Propagation on uncertainties: rewriting the expressions of the linear combinations

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This observation suggest that we can make use of the results obtained for linear combinations if we linearize the generic functions

$$Y_k = Y_k(\underline{X})$$

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We start making the expansion around the expected values of the X_i. It will be clear why this is the correct choice.

Linearization around the expected values

$$Y_k = Y_K\left(\mathrm{E}(\underline{X})\right) + \sum_i \left.\frac{\partial Y_k}{\partial X_i}\right|_{\mathrm{E}(\underline{X})} \cdot \left(X_i - \mathrm{E}(X_i)\right) + \ldots$$

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1. Expected values (neglecting hereafter the hight order terms)

$$\mathrm{E}(Y_k) \;=\; Y_K\,(\mathrm{E}(\underline{X})) + \mathbf{0}$$

because

- $Y_k\,(\mathrm{E}(\underline{X}))$ is just a number;
- $\mathrm{E}[X_i - \mathrm{E}(X_i)] = 0$.

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2. A conventient way to rewrite Y_k

$$Y_k = \sum_i \left.\frac{\partial Y_k}{\partial X_i}\right|_{\mathrm{E}(\underline{X})} \cdot X_i + Y_k^{(0)},$$

   with $Y_k^{(0)}$ including all terms non depending on X_i, and then irrelevant for variances and covariances of the Y_k

Linearization around the expected values

We have then reduced the problem to (approximatley) a linear combination

$$Y_k = \sum_{i=1}^{n} c_{ki}\, X_i + c_{k0}$$

with

$$c_{ki} = \left.\frac{\partial Y_k}{\partial X_i}\right|_{\mathrm{E}(\underline{X})}$$

$$c_{k0} = Y_k^{(0)} = Y_K\left(\mathrm{E}(\underline{X})\right) + \sum_{i=1}^{n} \left.\frac{\partial Y_k}{\partial X_i}\right|_{\mathrm{E}(\underline{X})} \cdot \mathrm{E}(X_i)$$

Linearization

exercise: Extending the A4 paper example

Imagine we have measured the two sides of an A4 paper, obtaining

$$\begin{aligned} a &= 29.73 \pm 0.03\,\mathrm{cm} \\ b &= 21.45 \pm 0.04\,\mathrm{cm}\,. \end{aligned}$$

Evaluate (expected values, standard uncertainty and correlation)

- perimeter, $p = 2\,a + 2\,b$;
- Area, $A = a\,b$;
- diagonal, $d = \sqrt{a^2 + b^2}$

assuming both $\rho(a, b) = 0$ and $\rho(a, b) = +0.8$.