Probabilistic Inference and Applications to Frontier Physics – Part 2 –

Giulio D’Agostini

Dipartimento di Fisica

Università di Roma La Sapienza

What to do?

$\Rightarrow$ 'Forward to past'

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$\Longrightarrow$ ‘Forward to past’

But benefitting of

- Theoretical progresses in probability theory
- Advance in computation (both symbolic and numeric)

$\longrightarrow$ many frequentistic ideas had their *raison d’être* in the computational barrier (and many simplified – often simplistic – methods were ingeniously worked out)

$\longrightarrow$ no longer an excuse!

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$\Rightarrow$ Use consistently probability theory

- “It’s easy if you try”
- But first you have to recover the intuitive concept of probability.

Probability

What is probability?

Standard textbook definitions

$$p = \frac{\#\text{ favorable cases}}{\#\text{ possible equiprobable cases}}$$

$$p = \frac{\#\text{ times the event has occurred}}{\#\text{ independent trials under same conditions}}$$

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Laplace: *"lorsque rien ne porte à croire que l'un de ces cas doit arriver plutot que les autres"*

Pretending that replacing 'equi-probable' by 'equi-possible' is just cheating students (as I did in my first lecture on the subject. . .).

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Future ⇔ Past (believed so)

$n \longrightarrow \infty$: $\longrightarrow$ *"usque tandem?"*

$\longrightarrow$ *"in the long run we are all dead"*

$\longrightarrow$ It limits the range of applications

Definitions $\longrightarrow$ evaluation rules

Very useful evaluation rules

$$A) \quad p = \frac{\#\,\text{favorable cases}}{\#\,\text{possible equiprobable cases}}$$

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If the implicit beliefs are well suited for each case of application.

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- Rule A will be recovered immediately (under the assumption of equiprobability, when it applies).
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It is what everybody knows what it is before going at school

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- → how much we are confident that something is true
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- → "A measure of the degree of belief that an event *will* occur"

[Remark: 'will' does not imply future, but only uncertainty.]

Or perhaps you prefer this way…

"Given the state of our knowledge about everything that could possible have any bearing on the coming true[1]*. . . ,*

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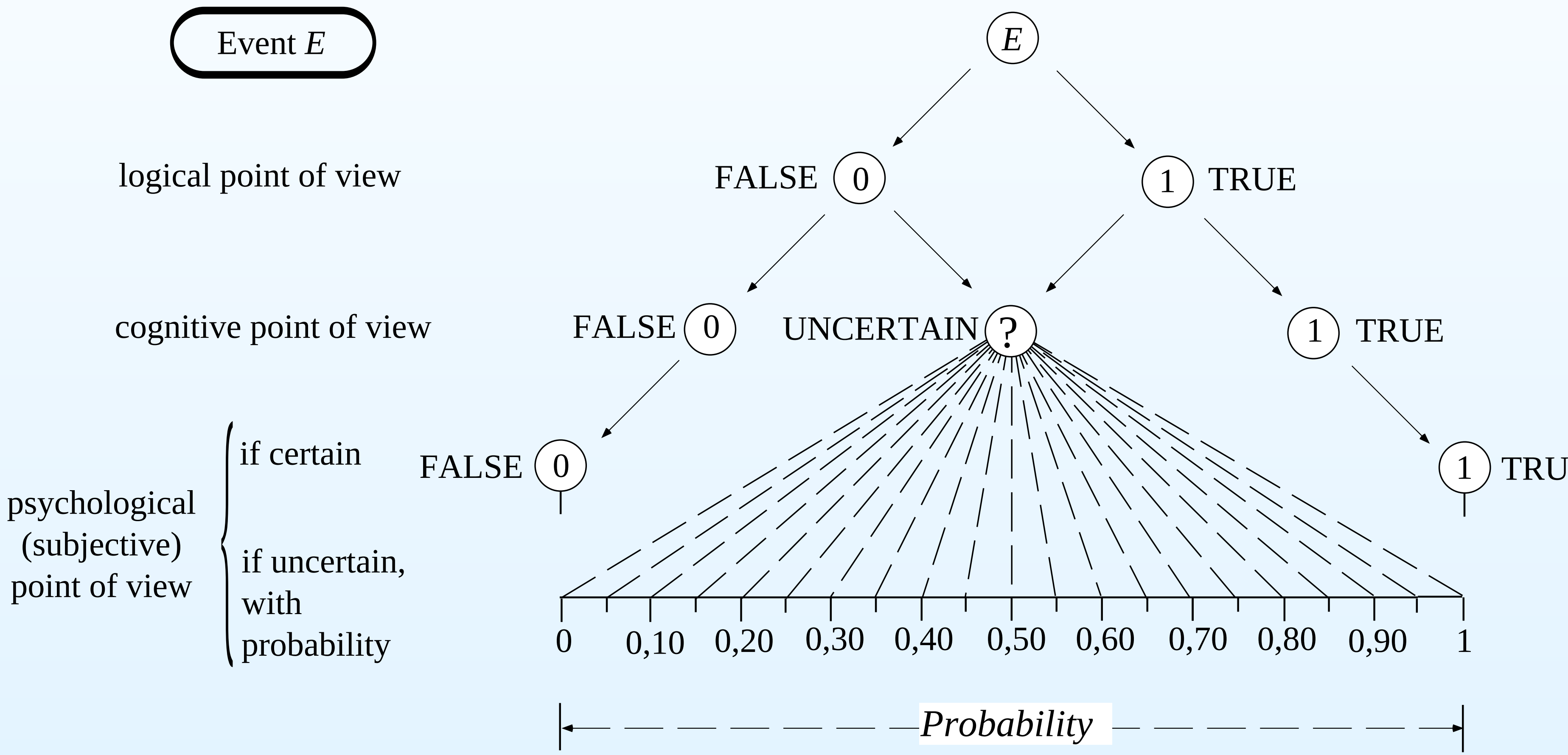
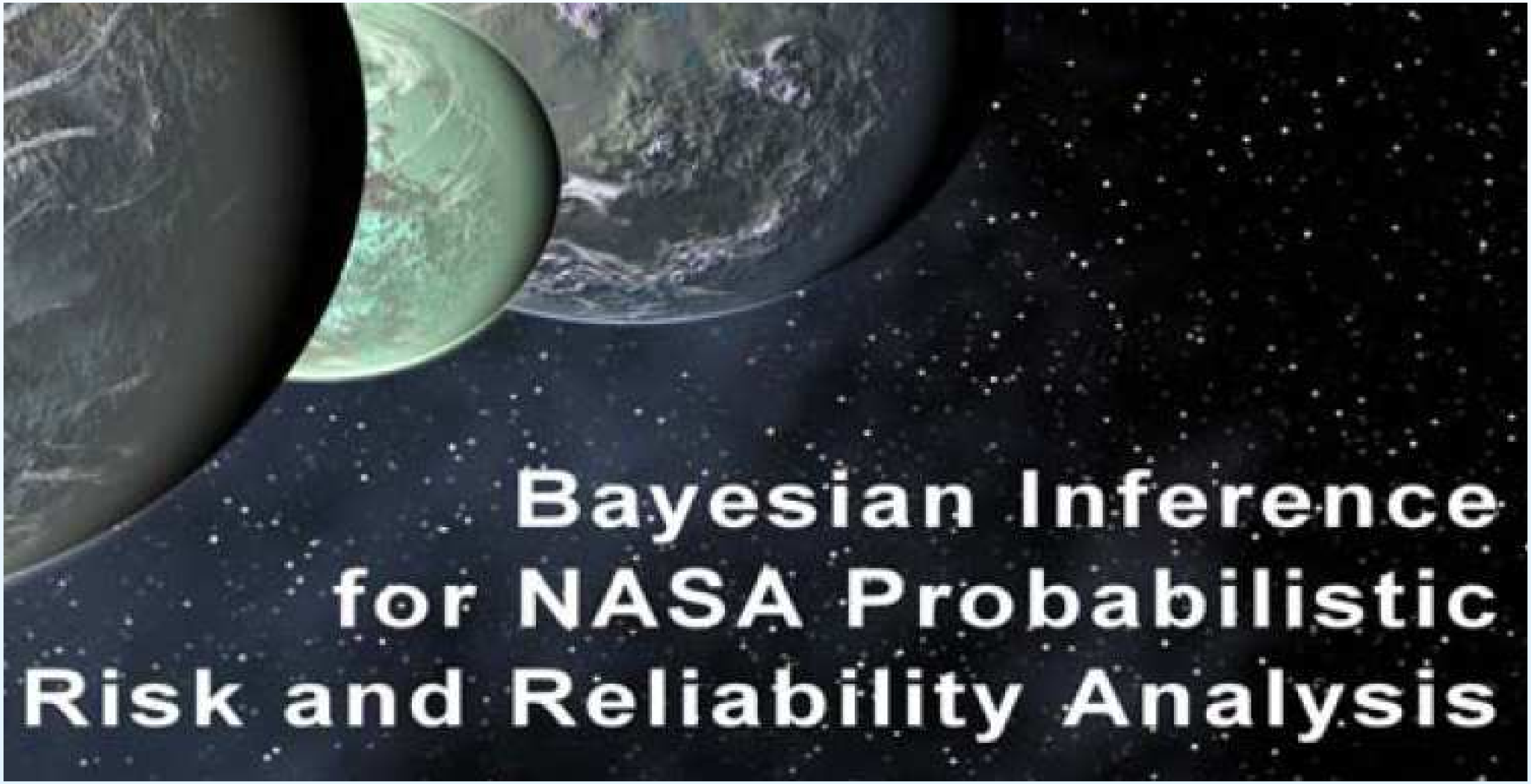
False, True and probable



An helpful diagram

The previous diagram seems to help the understanding of the concept of probability



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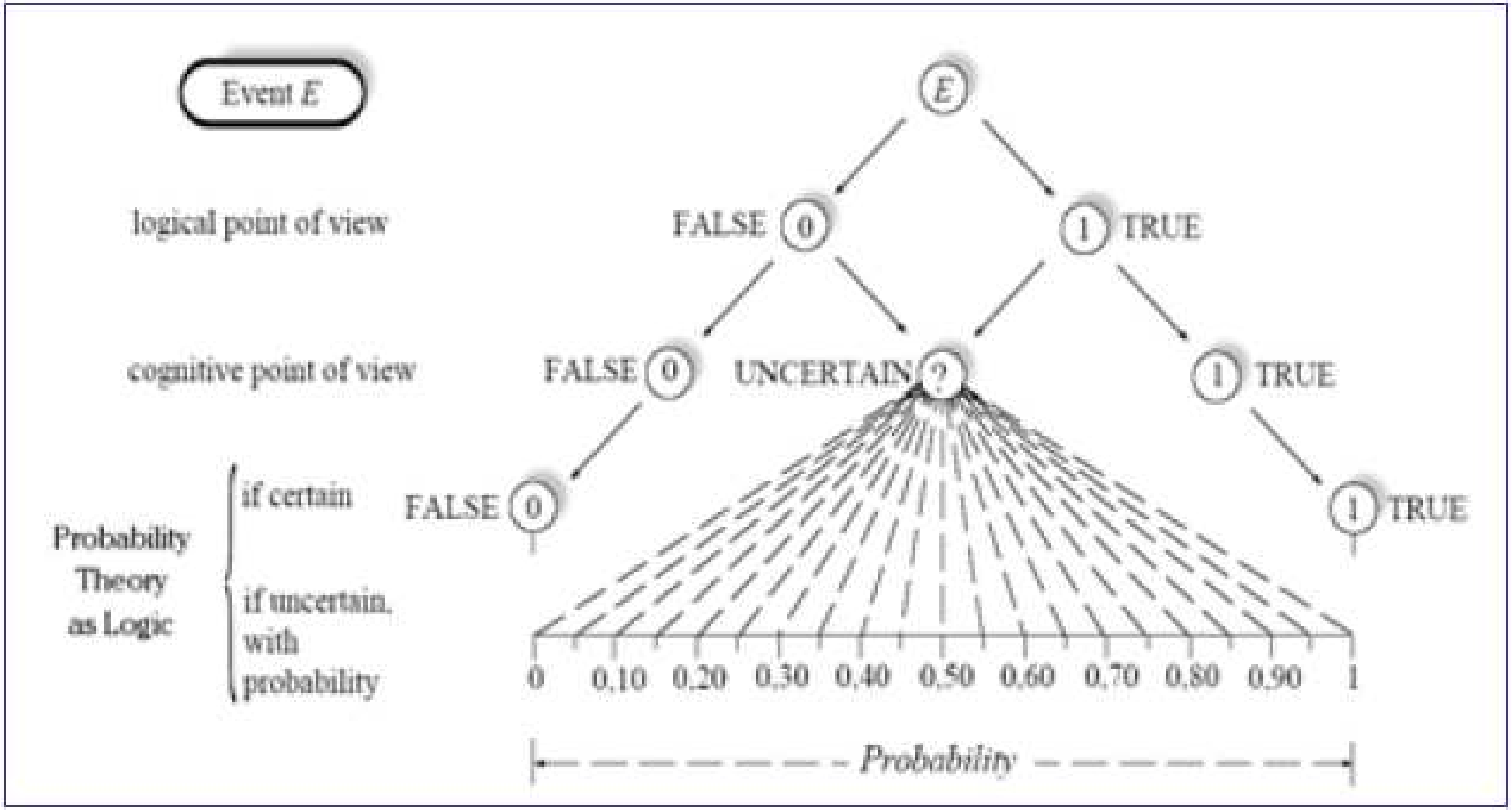

• Figure 2-1. Graphical abstraction of probability as a measure of information (adapted from "Probability and Measurement Uncertainty in Physics" by D'Agostini, [1995]).

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Uncertainty $\longrightarrow$ probability

Probability is related to uncertainty and not (only) to the results of repeated experiments

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"If we were not ignorant there would be no probability, there could only be certainty. But our ignorance cannot be absolute, for then there would be no longer any probability at all. Thus the problems of probability may be classed according to the greater or less depth of our ignorance." (Poincaré)

Uncertainty → probability

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The state of information can be different from subject to subject

⇒ intrinsic subjective nature.

- No negative meaning: only an acknowledgment that several persons might have different information and, therefore, necessarily different opinions.

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"How much we believe something"

Versione velocizzata per MAPSES 2011
→ slide mancanti sulla pagina web dedicata

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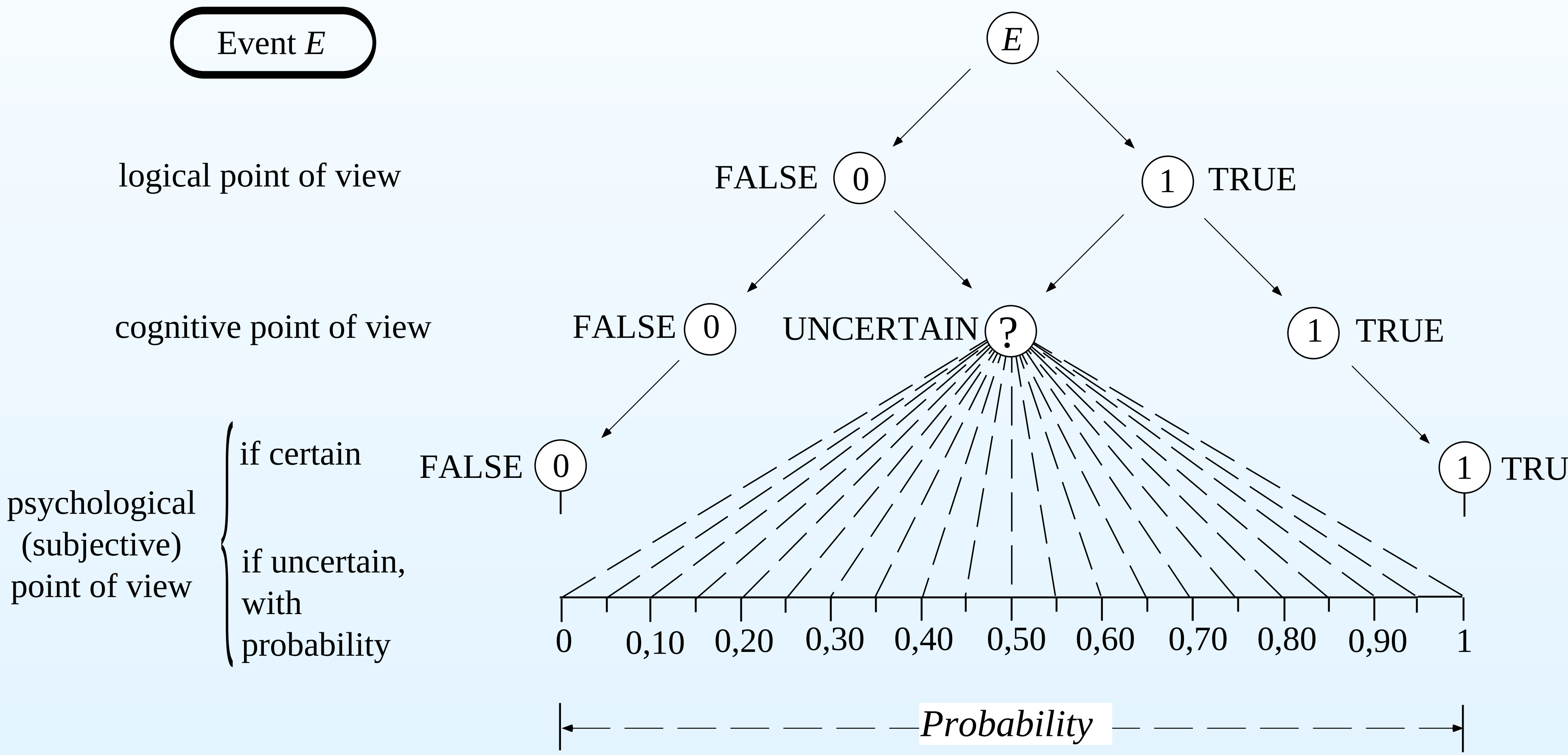

A reminder

Forse vale la pena di ricordare la famosa citazione di Einstein

La geometria, quando è certa, non dice

nulla del mondo reale,

e, quando dice qualcosa a proposito della

nostra esperienza, è incerta.

Chi vuole attenersi al regno del certo è meglio che si occupi di matematica che di fisica.

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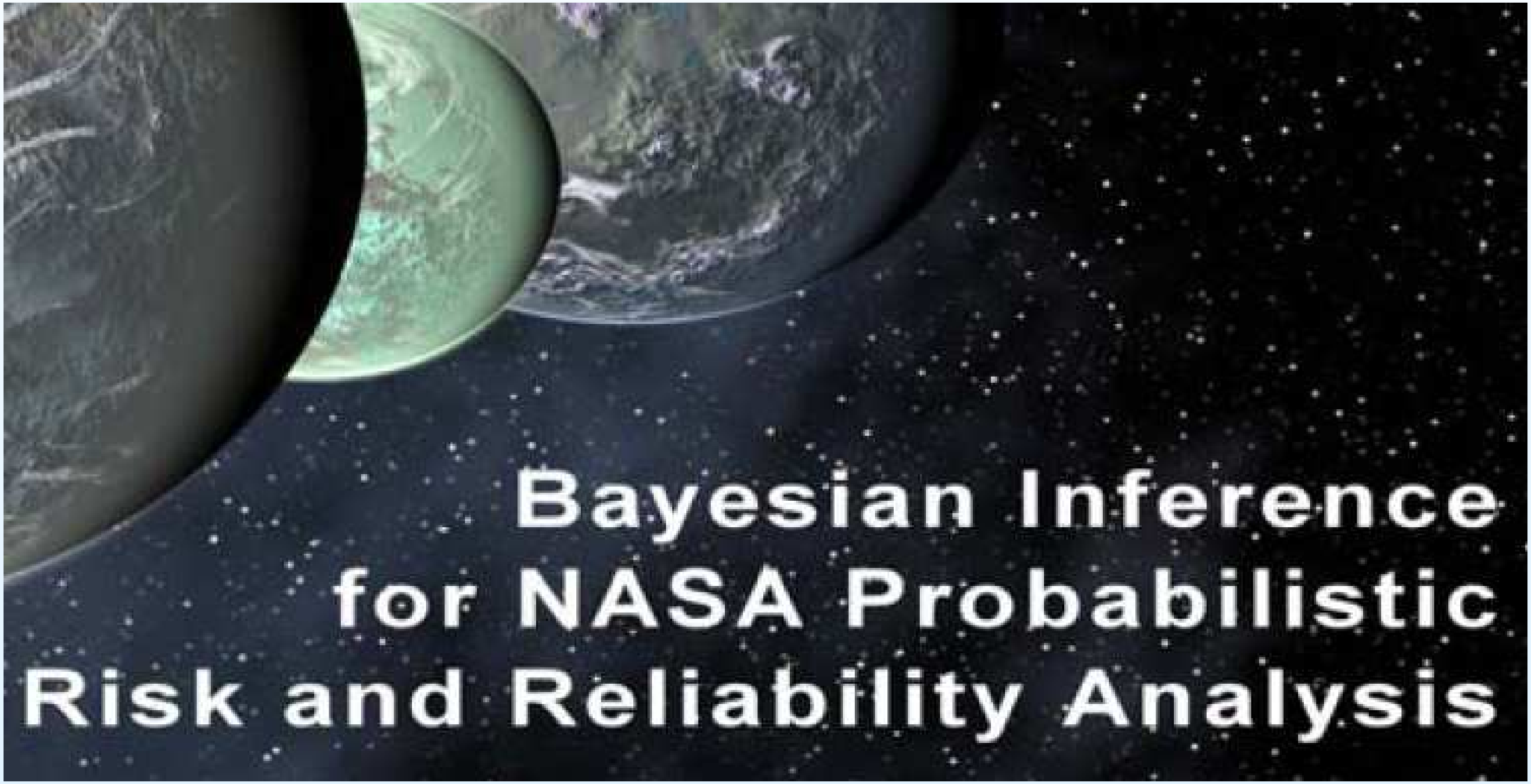

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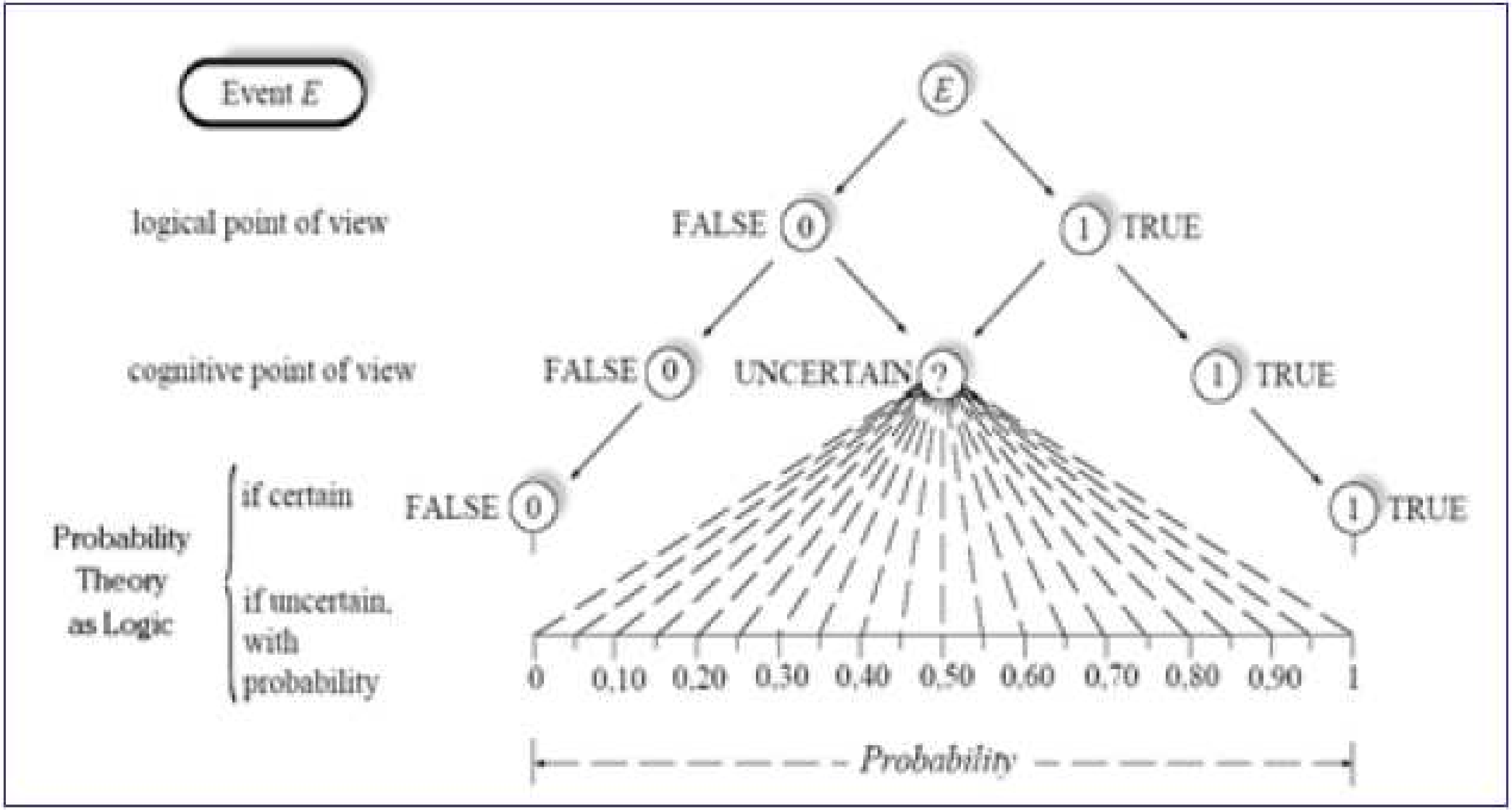

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Inference

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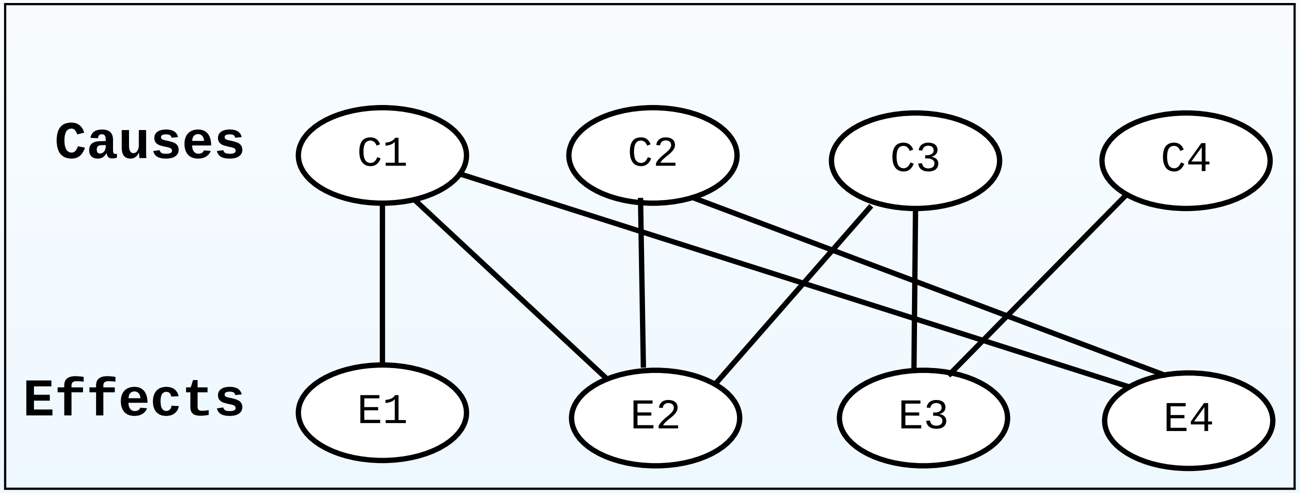
$\Rightarrow$ How do we learn from data in a probabilistic framework?

From causes to effects and back

Our original problem:



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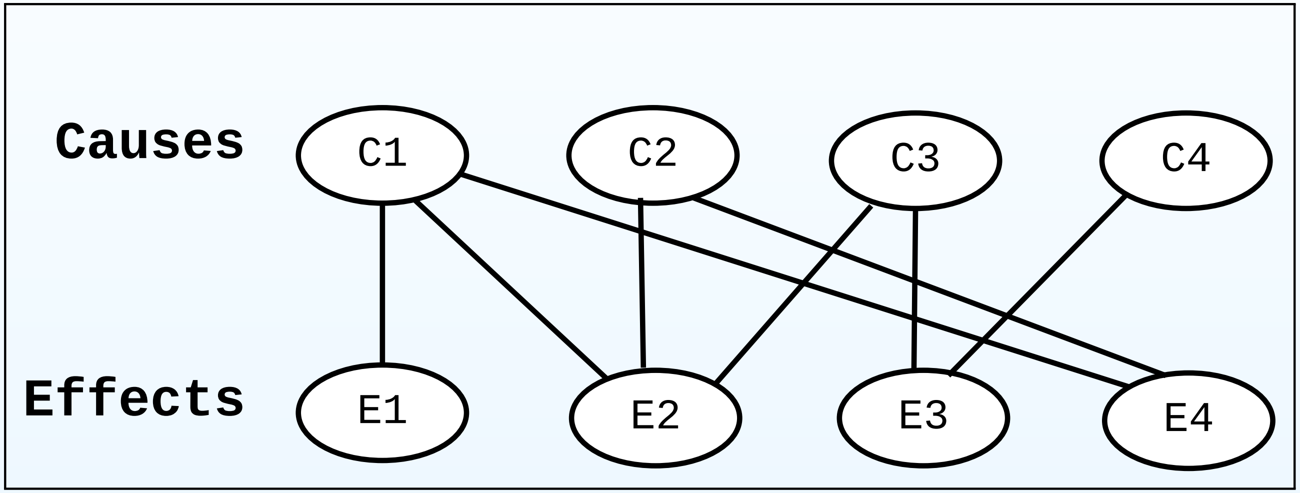

Our conditional view of probabilistic causation

$$P(E_i \,|\, C_j)$$

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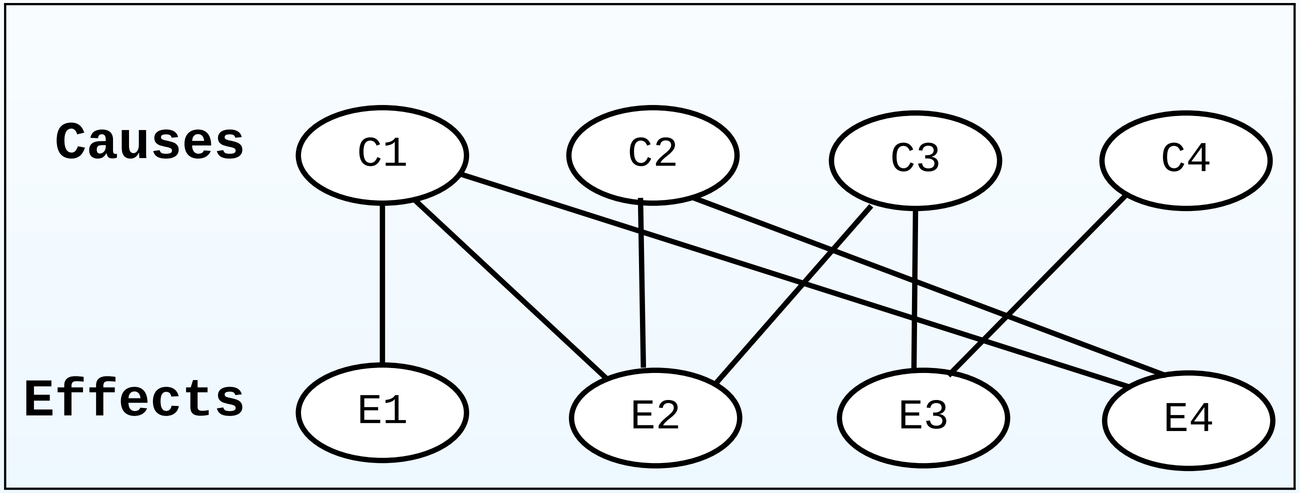

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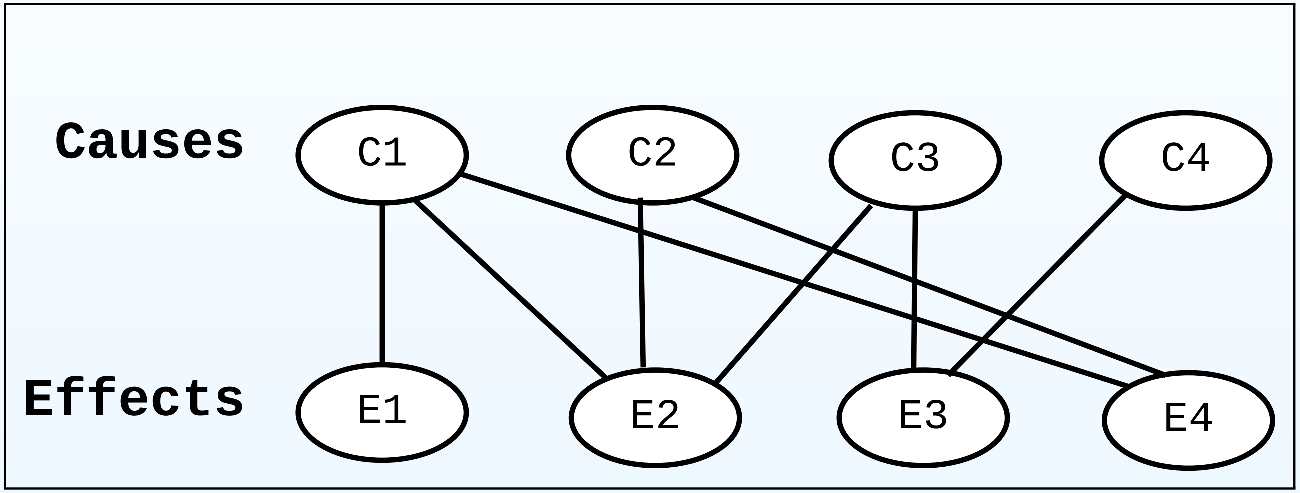

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$$\boxed{P(C_j \,|\, E_i)}$$

The fourth basic rule of probability:

$$\boxed{P(C_j, E_i) = P(E_i \,|\, C_j)\, P(C_j) = P(C_j \,|\, E_i)\, P(E_i)}$$

Symmetric conditioning

Let us take basic rule 4, written in terms of hypotheses H_j and effects E_i, and rewrite it this way:

$$\frac{P(H_j \,|\, E_i)}{P(H_j)} = \frac{P(E_i \,|\, H_j)}{P(E_i)}$$

"The condition on E_i changes in percentage the probability of H_j as the probability of E_i is changed in percentage by the condition H_j."

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Got 'after' Calculated 'before'

(where 'before' and 'after' refer to the knowledge that E_i is true.)

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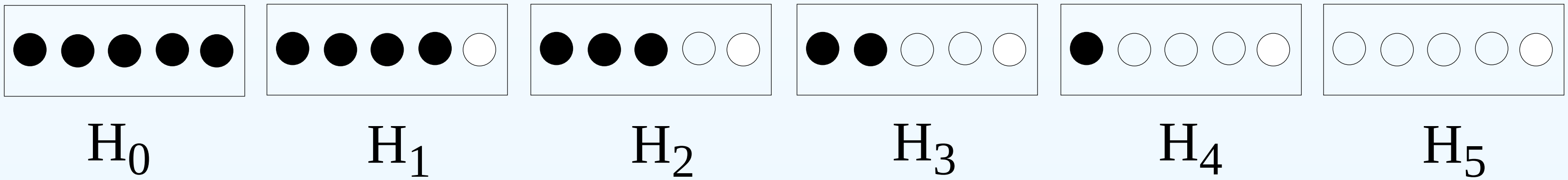
"post illa observationes" *"ante illa observationes"*

(Gauss)

Application to the six box problem



Remind:

- $E_1 =$ White
- $E_2 =$ Black

Collecting the pieces of information we need

Our tool:

$$P(H_j \,|\, E_i, I) = \frac{P(E_i \,|\, H_j, I)}{P(E_i \,|\, I)} \, P(H_j \,|\, I)$$

Collecting the pieces of information we need

Our tool:

$$P(H_j \mid E_i, I) = \frac{P(E_i \mid H_j, I)}{P(E_i \mid I)} P(H_j \mid I)$$

- $P(H_j \mid I) = 1/6$

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Our prior belief about H_j

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Probability of E_i under a well defined hypothesis H_j
It corresponds to the 'response of the apparatus in measurements.
$\rightarrow$ likelihood (traditional, rather confusing name!)

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Probability of E_i taking account all possible H_j
$\rightarrow$ How much we are confident that E_i will occur.
Easy in this case, because of the symmetry of the problem.
But already after the first extraction of a ball our opinion about the box content will change, and symmetry will break.

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'decomposition law': $P(E_i \,|\, I) = \sum_j P(E_i \,|\, H_j,\, I) \cdot P(H_j \,|\, I)$

($\rightarrow$ Easy to check that it gives $P(E_i \,|\, I) = 1/2$ in our case).

Collecting the pieces of information we need

Our tool:

$$P(H_j\,|\,E_i, I) = \frac{P(E_i\,|\,H_j, I)\cdot P(H_j\,|\,I)}{\sum_j P(E_i\,|\,H_j, I)\cdot P(H_j\,|\,I)}$$

- $P(H_j\,|\,I) = 1/6$
- $P(E_i\,|\,I) = \sum_j P(E_i\,|\,H_j,\, I) \cdot P(H_j\,|\,I)$
- $P(E_i\,|\,H_j,\, I)$:

$$\begin{aligned} P(E_1\,|\,H_j,\, I) &= j/5 \\ P(E_2\,|\,H_j,\, I) &= (5-j)/5 \end{aligned}$$

We are ready!

$\longrightarrow$ Let's play with our toy

Naming the method

Some ‘remarks’ on formalism and notation.

(But nothing deep!)

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(But nothing deep!)

From now on it is only a question of

- experience and good sense to model the problem;
- patience;
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- experience and good sense to model the problem;
- patience;
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- computer skill.

Moving to continuous quantities:

- transitions discrete$\rightarrow$continuous rather simple;
- prob. functions $\longrightarrow$ pdf
- learn to summarize the result in *‘a couple of meaningful numbers’* (but remembering that the full answer is in the *final pdf*).

Bayes theorem

The formulae used to *infer* H_i and
to *predict* $E_j^{(2)}$ are related to the name of Bayes

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$$\frac{P(H_j \,|\, E_i)}{P(H_j)} \quad = \quad \frac{P(E_i \,|\, H_j)}{P(E_i)}$$

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Different ways to write the

Bayes' Theorem

Updating the knowledge by new observations

Let us repeat the experiment:

Sequential use of Bayes theorem

Old posterior becomes new prior, and so on

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$$P(H_j \,|\, E^{(1)}, E^{(2)}) \;\propto\; P(E^{(2)} \,|\, H_j, E^{(1)}) \cdot P(H_j \,|\, E^{(1)})$$

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&\propto P(E^{(1)}, E^{(1)} \,|\, H_j) \cdot P_0(H_j) \\
P(H_j \,|\, \text{data}) &\propto P(\text{data} \,|\, H_j) \cdot P_0(H_j)
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P(H_j \,|\, \text{data}) &\propto P(\text{data} \,|\, H_j) \cdot P_0(H_j)
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$$

Bayesian inference

Updating the knowledge by new observations

Let us repeat the experiment:

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$$

Learning from data using probability theory

Exercises and discussions

- Continue with six box problem [$\rightarrow$ *AJP* 67 (1999) 1260]
  $\longrightarrow$ Slides
- Home work 1: AIDS problem $\rightarrow P(\text{HIV}\,|\,\text{Pos})$?

$$\begin{aligned} P(\text{Pos}\,|\,\text{HIV}) &= 100\% \\ P(\text{Pos}\,|\,\overline{\text{HIV}}) &= 0.2\% \\ P(\text{Neg}\,|\,\overline{\text{HIV}}) &= 99.8\% \end{aligned}$$

- Home work 2: Particle identification:
  A particle detector has a μ identification efficiency of $95\,\%$, and a probability of identifying a π as a μ of $2\,\%$. If a particle is identified as a μ, then a trigger is fired. Knowing that the particle beam is a mixture of $90\,\%$ π and $10\,\%$ μ, what is the probability that a trigger is really fired by a μ? What is the signal-to-noise (S/N) ratio?

Odd ratios and Bayes factor

$$\frac{P(\text{HIV}\,|\,\text{Pos})}{P(\overline{\text{HIV}}\,|\,\text{Pos})} = \frac{P(\text{Pos}\,|\,\text{HIV})}{P(\text{Pos}\,|\,\overline{\text{HIV}})} \cdot \frac{P_\circ(\text{HIV})}{P(\overline{\text{HIV}})}$$

$$= \frac{\approx 1}{0.002} \times \frac{0.1/60}{\approx 1} = 500 \times \frac{1}{600} = \frac{1}{1.2}$$

$$\Rightarrow P(\text{HIV}\,|\,\text{Pos}) = 45.5\%\,.$$

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There are some advantages in expressing Bayes theorem in terms of odd ratios:

- There is no need to consider all possible hypotheses (how can we be sure?)
  We just make a comparison of any couple of hypotheses!

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\end{aligned}$$

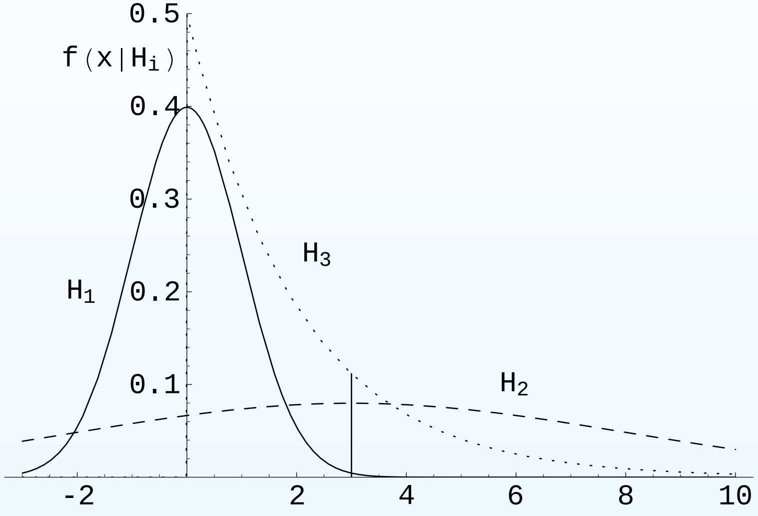
There are some advantages in expressing Bayes theorem in terms of odd ratios:

- There is no need to consider all possible hypotheses (how can we be sure?)
  We just make a comparison of any couple of hypotheses!
- Bayes factor is usually much more inter-subjective, and it is often considered an 'objective' way to report how much the data favor each hypothesis.

Further comments on first meeting

The three models example

Choose among H_1, H_2 and H_3 having observed $x = 3$:



In case of 'likelihoods' given by pdf's, the same formulae apply: "$P(\text{data} \,|\, H_j)$" $\longleftrightarrow$ "$f(\text{data} \,|\, H_j)$".

$$BF_{j,k} = \frac{f(x\!=\!3 \,|\, H_j)}{f(x\!=\!3 \,|\, H_k)}$$

$BF_{2,1} = 18$, $BF_{3,1} = 25$ and $BF_{3,2} = 1.4$ $\rightarrow$ data favor model H_3 (as we can see from figure!), but if we want to state how much we believe to each model we need to 'filter' them with priors.

Assuming the three models initially equally likely, we get final probabilities of 2.3%, 41% and 57% for the three models.

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- for a ‘serious’ probabilistic model comparisons,
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- p-values (e.g. ‘χ^2 tests) have to be considered very useful starting points to understand if further investigation is worth [Yes, I also use χ^2 to get an idea of the “distance” between a model and the experimental data – but not more than that].

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- But until you don't have an alternative and credible model to explain the data, there is little to say about the "chance that the data come from the model", unless the data are really impossible.
- Why do frequentistic test often work? → Think about. . . (Just by chance – no logical necessity)

The hidden uniform

What was the mistake of people saying $P(\overline{\text{HIV}}\,|\,\text{Pos}) = 0.2$?

We can easily check that this is due to have set $\frac{\mathrm{P}_\circ(\text{HIV})}{\mathrm{P}_\circ(\overline{\text{HIV}})} = 1$, that, hopefully, does not apply for a randomly selected Italian.

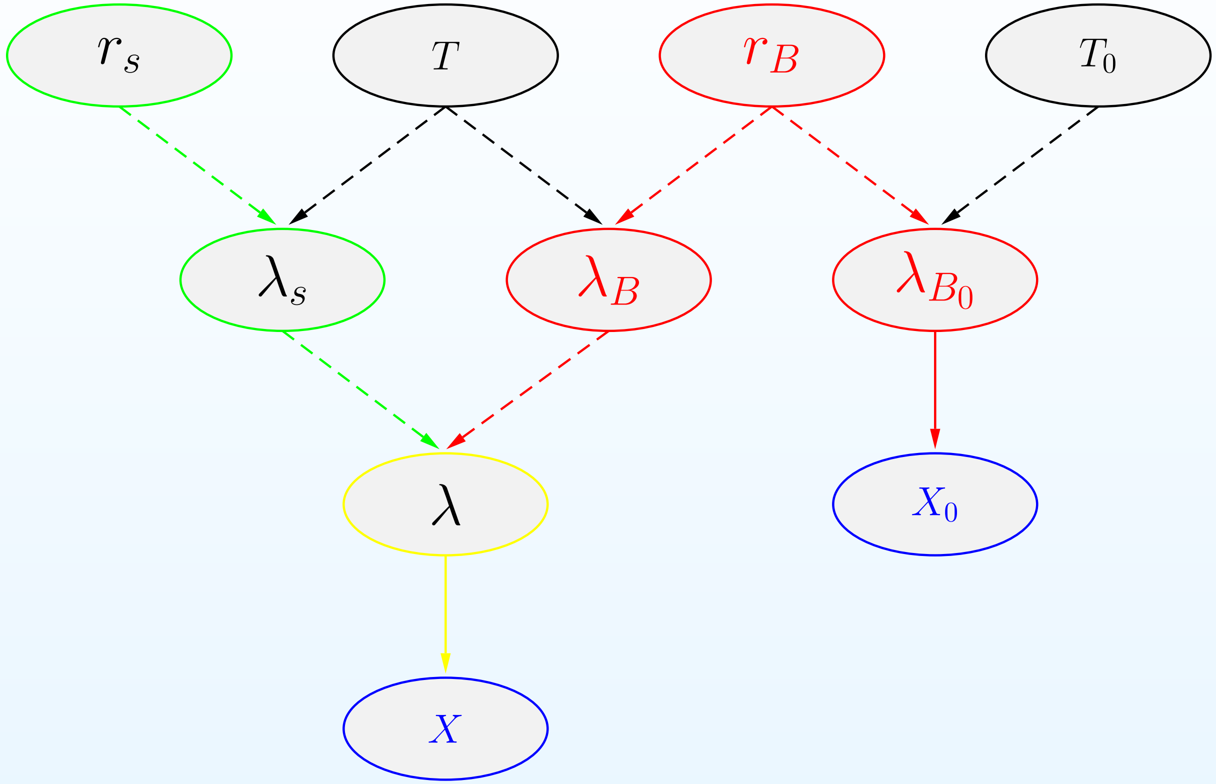
- This is typical in arbitrary inversions, and often also in frequentistic prescriptions that are used by the practitioners to form their confidence on something:

$\longrightarrow$ "absence of priors" means in most times uniform priors over the all possible hypotheses

- but they criticize the Bayesian approach because it takes into account priors explicitly !

Better methods based on 'sand' than methods based on nothing!

Inferring a rate of a Poisson process



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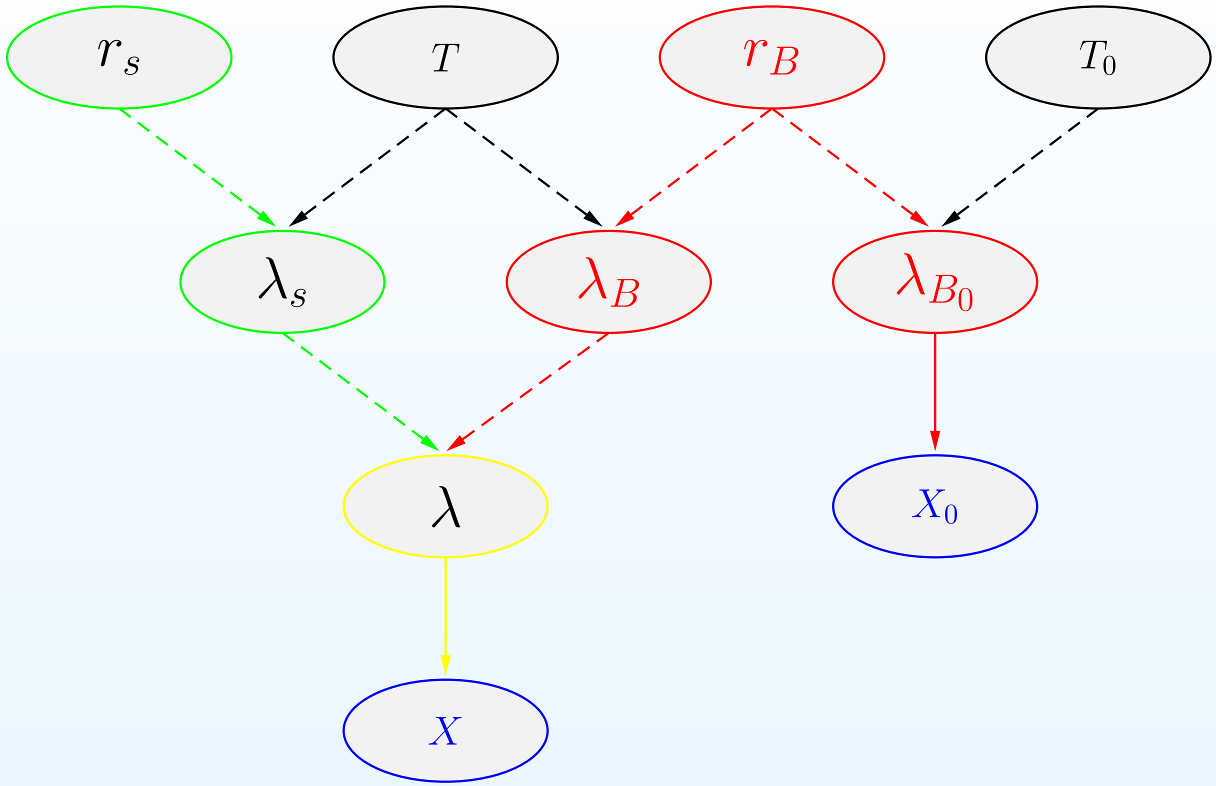

$$f(r_s, r_b \,|\, x, x_0, T, T_0) \;\propto\; f(x, x_0 \,|\, r_s, rb, T, T_0) \cdot f_0(r_s, r_b)$$

$$\propto\; f(x \,|\, (r_s + r_b) \cdot T) \cdot f(x_0 \,|\, r_b \cdot T_0) \cdot f_0(r_s) \cdot f_0(r_b$$

$$f(r_s \,|\, x, x_0, T, T_0) \;\propto\; \int_0^{\infty} f(r_s, r_b \,|\, x, x_0, T, T_0)\, dr_b$$

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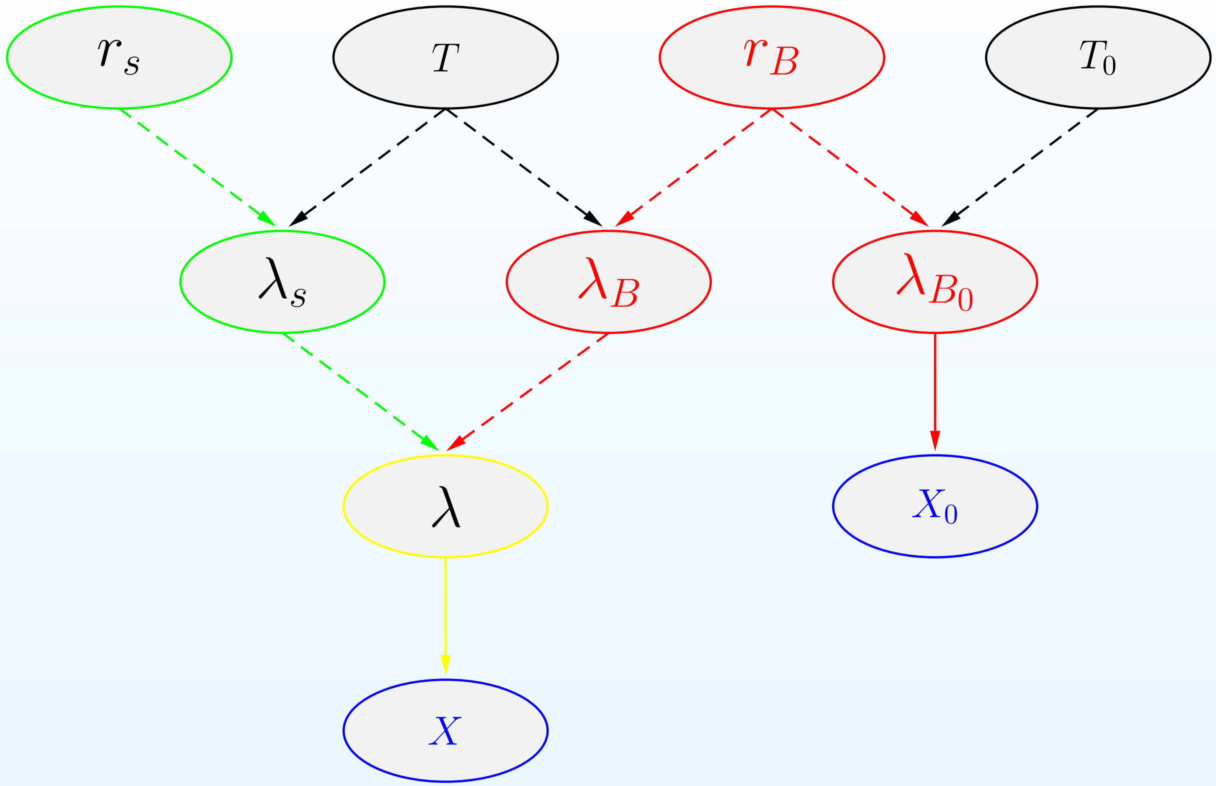

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$$f(r_s \,|\, x, x_0, T, T_0) \propto \int_0^\infty f(r_s, r_b \,|\, x, x_0, T, T_0)\, dr_b \quad \Rightarrow \textit{JAGS}$$

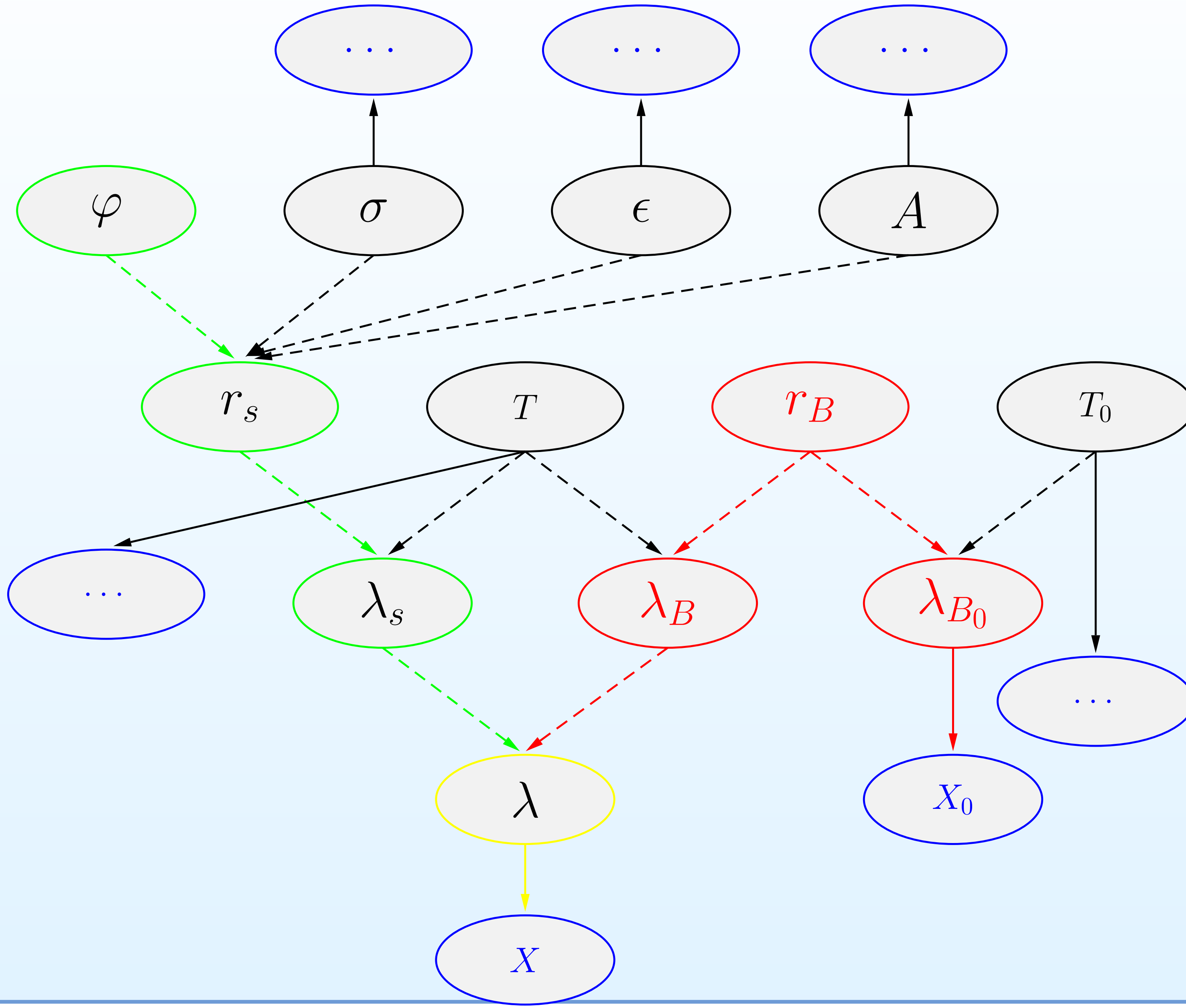
BUGS (Jags) code to model the problem

```
model {
   X ~ dpois(lambda)
   lambda <- ls + lB
   ls <- r * T
   r ~ dgamma(1, 0.00001)  # gamma, ma esattamente come dexp(0.00001)
   lB <- rB * T

   # info sperimentali sul background
   lB0 <- rB * T0
   XB ~ dpois(lB0)
   rB ~ dgamma(1, 0.00001) # anche sul backgrond mettiamo prior vaga
}
```

prova **prova** *prova* `prova`

Making the model more realistic



Upper/lower limits

"Ogni limite ha una pazienza" (Totò)

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“Ogni limite ha una pazienza” (Totò)

A very simple problem:

- counting experiment described by a binomial of unkown p;
- our aim is to ‘get’ p, in the sense of evaluating $f(p\,|\,\text{data})$;
- we make n trials and get $x = 0$ successes.

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- our aim is to ‘get’ p, in the sense of evaluating $f(p\,|\,\text{data})$;
- we make n trials and get $x = 0$ successes.

Bayes’ theorem:

$$f(p\,|\,n, x=0, \mathcal{B}) \;=\; \frac{f(x=0\,|\,n, \mathcal{B})\; f_0(p)}{\int_0^1 f(x=0\,|\,n, \mathcal{B})\; f_0(p)\, dp}$$

with

$$f(x=0\,|\,n, \mathcal{B}) \;=\; (1-p)^n$$

Bernoulli trials $\Rightarrow N$ boxes $\longrightarrow \infty$

Conceptually exactly equivalente to the 6-box problem:

- "success" $\leftrightarrow$ "white ball"
- $p \leftrightarrow$ "proportion of white balls"
- $f(p\,|\,x,n) \;\leftrightarrow\; P(H_i\,|\,x,n)$

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- as log as we continue to extract only black boxes we get more and more convinced ('confident') that Nature has presented us H_0, although we cannot exclude H_1, a bit less H_2, etc.
  $\Rightarrow$ Rigorously speaking, only H_N gets falsified!

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  $\Rightarrow$ Rigorously speaking, only H_N gets falsified!

$$P(H_N\,|\,n, x=0) = 0 \quad \leftrightarrow \quad f(p=1\,|\,n, x=0) = 0$$

Inference about p from 0 counts

Using flat prior, i.e. $f_0(p) = k$

$$
\begin{aligned}
f(p\,|\,n, x=0, \mathcal{B}) &= (n+1)\,(1-p)^n \\
p_{max} &= 0 \\
E(p) &= \frac{1}{n+2} \longrightarrow \frac{1}{n} \\
\sigma(p) &= \sqrt{\frac{(n+1)}{(n+3)(n+2)^2}} \longrightarrow \frac{1}{n} \\
p_{95\%UL} &= 1 - \sqrt[n+1]{0.05}.
\end{aligned}
$$

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As n increases, we get more and more convinced that p has to be very small

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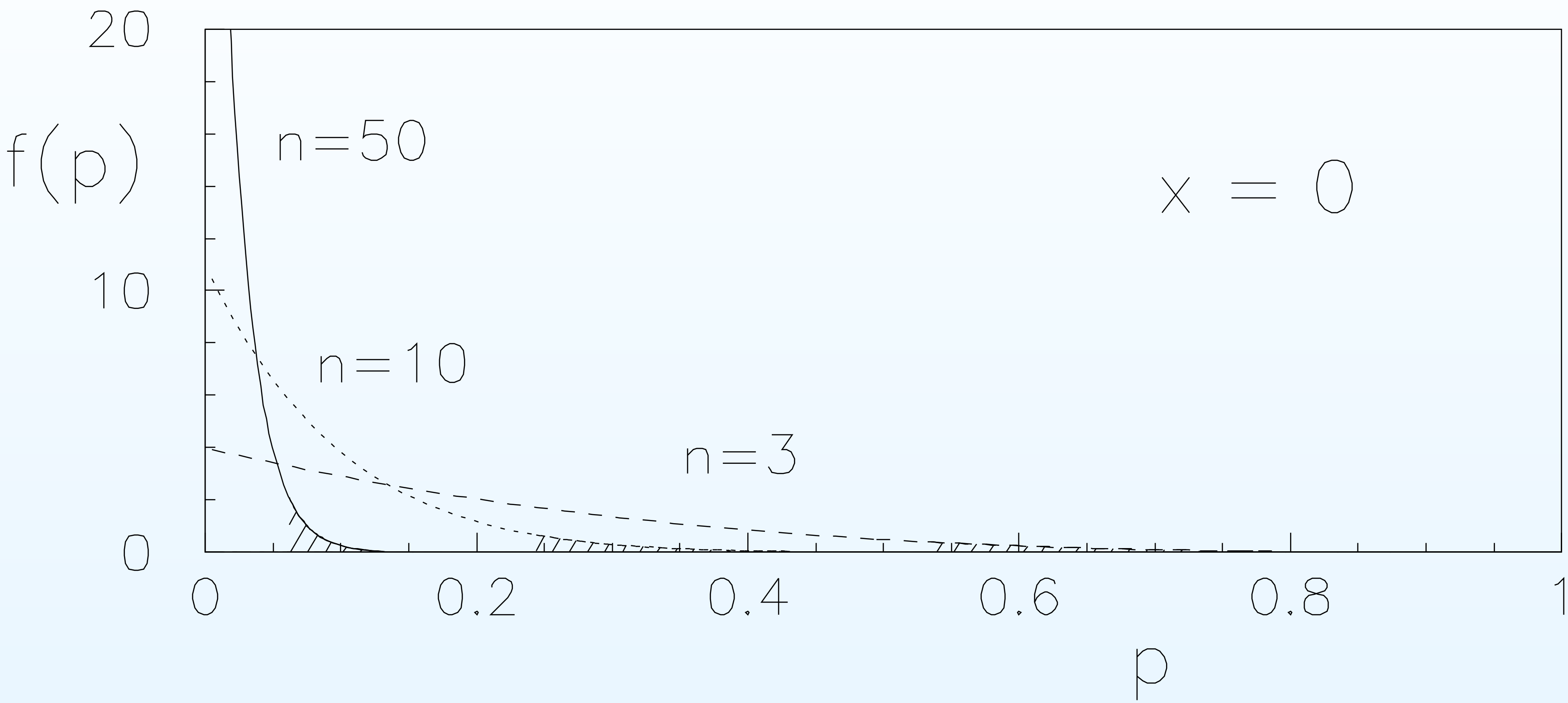

$$f(p \,|\, n, x=0, \mathcal{B}) \;=\; (n+1)\,(1-p)^n$$

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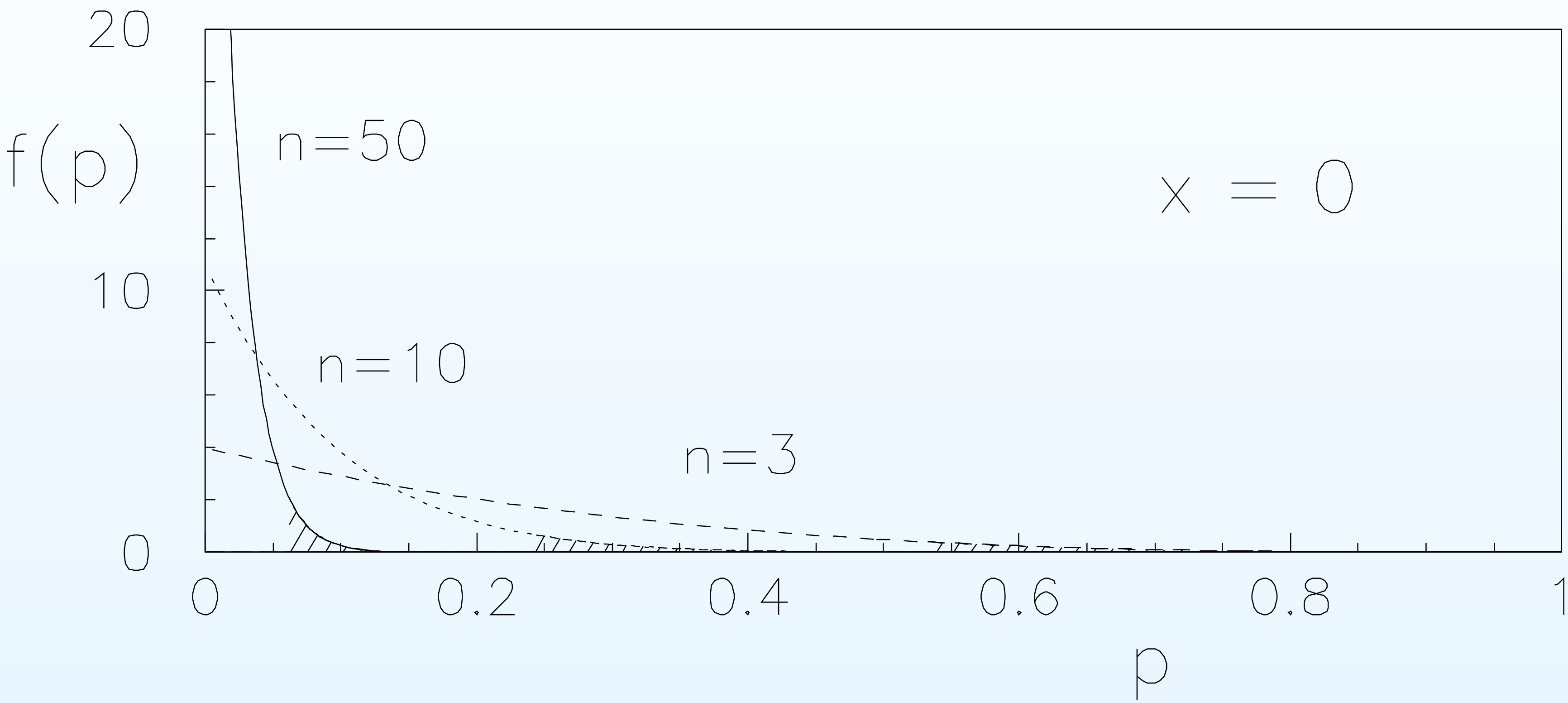

Seems not problematic at all, but we have to remember that it relies on

$$f(x=0\,|\,n,\mathcal{B}) = (1-p)^n$$
$$f_0(p) = k$$

When likelihoods are non ‘closed’

Where is the problem? (Flat priors are regulary used, and are often assumed in other approaches, e.g. ML methods)

When likelihoods are non 'closed'

The major problem is not in $f_0(p)$, but rather in the likelihood $f(x = 0, | \, n, \mathcal{B})$ that does not go to zero on both sides!

When likelihoods are non ‘closed’

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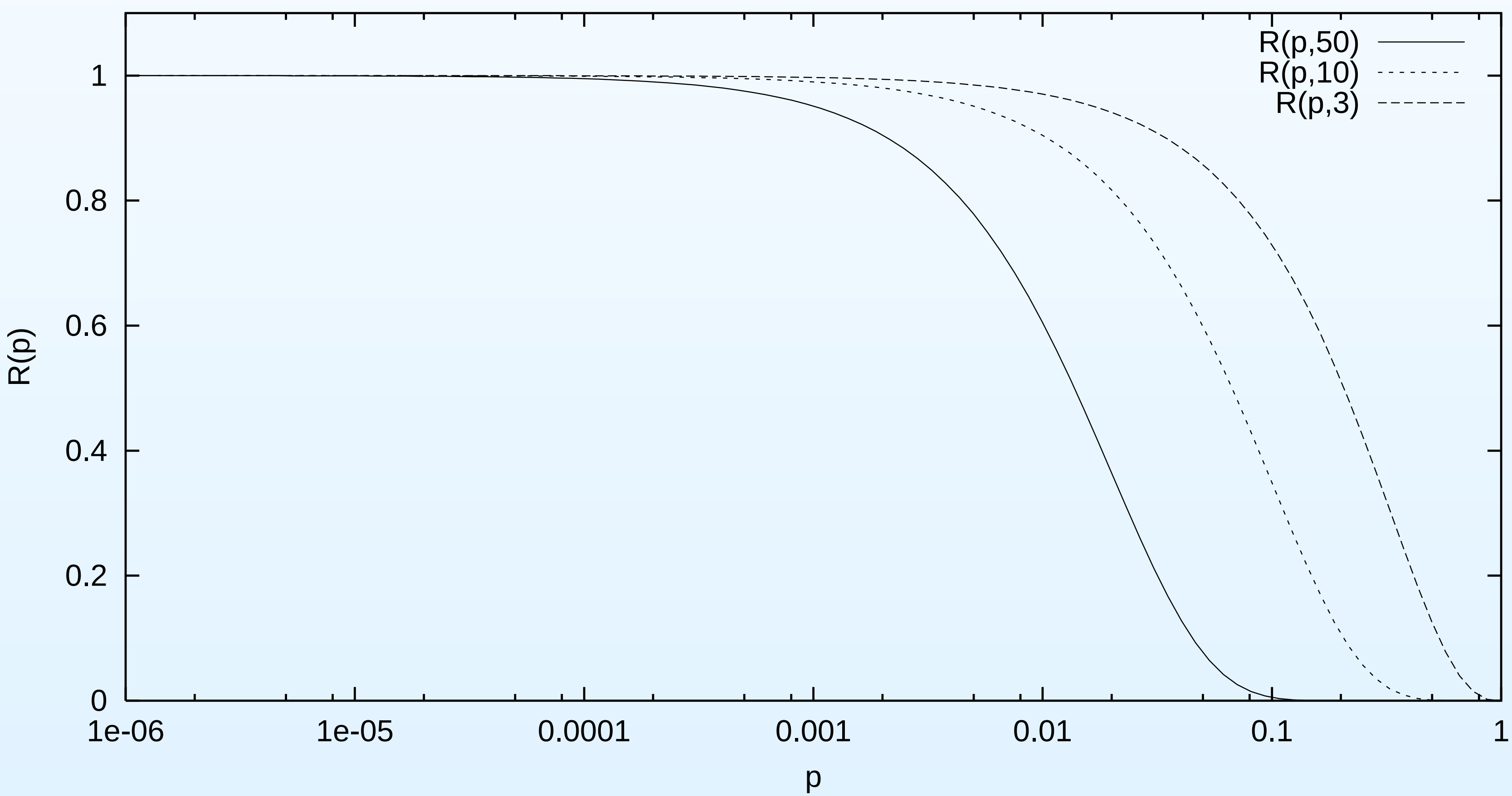
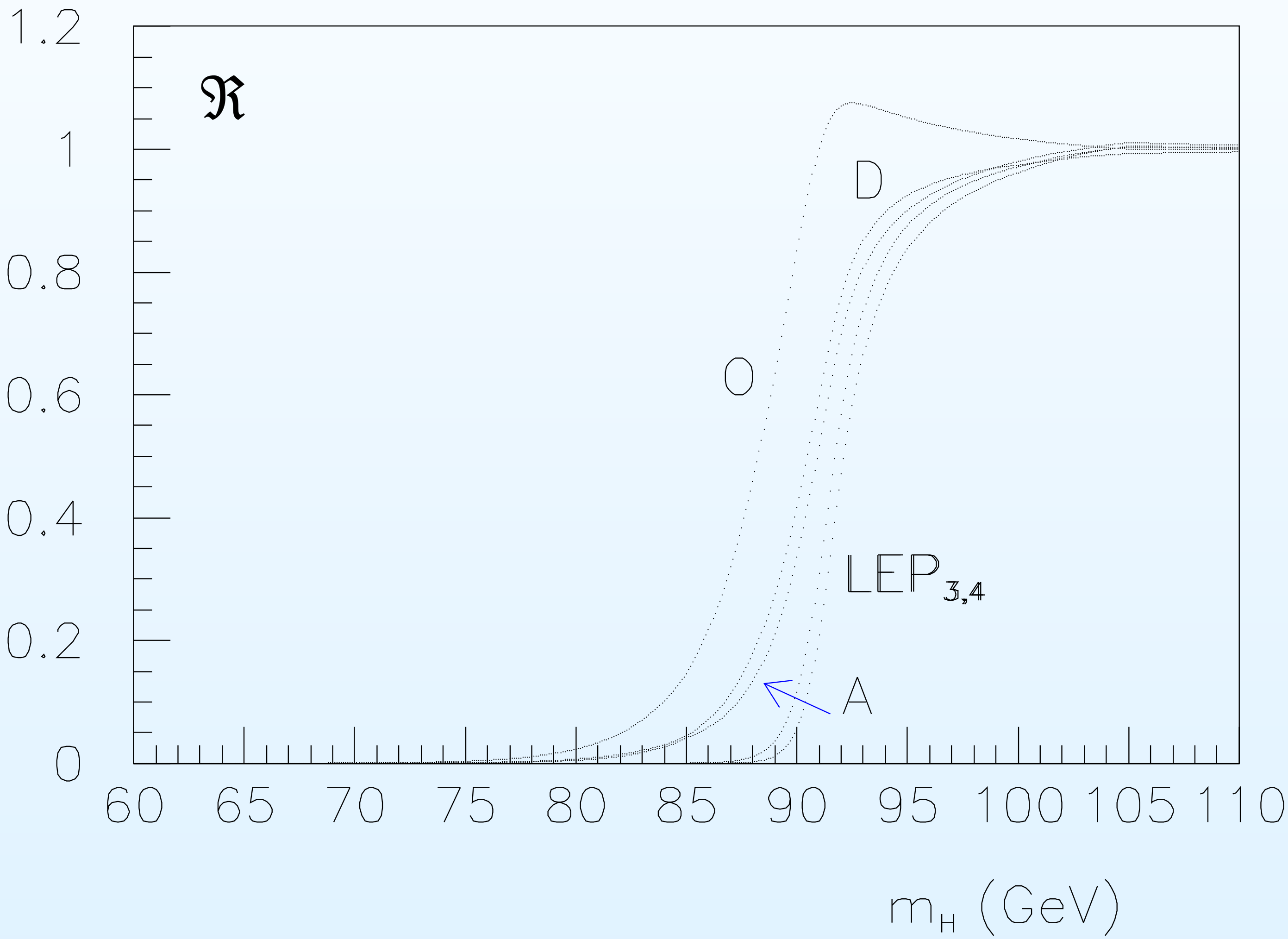
A different representation of the likelihood (properly rescaled) helps:



A probabilistic lower bound for the Higgs?

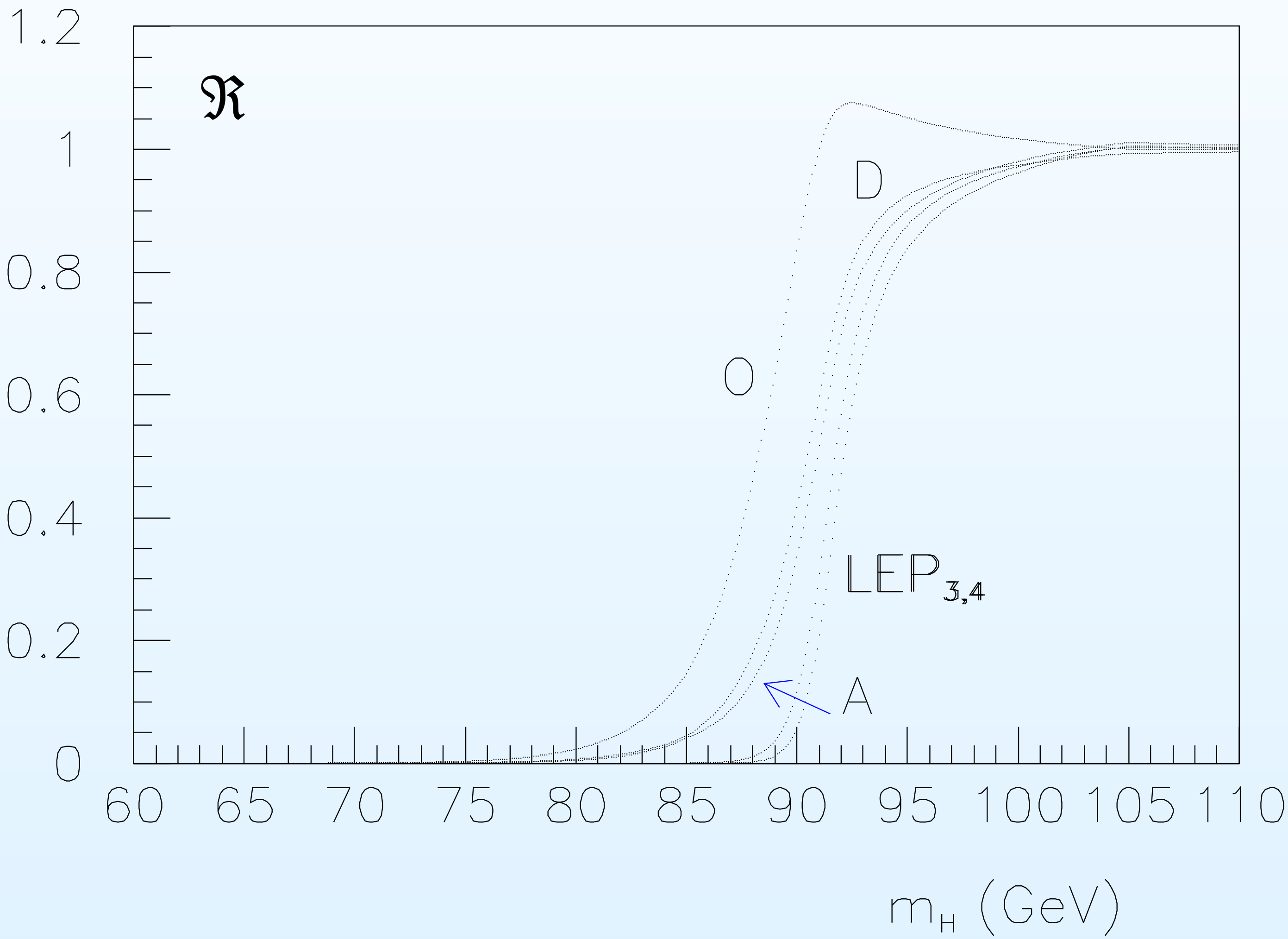
A similar think happens with the direct searches of the Higgs particle at LEP



(1999 figure, but substance unchanged)

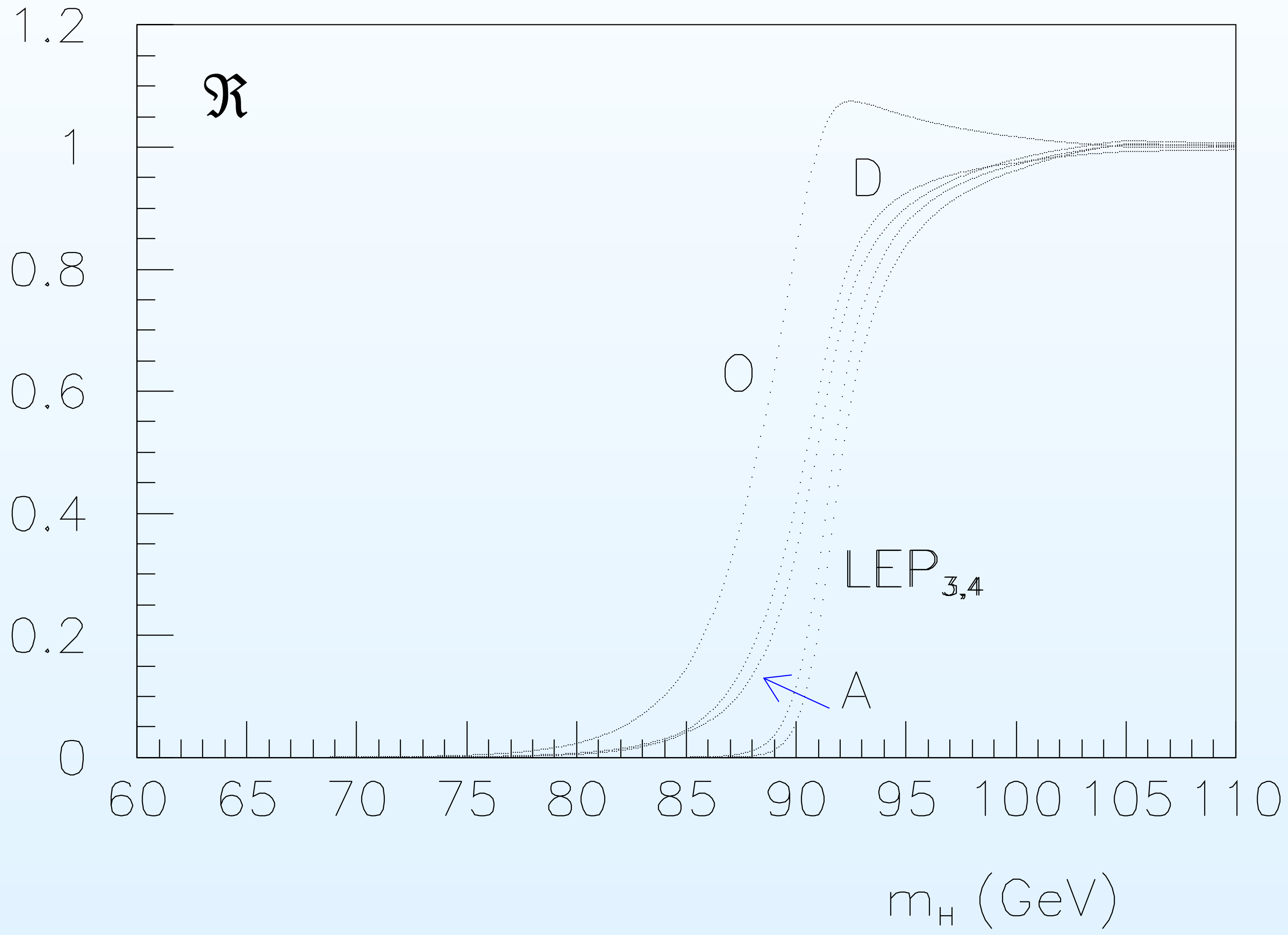
A probabilistic lower bound for the Higgs?

Impossible to express our confidence in probabilistic terms,
unless we define an upper cut!



A probabilistic lower bound for the Higgs?

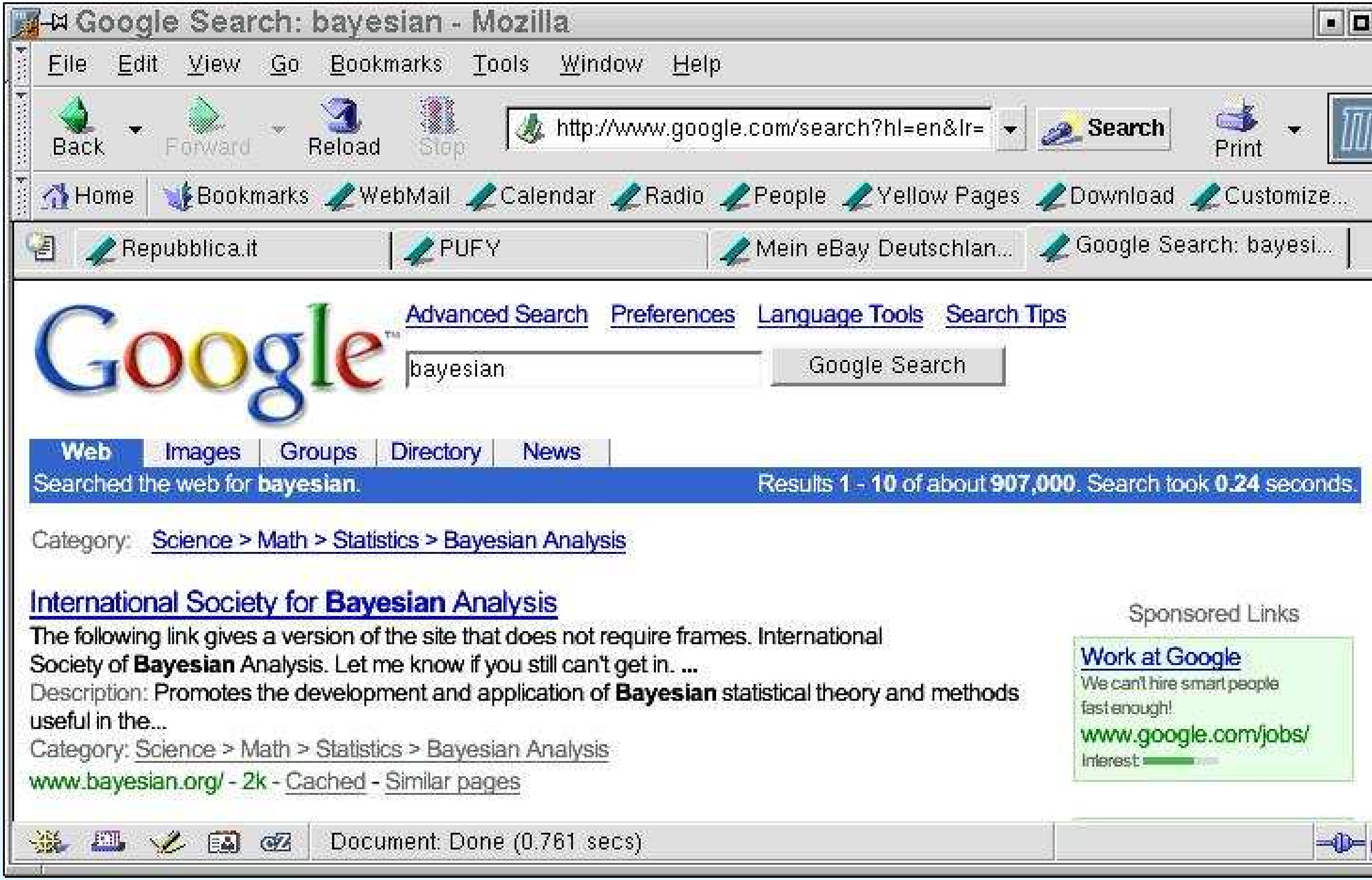
Confidence limit $\Rightarrow$ Sensitivity bound



Conclusions

- Probabilistic reasoning helps . . .
  . . . at least to avoid conceptual errors.
- Probabilistic statements can attributed, quantitaively and consistently, to all 'objects' respect to which we are in condition of uncertainty
- . . . allowing us to make meaninful statements concerning true values.
- In particular uncertainties due to systematic errors can be easily included
- Several 'standard' methods (like Least Square, etc.) can be easily recovered under well defined assumptions.
- But if this is not the case, nowdays there are no longer excuses to avoid the more general approach.
- Bayesian networks are a powerful conceptual and computational tool.

Are Bayesians ‘smart’ and ‘brilliant’?



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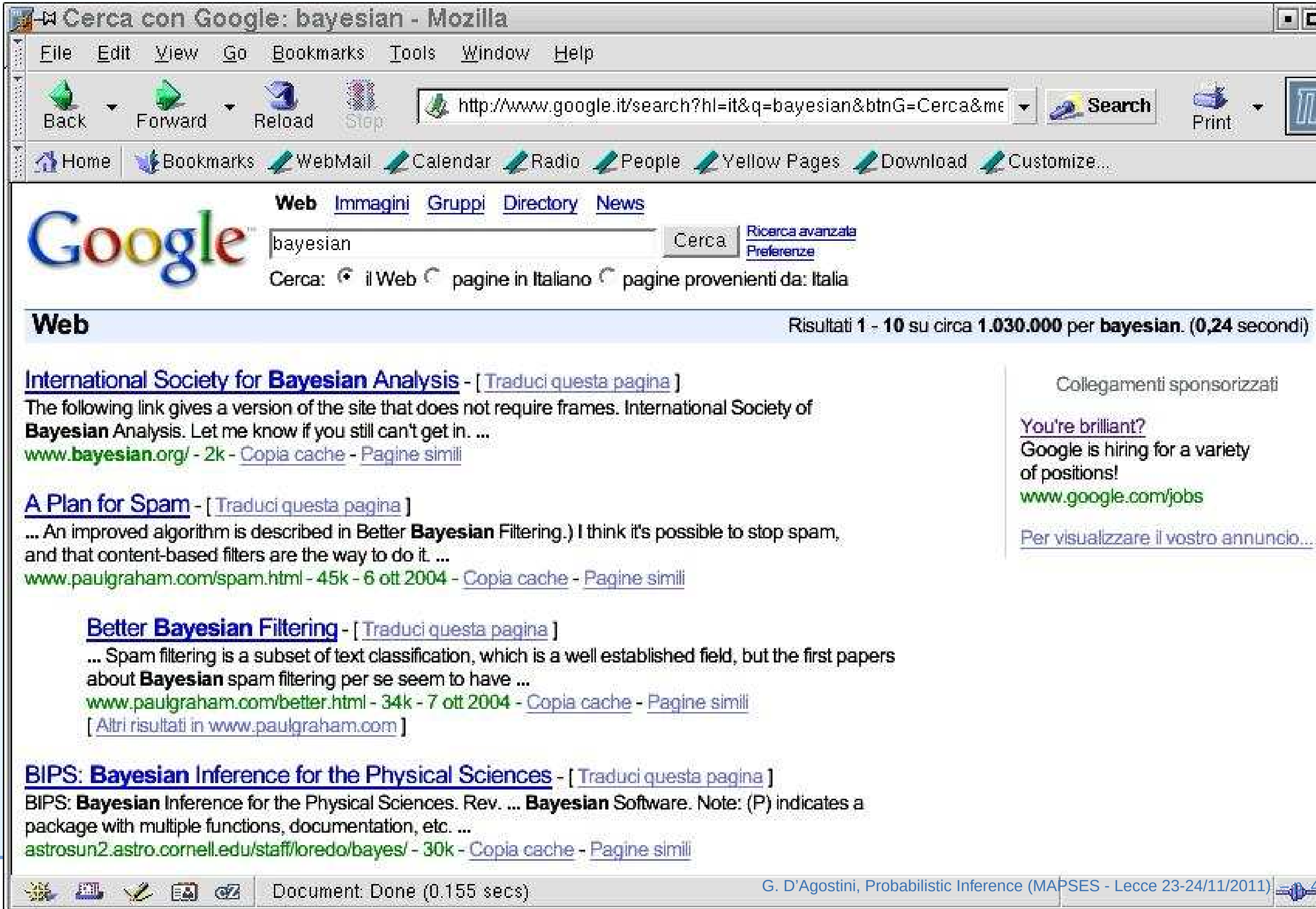

End

FINE