Poisson distribution

One of the best known distributions by physicists.

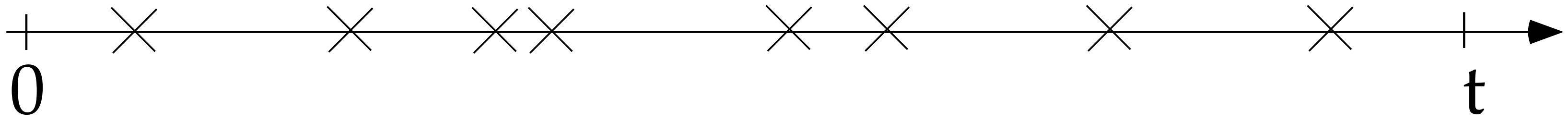
For a while, just take a mathematical approach:

$$f(x \,|\, \mathcal{P}_\lambda) = \frac{\lambda^x}{x!}\, e^{-\lambda} \qquad \left\{ \begin{array}{l} 0 < \lambda < \infty \\ x = 0, 1, \ldots, \infty \end{array} \right. .$$

Reminding also the well known property

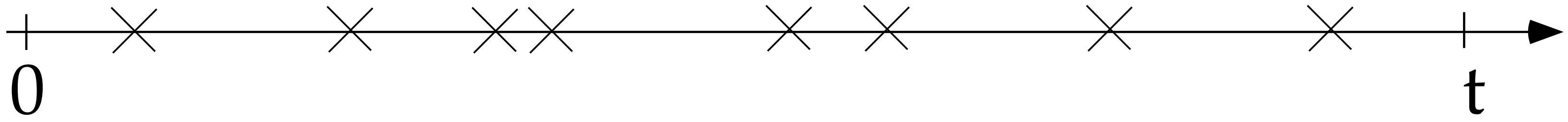
$$\mathcal{B}_{n,p} \xrightarrow[\substack{n \to \infty \\ p \to 0 \\ (n\,p = \lambda)}]{} P_\lambda \,.$$

Poisson process



Let us consider some phenomena that might happen at a give instant

Poisson process



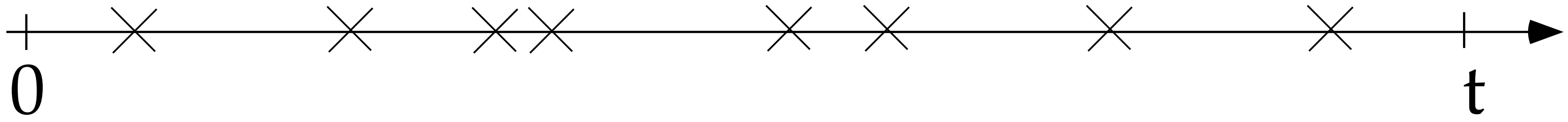
Let us consider some phenomena that might happen at a give instant, such that

- Probability of 1 count in ΔT is proportional to ΔT, with ΔT 'small'.

  $$p = P(\text{“1 count in } \Delta T\text{''}) = r\,\Delta T$$

  where r is the intensity of the process'

Poisson process



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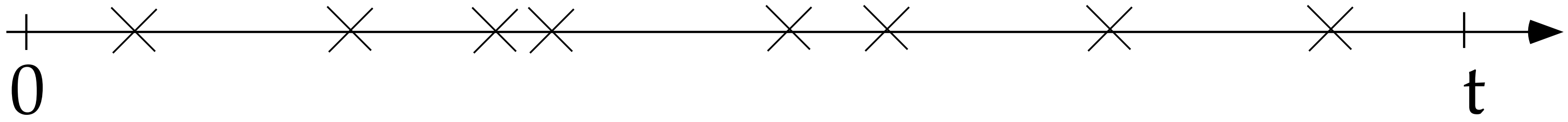

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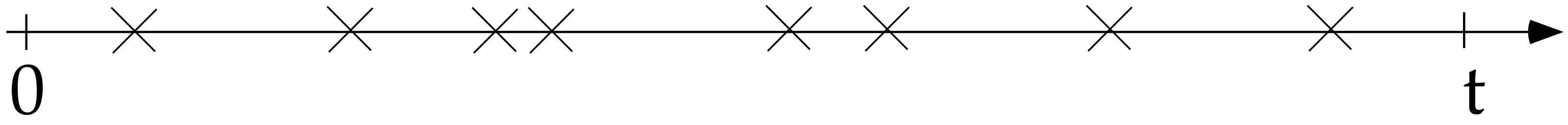

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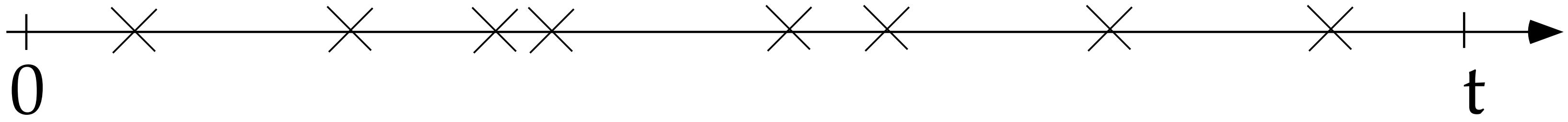
Let us divide a finite interval T in n small intervals,
i.e. $T = n\,\Delta T$, and $\Delta T = T/n$.

Poisson process $\rightarrow$ Poisson distribution



Considering the possible occurrence of a count in each small interval ΔT an independent Bernoulli trial, of probability

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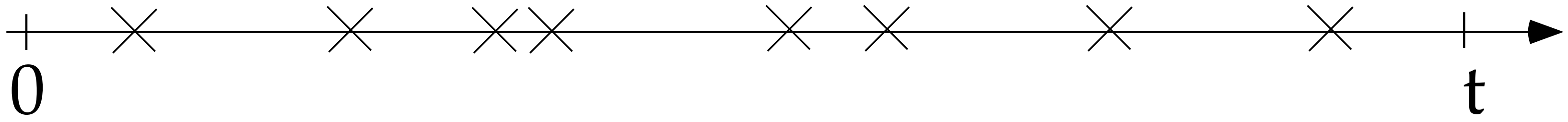

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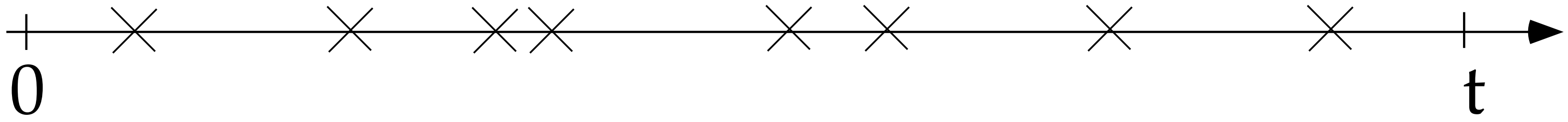

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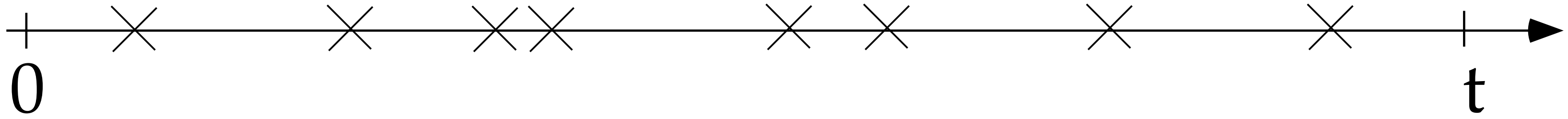
But $n \rightarrow \infty$ and $p \rightarrow 0$ $\quad \Rightarrow \mathcal{B}_{n,p} \rightarrow \mathcal{P}_{\lambda}$ where $\lambda = n\,p = r\,T$

$\Rightarrow$ λ depends only on the intensity of the process and on the finite time of observation.

Poisson process → waiting time



Another interesting problem: how long do we have to wait for the first count? (Starting from any arbitrary time)

Problem analogous to the Geometric, but now it makes no sense to talk about the i-th small interval the counts will occur!

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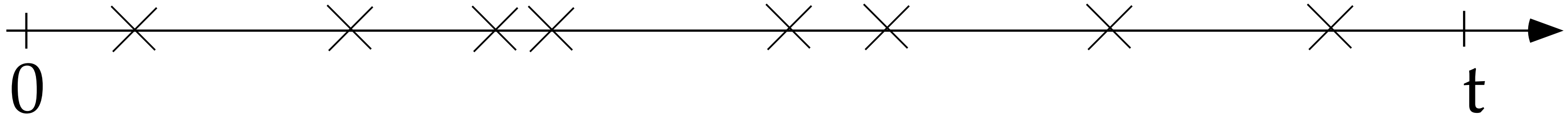

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Let us restart from the Geometric and calculate $P(X > x)$:

$$P(X > x) = \sum_{i>x} f(i \,|\, \mathcal{G}_p) = (1-p)^x$$

(The count will not occur in the first x trials).

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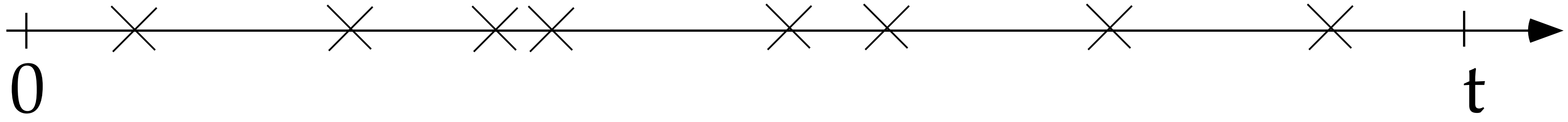

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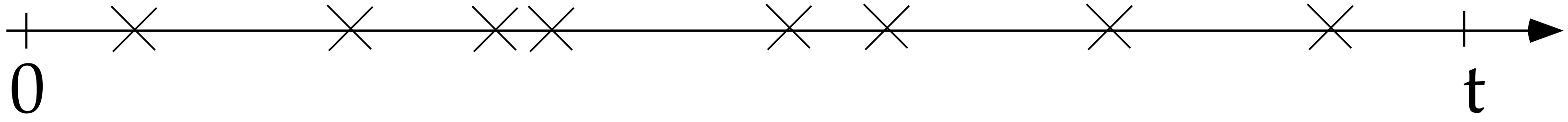

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$$P(T > t) = (1 - p)^n \quad = \quad \left(1 - \frac{r\,t}{n}\right)^n \xrightarrow[n \to \infty]{} e^{-r\,t}$$

Poisson process $\rightarrow$ Exponential distribution

Knowing $P(T > t)$ we get easily the cumulative $F(t)$:

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$\rightarrow$ This leads us to define a probability density function (pdf) for continuous variables:

$$f(t) = \frac{d\,F(t)}{d\,t}\,.$$

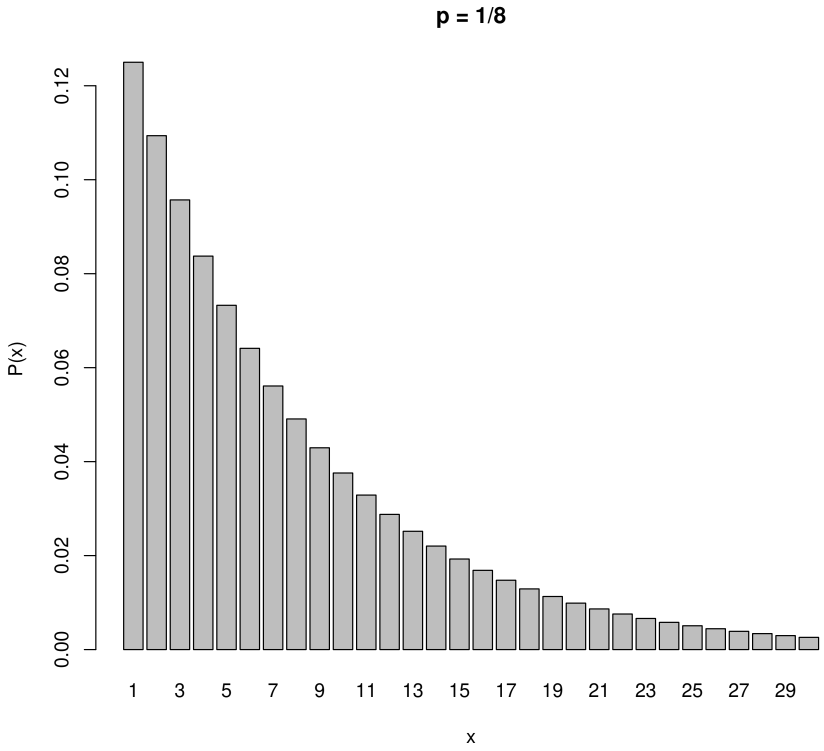
- In this case $f(t) = r\,e^{-r\,t} = \frac{1}{\tau}\,e^{-t/\tau}$

$\rightarrow$ Exponential distribution ($\tau = 1/r$): $\mathrm{E}[T] = \sigma(T) = \tau$.

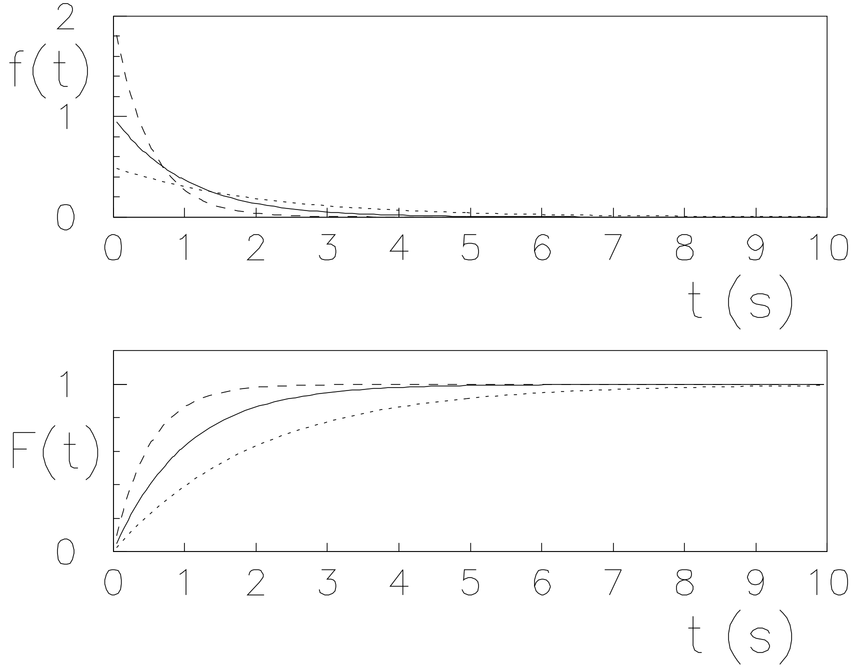
(⇒ Properties of pdf assumed to be known for the moment.)

Geometric ↔ Exponential

Geometric



Exponential



Exponential is just the limit to the continuum of the Geometric.

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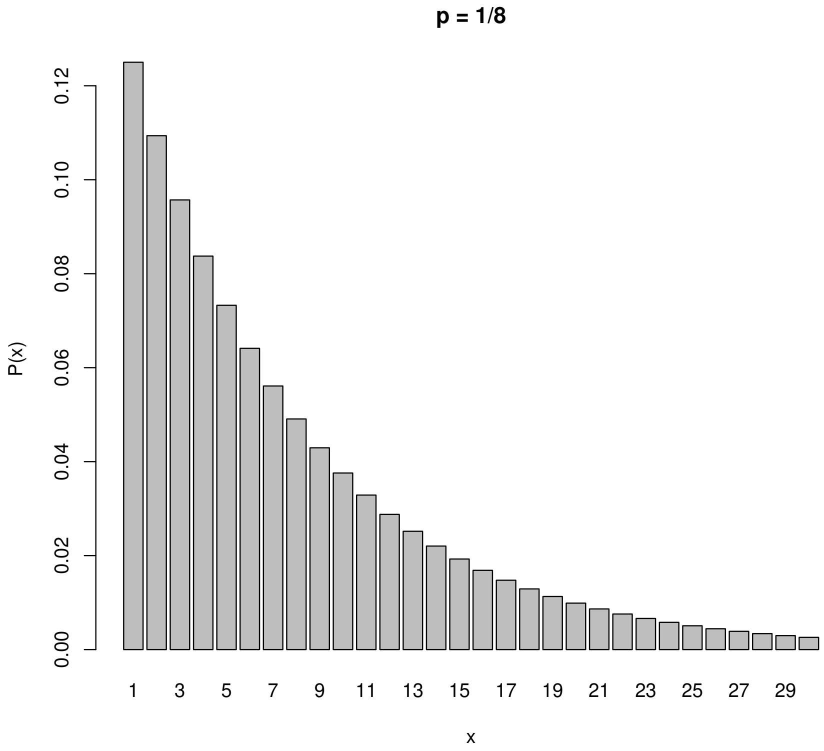

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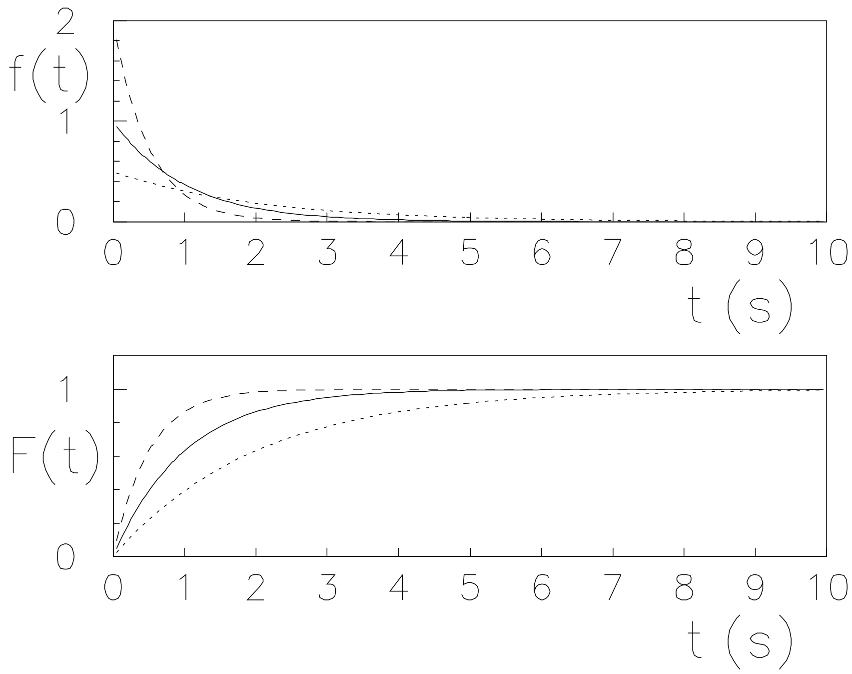

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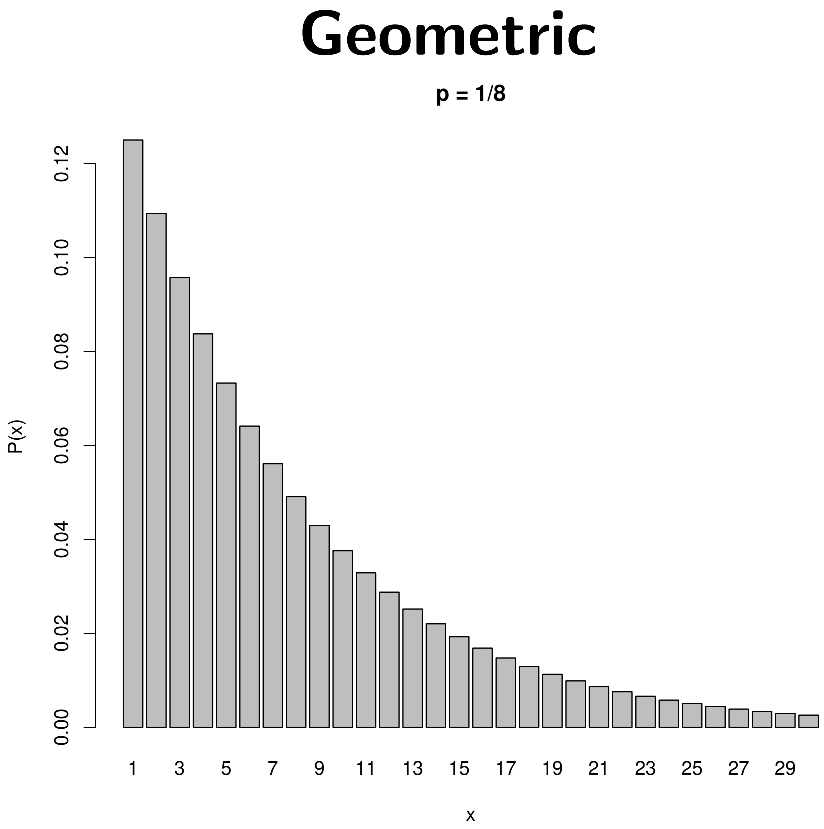
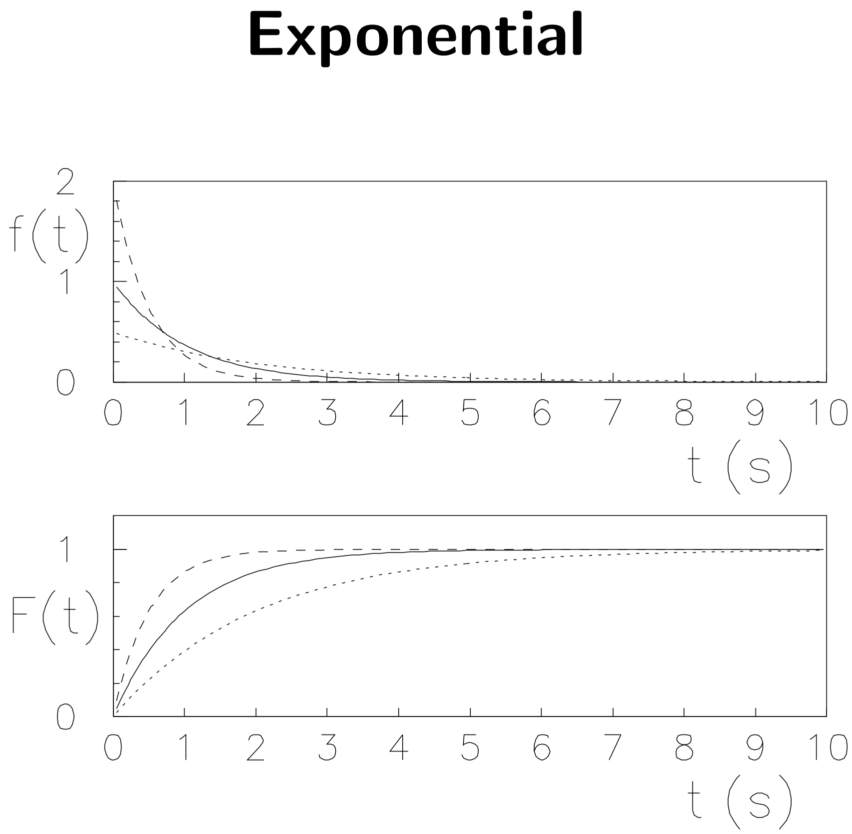
'No memory' property for both:

Geometric $\leftrightarrow$ Exponential





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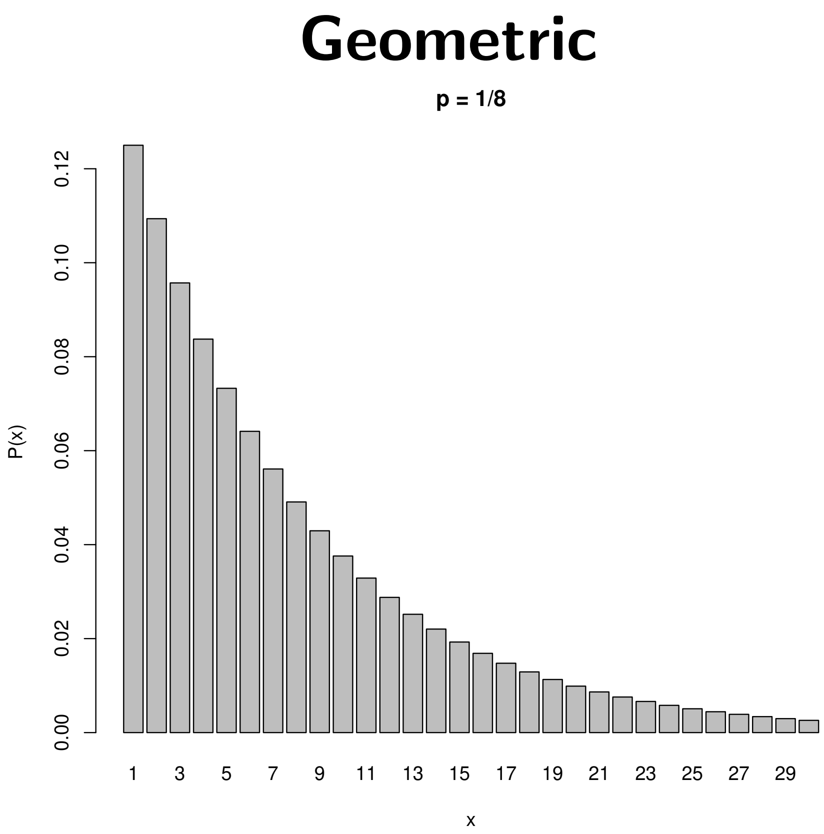
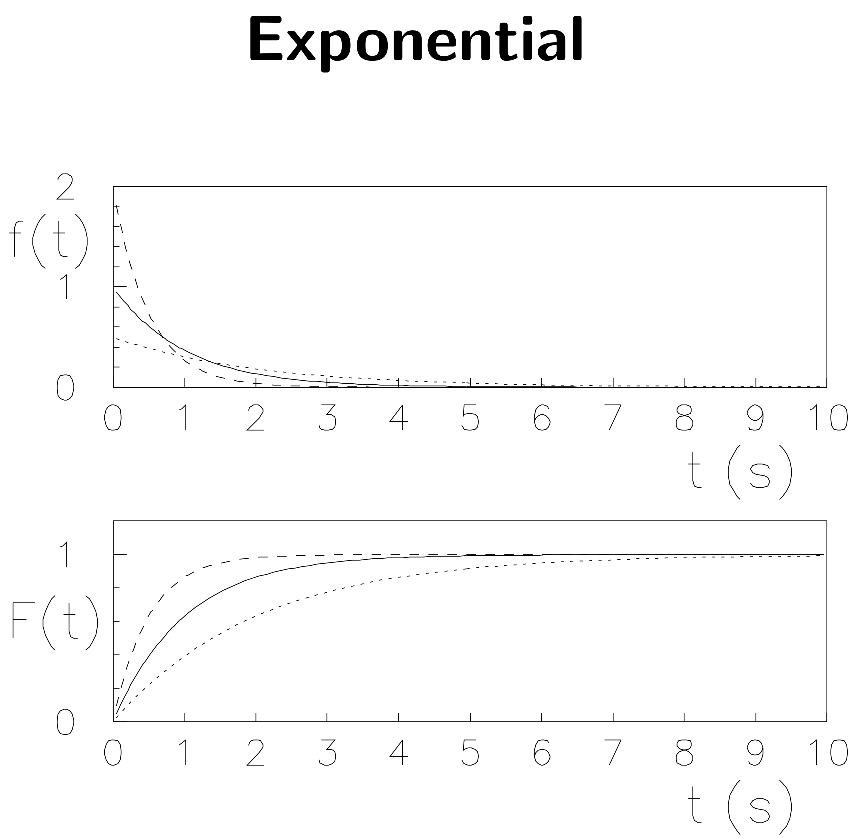
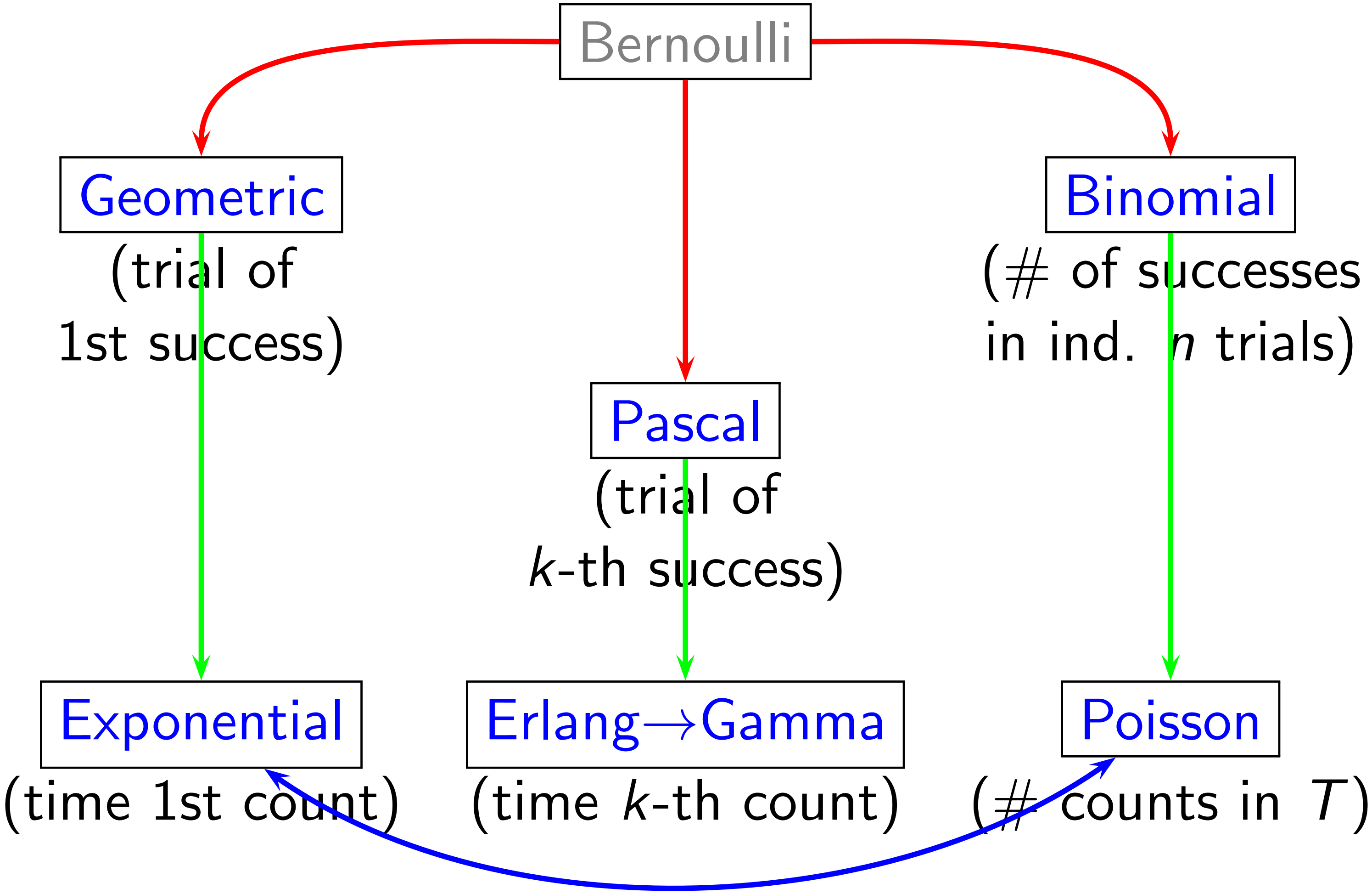




Exponential is just the limit to the continuum of the Geometric. 'No memory' property for both: Assuming that a success (or a count) has not happened until a certain trial (or time), the distributions restart from there. No need to know the instant of particle creation to measure 'life time' ($\rightarrow$ the "10^{25} year old" proton!).

Distributions derived from the Bernoulli process



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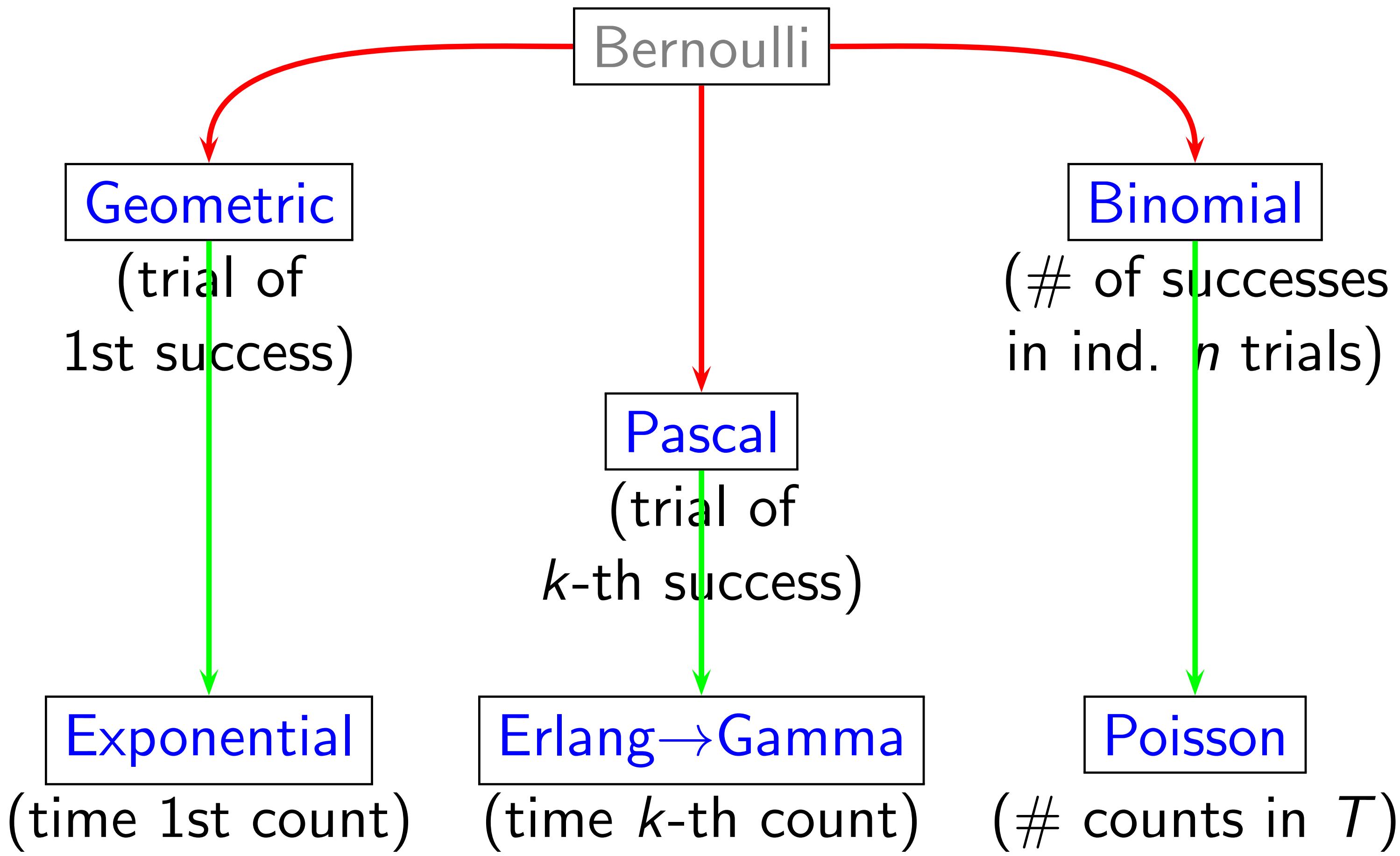

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