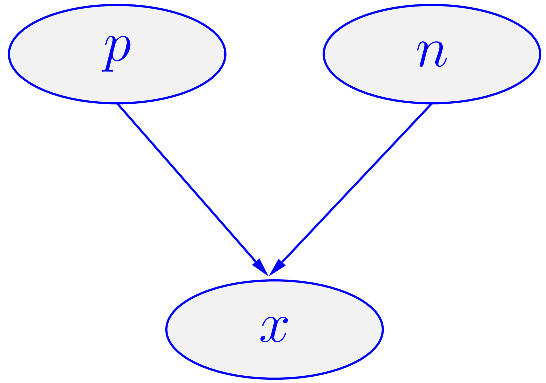

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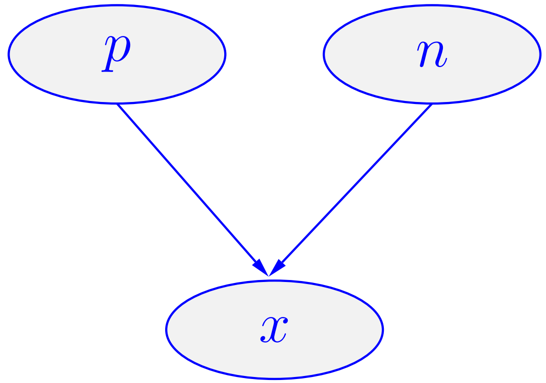

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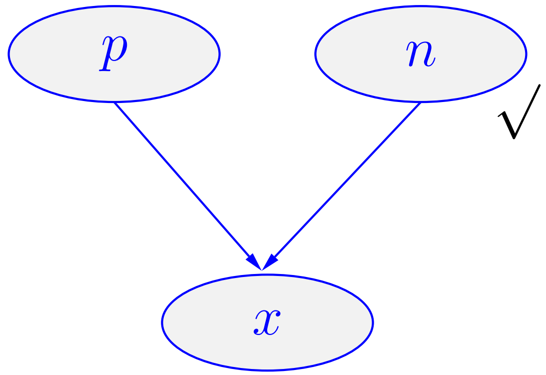
&= f(x \,|\, p, n) \cdot f(p) \cdot f(n)
\end{aligned}
$$

(n and p are independent)

n independent Bernoulli processes

Usual case → n fixed (for the moment)

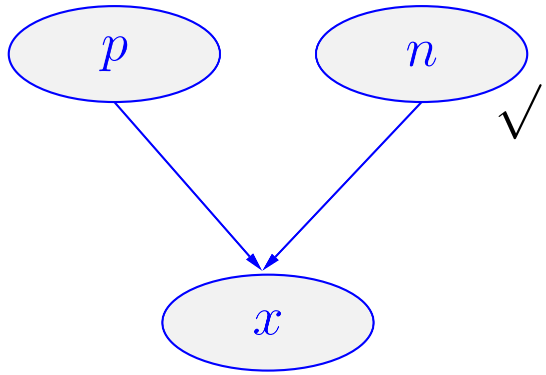
Model



n independent Bernoulli processes

Usual case → n fixed (for the moment)

Model



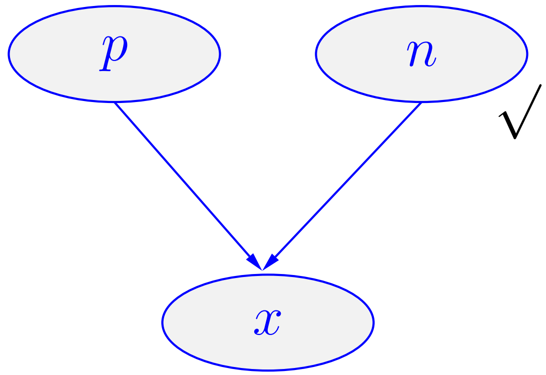
Joint pdf

$$f(x, p \,|\, n) \;=\; f(x \,|\, p, n) \cdot f(p)$$

n independent Bernoulli processes

Usual case → n fixed (for the moment)

Model



Joint pdf

$$f(x, p \,|\, n) \;=\; f(x \,|\, p, n) \cdot f(p)$$

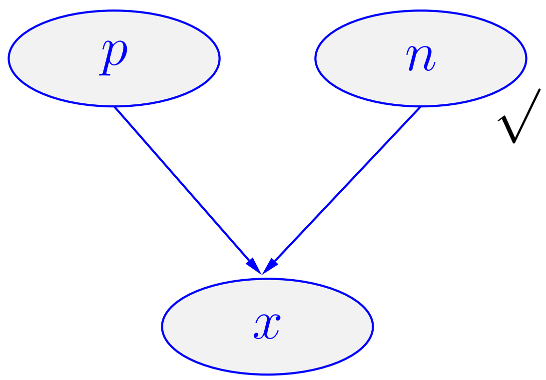
Typical problems

- p is assumed → interested in $f(x \,|\, n, p)$
  → well known binomial;

n independent Bernoulli processes

Usual case → n fixed (for the moment)

Model



Joint pdf

$$f(x, p \mid n) \;=\; f(x \mid p, n) \cdot f(p)$$

Typical problems

- p is assumed → interested in $f(x \mid n, p)$
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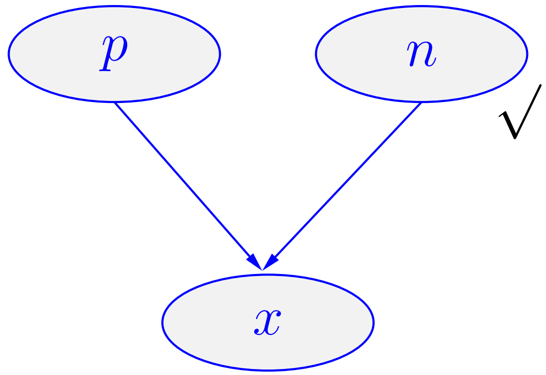
- x is assumed (‘observed’)

n independent Bernoulli processes

Usual case → n fixed (for the moment)

Model



Joint pdf

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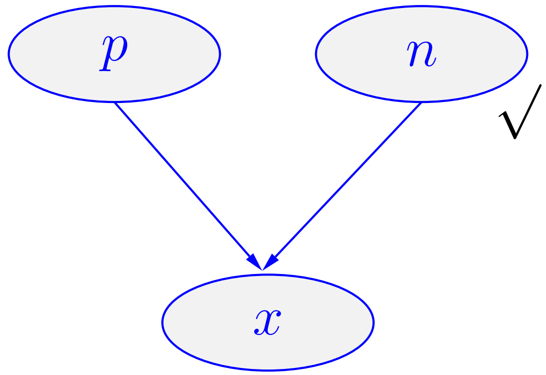
Typical problems

- p is assumed → interested in $f(x \mid n, p)$
  → well known binomial;
- x is assumed ('observed') → $f(p \mid n, x)$

n independent Bernoulli processes

Usual case $\rightarrow$ n fixed (for the moment)

Model



Joint pdf

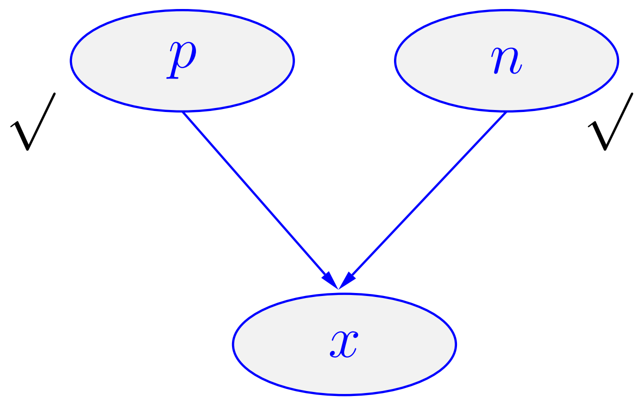
$$f(x, p \,|\, n) \;=\; f(x \,|\, p, n) \cdot f(p)$$

Typical problems

- p is assumed $\rightarrow$ interested in $f(x \,|\, n, p)$
  $\rightarrow$ well known binomial;
- x is assumed ('observed') $\rightarrow$ $f(p \,|\, n, x)$:
  $\rightarrow$?

n independent Bernoulli processes

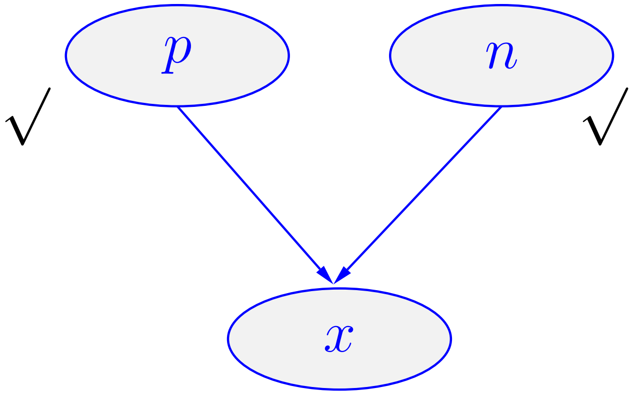
Graphical models of the typical problems



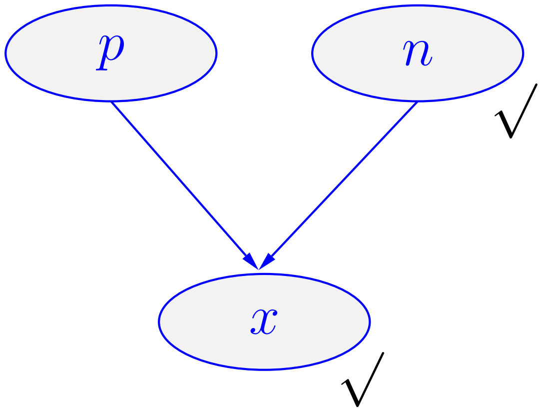
$$\rightarrow f(x \mid n, p)$$

n independent Bernoulli processes

Graphical models of the typical problems



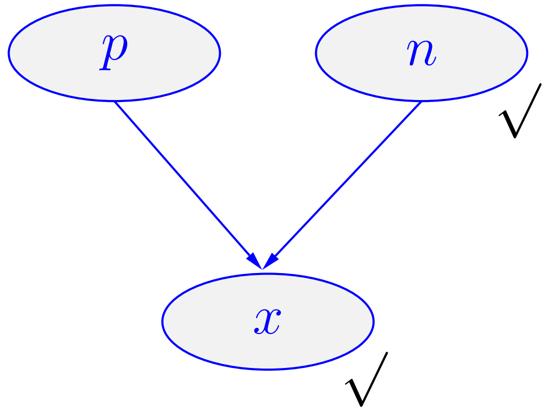
$\rightarrow\ f(x\,|\,n,p)$



$\rightarrow\ f(p\,|\,n,x)$

n independent Bernoulli processes

Inferring p



n independent Bernoulli processes

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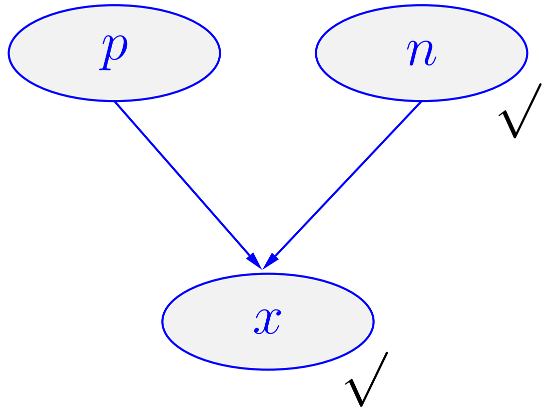
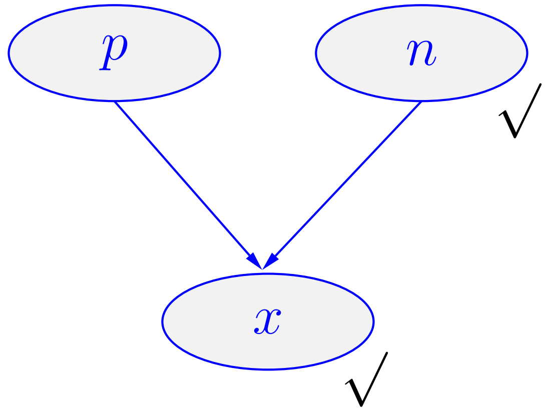


$$f(p\,|\,x,n) \;=\; \frac{f(p,x\,|\,n)}{f(x\,|\,n)}$$

n independent Bernoulli processes

Inferring p



$$
\begin{aligned}
f(p \mid x, n) &= \frac{f(p, x \mid n)}{f(x \mid n)} \\
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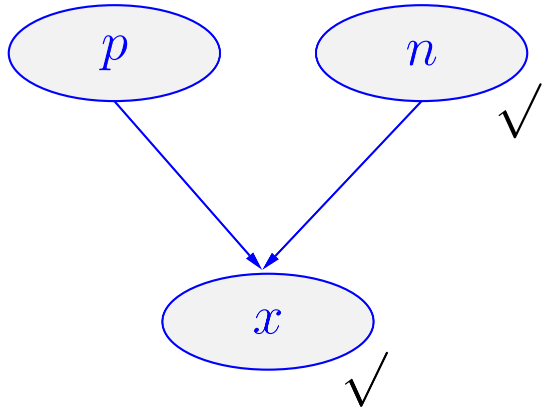
\end{aligned}
$$

n independent Bernoulli processes

Inferring p



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n independent Bernoulli processes

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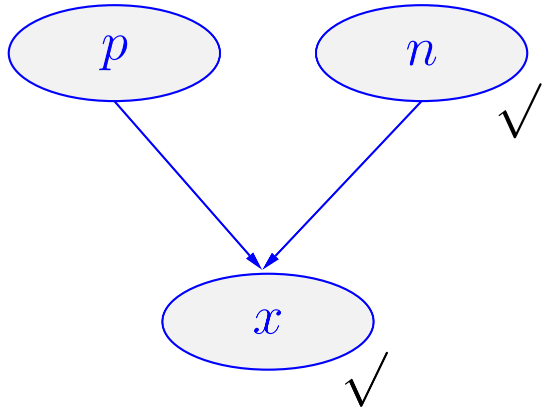

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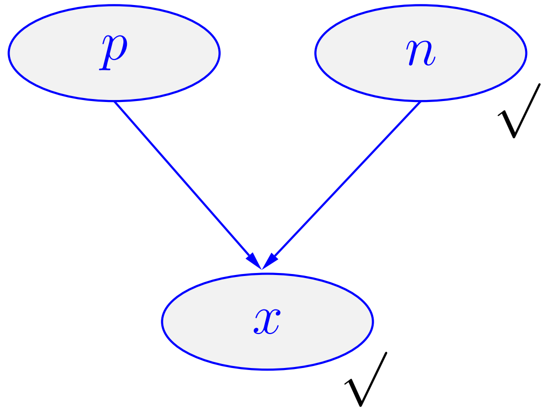
&\propto f(x\,|\,n,p)\cdot f_0(p)
\end{aligned}
$$

n independent Bernoulli processes

Inferring p



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&\propto f(x \mid n, p) \cdot f_0(p)
\end{aligned}
$$

(denominator just normalization!)

Inferring "Bernoulli's p"

We just need to make explicit $f(x \,|\, n, p)$:

$$f(x \,|\, n, p) \;=\; \binom{n}{x}\, p^x\, (1-p)^{n-x}$$

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We get then, including normalization:

$$f(p\,|\,x,n) \;=\; \frac{\frac{n!}{(n-x)!\,x!}\,p^x\,(1-p)^{n-x}\,f_\circ(p)}{\int_0^1 \frac{n!}{(n-x)!\,x!}\,p^x\,(1-p)^{n-x}\,f_\circ(p)\,\mathrm{d}p}$$

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$$=\; \frac{p^x (1-p)^{n-x} f_\circ(p)}{\int_0^1 p^x (1-p)^{n-x} f_\circ(p)\,\mathrm{d}p}$$

(The binomial coefficient is irrelevant, not depending on p)

Inferring "Bernoulli's p"

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For teaching purposes we start from a **uniform prior**, i.e. $\boldsymbol{f_\circ(p)} = 1$:

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- The integral at the denominator is the special function "β" (also defined for real values of x and n).

Inferring "Bernoulli's p"

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- The integral at the denominator is the special function "β" (also defined for real values of x and n).
- In our case these two numbers are integer and the integral becomes equal to

$$\frac{x!\,(n-x)!}{(n+1)!}$$

Inferring "Bernoulli's p"

Solution for uniform prior (think to **Bayes' billard**)

$$f(p\,|\,x,n) \quad = \quad \frac{(n+1)!}{x!\,(n-x)!}\,p^x\,(1-p)^{n-x}$$

Inferring "Bernoulli's p"

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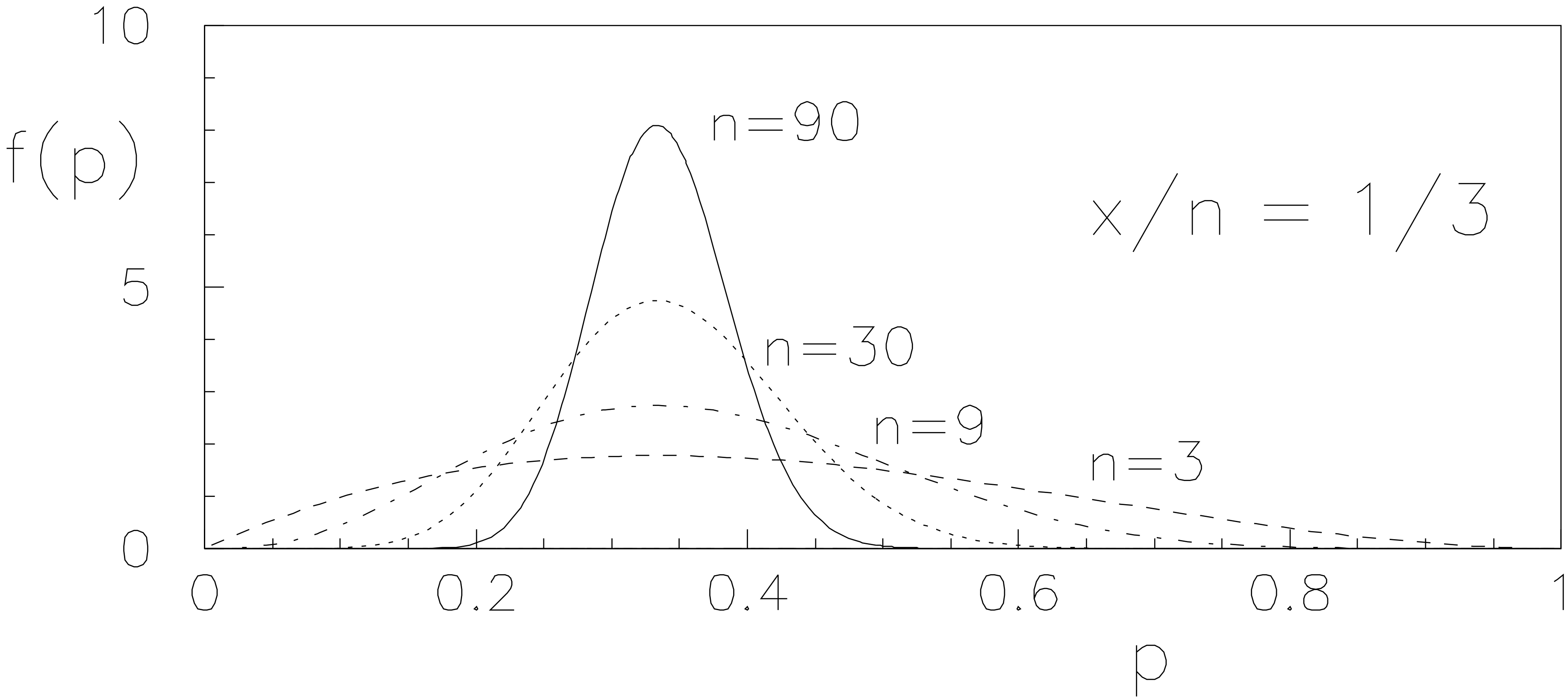

Inferring "Bernoulli's p"

Summaries of the **posterior distribution**

$$p_m = \text{mode}(p) \quad = \quad \frac{x}{n}$$

Inferring "Bernoulli's p"

Summaries of the **posterior distribution**

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$$E(p) = \frac{x+1}{n+2}$$

Inferring "Bernoulli's p"

Summaries of the **posterior distribution**

$$p_m = \text{mode}(p) = \frac{x}{n}$$

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"recursive Laplace formula"

("Laplace's rule of succession")

Inferring "Bernoulli's p"

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$$p_m = \text{mode}(p) = \frac{x}{n}$$

$$E(p) = \frac{x+1}{n+2}$$

"recursive Laplace formula"

("Laplace's rule of succession")

$$\begin{aligned}\text{Var}(p) &= \frac{(x+1)(n-x+1)}{(n+3)(n+2)^2} \\ &= \frac{x+1}{n+2}\left(\frac{n+2}{n+2} - \frac{x+1}{n+2}\right)\frac{1}{n+3} \\ &= E(p)\,(1-E(p))\,\frac{1}{n+3}\end{aligned}$$

Inferring the "Bernoulli's p"

About the meaning of E(p)

- We have used the "first"$^{(*)}$ n trials to learn about "p".

  [$^{(*)}$ "First" does not imply time order, but just order in usage.]

Inferring the "Bernoulli's p"

About the meaning of E(p)

- We have used the "first"$^{(*)}$ n trials to learn about "p".
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- What will be the probability of other trials?

$$P(E_{i>n}) = ??$$

Inferring the "Bernoulli's p"

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Inferring the “Bernoulli's p”

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Inferring the "Bernoulli's p"

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$$P(E_{i>n} \,|\, x, n) \;=\; \int_0^1 P(E_i \,|\, p) \cdot f(p \,|\, x, n)\, \mathrm{d}p$$

Inferring the "Bernoulli's p"

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Inferring the "Bernoulli's p"

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Inferring the "Bernoulli's p"

About the meaning of $\mathrm{E}(p)$

- We have used the "first"$^{(*)}$ n trials to learn about "p".
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$$\begin{aligned} P(E_{i>n}\,|\,x,n) &= \int_0^1 P(E_i\,|\,p)\cdot f(p\,|\,x,n)\,\mathrm{d}p \\ &= \int_0^1 p\cdot f(p\,|\,x,n)\,\mathrm{d}p = \mathrm{E}(p) \quad (!!) \end{aligned}$$

$\mathrm{E}(p)$ (and not the mode!) is the probability of every 'future' event which is believed to have the **same p** of the 'previous' ones.

Inferring the “Bernoulli’s p”

About the meaning of $\mathrm{E}(p)$

- We have used the “first”$^{(*)}$ n trials to learn about “p”.
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$$P(E_i \,|\, p) = p$$

- But since we are uncertain about it, we have to take into account all possible values, weighing them with our degree of belief.

$$\begin{aligned} P(E_{i>n} \,|\, x, n) &= \int_0^1 P(E_i \,|\, p) \cdot f(p \,|\, x, n) \, \mathrm{d}p \\ &= \int_0^1 p \cdot f(p \,|\, x, n) \, \mathrm{d}p = \mathrm{E}(p) \quad (!!) \end{aligned}$$

$\mathrm{E}(p)$ (and not the mode!) is the probability of every ‘future’ event which is believed to have the **same p** of the ‘previous’ ones.

(But keep in mind the **inductivist turkey**!)

Inferring the "Bernoulli's p"

Large number behaviour

When the number of successes <u>and</u> the number of failures become 'large' (x large is not enough, as it can be easily understood from the simmetric properties of the binomial $p \leftrightarrow q$):

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(Similarly to Bernoulli's theorem, it is not a 'mathematical' limit!)

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— **No need of ‘frequentistic definition’!**

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Frequency and probability are **related** in probability theory:

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- It does not justify the frequentistic definition.