Propagating uncertainties

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"Two and two equal four.
The trouble is, in that world of shadows and distorting mirrors
what may or may not appear to be two,
when multiplied by a factor that may or may not be two,
could possibly come out at four but probably will not."

(Frederick Forsyth – *The fist of God*)

Propagation of uncertainties

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$\Rightarrow$ Therefore, the famous problem of propagation of uncertainty is straightforward in a probabilistic approach: just use probability theory.

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The general problem:

$$f(x_1, x_2, \ldots, x_n) \xrightarrow[Y_j = Y_j(X_1, X_2, \ldots, X_n)]{} f(y_1, y_2, \ldots, y_m)\,.$$

This calculation can be quite challenging, but it can be easily performed by Monte Carlo techniques.

Minimal solution of uncertainty propagation

The most general problem:

$$f(x_1, x_2, \ldots, x_n) \xrightarrow[Y_j = Y_j(X_1, X_2, \ldots, X_n)]{} f(y_1, y_2, \ldots, y_m)\,.$$

The 'minimal' solution: linear combinations, i.e.

$$\left\{ \begin{array}{l} \mathrm{E}(X_i) \\ \sigma(X_i) \\ \rho(X_i, X_{i'}) \end{array} \right. \xrightarrow[Y_j = c_{j0} + c_{j1}X_1 + c_{j2}X_2 + \cdots + c_{jn}X_n]{} \left\{ \begin{array}{l} \mathrm{E}(Y_j) \\ \sigma(Y_j) \\ \rho(Y_j, Y_{j'}) \end{array} \right.$$

But not forgetting the correlations!

Linear combination of independent variables

Simple, but instructive and important case:

- One (*output*) variable (Y) depending from many (*input*) quantities X_i, $i = 1, 2, \ldots, n$.

$$\begin{aligned} Y &= c_0 + c_1 X_1 + c_2 X_2 + \cdots + c_n X_n] \\ &= c_0 + \sum_i c_i\, X_i \end{aligned}$$

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Less general (it holds only if X_i are independent) property:

$$\begin{aligned} \mathrm{Var}[Y] &= c_1^2 \mathrm{Var}[X_1] + c_2^2 \mathrm{Var}[X_2] + \cdots + c_n^2 \mathrm{E}[X_n] \\ &= \sum_i c_i^2 \mathrm{Var}[X_i] \end{aligned}$$

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These two properties (and the extension of the second in the case of *correlated input variable*) are the main reason to prefer, as mostly representative summaries of distributions,

- expected value
- standard deviation ($= \sqrt{\text{Var}}$)

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Linear combination of Gaussian variables

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"A linear combinations of Gaussians
is still Gaussian"!

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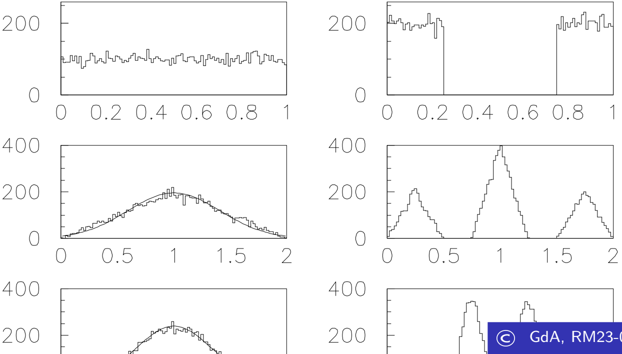
Central Limit Theorem:

$$``n \rightarrow \infty" \Longrightarrow Y \sim \mathcal{N}\left(\sum_{i=1}^{n} c_i\, \mathrm{E}(X_i), \left(\sum_{i=1}^{n} c_i^2\, \sigma_i^2\right)^{\frac{1}{2}}\right)$$

if $c_i^2\, \sigma_i^2 << \sum_{i=1}^{n} c_i^2\, \sigma_i^2$ for all X_i not described by a Gaussian!
(i.e. a single non-Gaussian variable has not to dominate the uncertainty about Y.)

Central Limit Theorem: a cartoon ‘proof’



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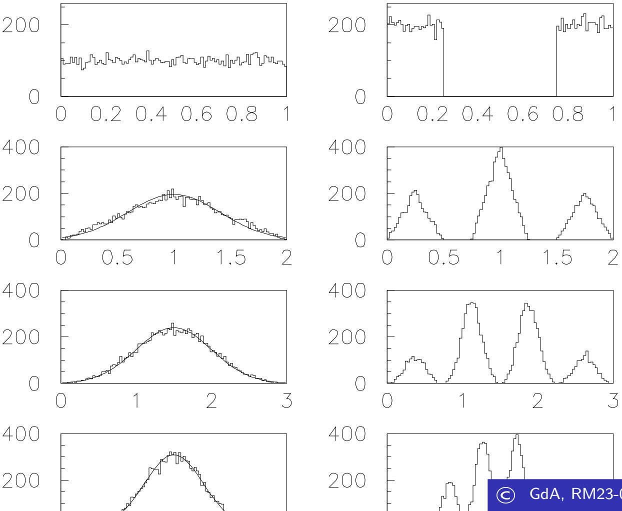

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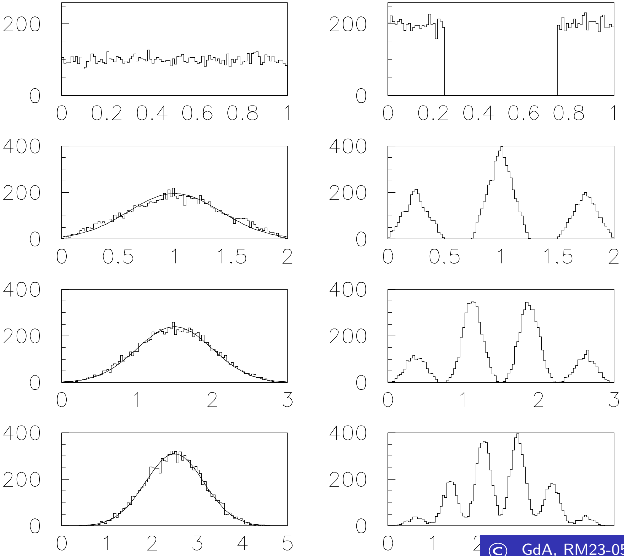

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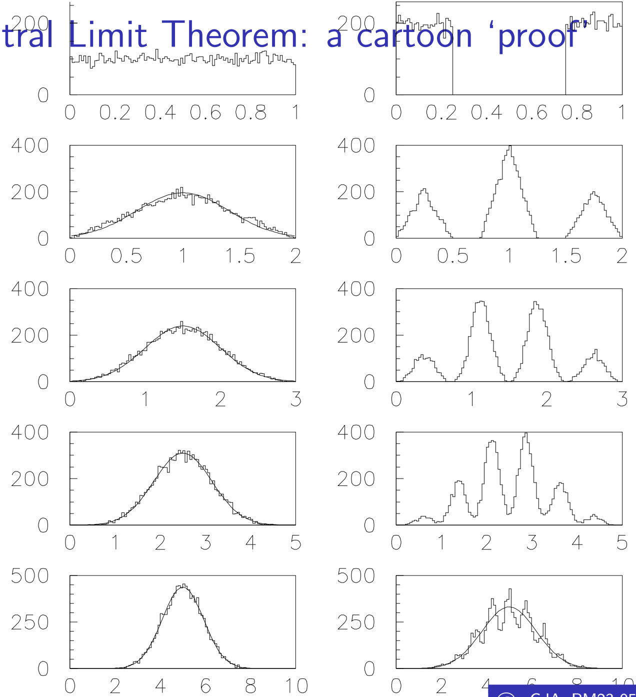

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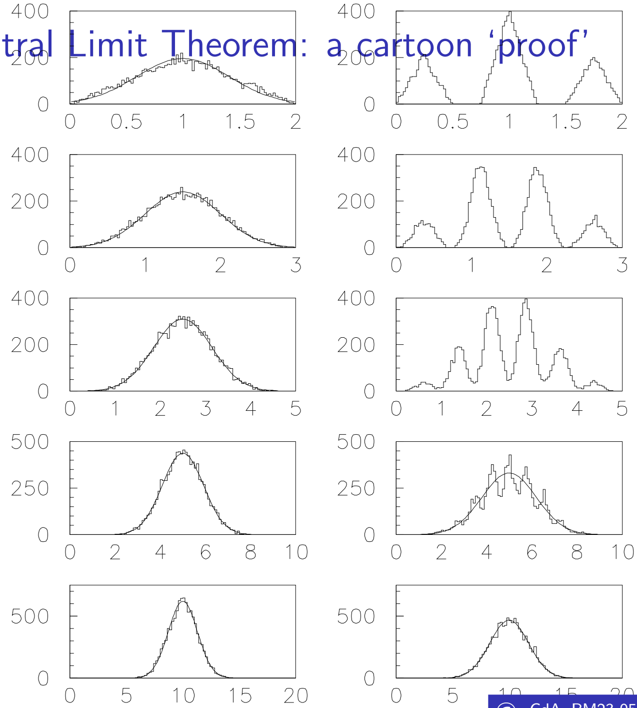

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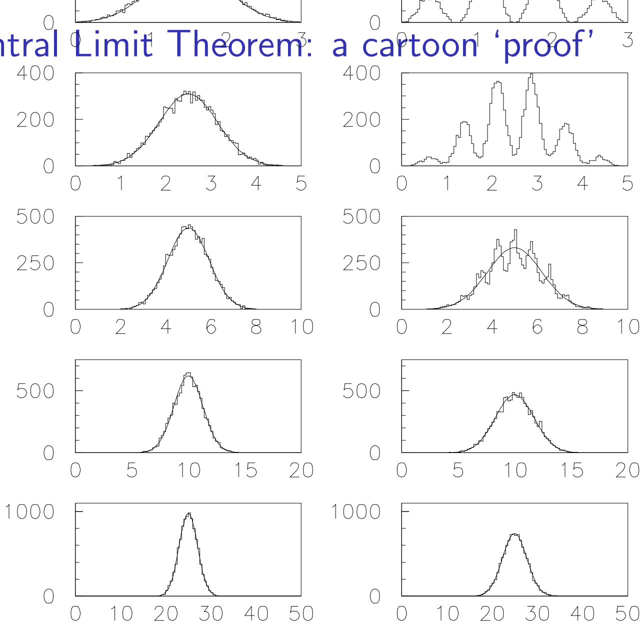
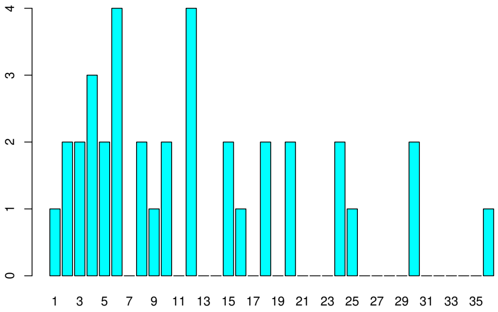


Central Limit Theorem: another animated demonstration

Starting from the distribution of the product of the outcomes of two dice



(Unnormalized distribution)

⇒ R script

Sum of product of outcomes of *n* pairs of dice

```
pausa <- function() {
  cat ("\n >> press Enter to continue\n")
  scan()
}

outcomes <- as.vector(outer(1:6,1:6))
N <- 100
n <- 10000
sx <- rep(0, n)
for(ns in 1:N) {
  xi <- sample(outcomes, n, replace=TRUE)
  sx <- sx + xi
  hist(sx, nc=100, col='cyan', xlab='Sum X', freq=FALSE,
       main=sprintf("ns = %d; mean = %.1f, std = %.1f ",
                    ns, mean(sx), sd(sx)))
  pausa()
}
```