Propagating uncertanties via linearization

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Propagation on uncertainties: rewriting the expressions of the linear combinations

Let's take linear combination $\underline{Y}$ of n *input* variables $\underline{X}$:

$$Y_k = \sum_i c_{ki}\, X_i$$

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The coefficients c_{ki} have the trivial interpretation of partial derivarives, that is

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This observation suggest that we can make use of the results obtained for linear combinations if we linearize the generic functions [Note: $Y_k()$ stands for the k-th function.]

$$Y_k = Y_k(\underline{X})$$

Linearization

Some premizes:

- it is important to understand around which point we have to make the linear expansion;
- the probability *mass* of the X_i has to concentrate in the region in which the linearization is reasonable.

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We start making the expansion around the expected values of the X_i. It will be clear why this is the correct choice.

Linearization around the expected values

$$Y_k \quad = \quad Y_K\left(\mathrm{E}(\underline{X})\right) \;+\; \sum_i \left.\frac{\partial Y_k}{\partial X_i}\right|_{\mathrm{E}(\underline{X})} \cdot \left(X_i - \mathrm{E}(X_i)\right) + \ldots$$

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1. Expected values (neglecting hereafter higher order terms)

$$\mathrm{E}(Y_k) \;=\; Y_K\left(\mathrm{E}(\underline{X})\right) + \mathbf{0}$$

because

- $Y_k\left(\mathrm{E}(\underline{X})\right)$ is just a number;
- $\mathrm{E}[X_i - \mathrm{E}(X_i)] = 0$.

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2. A conventient way to rewrite Y_k

   $$Y_k = \sum_i \left.\frac{\partial Y_k}{\partial X_i}\right|_{\mathrm{E}(\underline{X})} \cdot X_i + Y_k^{(0)},$$

   with $Y_k^{(0)}$ including all terms non depending on X_i, and then irrelevant for variances and covariances of the Y_k

Linearization around the expected values

We have then reduced the problem to (approximatley) a linear combination

$$Y_k = \sum_{i=1}^{n} c_{ki}\, X_i + c_{k0}$$

with

$$\begin{aligned} c_{ki} &= \left.\frac{\partial Y_k}{\partial X_i}\right|_{\mathrm{E}(\underline{X})} \\ c_{k0} &= Y_k^{(0)} = Y_K\left(\mathrm{E}(\underline{X})\right) + \sum_{i=1}^{n} \left.\frac{\partial Y_k}{\partial X_i}\right|_{\mathrm{E}(\underline{X})} \cdot \mathrm{E}(X_i) \end{aligned}$$

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$\Rightarrow$ We apply the rule of expected value of linear combinations, in particular,

$$\mathrm{Var}(Y_k) = \sum_{i=1}^{n} c_{ki}^2\, \mathrm{Var}(X_i)$$

Linearization

exercise: measuring an A4 paper

Imagine we have measured the two sides of an A4 paper, obtaining

$$\begin{aligned} a &= 29.73 \pm 0.03\,\mathrm{cm} \\ b &= 21.45 \pm 0.04\,\mathrm{cm}\,. \end{aligned}$$

Evaluate (expected values, standard uncertainty and correlation)

- perimeter, $p = 2\,a + 2\,b$;
- Area, $A = a\,b$;
- diagonal, $d = \sqrt{a^2 + b^2}$.

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Matching the general notation:

- $\underline{X} = \{a, b\}$, $\underline{Y} = \{p, A, d\}$
- $\mathrm{E}[a] = 29.73\,\mathrm{cm}$; $\sigma(a) = 0.03\,\mathrm{cm}$; $\mathrm{Var}(a) = (0.03\,\mathrm{cm})^2$; etc...
- $\partial Y_1/\partial X_1 = \partial p/\partial a = 2$; $\quad \partial Y_1/\partial X_2 = \partial p/\partial b = 2$;
- $\partial Y_2/\partial X_1|_{\mathrm{E}(\underline{X})} = \partial A/\partial a|_{\mathrm{E}(\underline{X})} = b|_{\mathrm{E}(\underline{X})} = 21.45\,\mathrm{cm}$
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- etc. etc, ...

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(neglecting an irrelevant numerical factor).

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- linearization formulae should be used with care
- ... and possibly avoided!
- But, nevertheless, when used correctly they offer useful insights in the dependence of the final result on the *input quantities* in terms of relative uncertainties.

Linearization of monomial forms

The coefficients of the linear expansion around the expected values acquire a very simple and useful form

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&= \frac{\alpha_i}{X_i} \cdot X_1^{\alpha_1} \cdot X_2^{\alpha_2} \cdot \ldots \cdot X_i^{\alpha_i} \cdot \ldots \cdot X_n^{\alpha_n}
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In the case of several Y's the elements c_{ki} of the tranformation matrix **C** are therefore

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\end{aligned}
$$

Linearization of monomial forms: single Y

Special subcase: variance from independent variables

In the case of a single Y and independent $\underline{X}$, we get

$$\sigma^2(Y) \;\approx\; \sum_i \alpha_i^2 \left(\left.\frac{Y}{X_i}\right|_{\mathrm{E}(\underline{X})}\right)^2 \sigma^2(X_i)$$

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\left(\frac{\sigma(Y)}{|\mathrm{E}(Y)|} \right)^2 &\approx \sum_i \alpha_i^2 \cdot \left(\frac{\sigma(X_i)}{|\mathrm{E}(X_i)|} \right)^2
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\left(\frac{\sigma(Y)}{|\mathrm{E}(Y)|} \right)^2 &\approx \sum_i \alpha_i^2 \cdot \left(\frac{\sigma(X_i)}{|\mathrm{E}(X_i)|} \right)^2 \\
r_Y^2 &\approx \sum_i \alpha_i^2 \cdot r_{X_i}^2 \, ,
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r_Y^2 &\approx \sum_i \alpha_i^2 \cdot r_{X_i}^2\,, \qquad \left[\longrightarrow \sum_{ij} \rho_{ij}\, \alpha_i\, \alpha_j\, r_i\, r_j \right]
\end{aligned}
$$

having indicated with r the relative (standard) uncertainties $\sigma()/|E()|$.

Linearization of monomial expressions of independent variables

Exercise

Imagine we want to measure g with a pendulum:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

from which it follows

$$g = (2\pi)^2\, l\, T^{-2}$$

Q.: How precisely we have to measure l and T if we require they contribute equally to r_g, that we want to keep $\leq 1\,\%$?

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Exercise – solution

If the determinations of l and T are independent, then, in *percentages* (p):

$$p_g^2 \quad = \quad p_l^2 + 4\, p_T^2$$

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- Requirements

$$p_g \;\leq\; 1$$

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$$\begin{aligned} p_g &\leq 1 \\ p_l = 2p_T &\leq 1/\sqrt{2} = 0.71 \end{aligned}$$

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$$\begin{aligned} p_g &\leq 1 \\ p_I = 2p_T &\leq 1/\sqrt{2} = 0.71 \\ p_T &\leq 1/(2\,\sqrt{2}) = 0.35 \end{aligned}$$

Last remarks on linearization

- Linerization formulae rely on the fact that the trasformations are 'enough' linear in the regions where the probability mass of the input quantities are concentrated (around their expected values).
- Always check by Monte Carlo!