Table 1. Relative Efficiencies of δ_k^* to the Median

v	$n = 3$	$n = 5$	$n = 9$	$n = 19$
1	1.449	1.287	1.166	1.081
2	2.795	2.147	1.664	1.323
3	5.038	3.582	2.495	1.727
4	8.179	4.589	3.658	2.293
5	12.217	8.171	5.153	3.020

NOTE: The coefficient of variation, v, is for a normal population and the relative efficiency is calculated for the refined estimator (1) with respect to the median of a sample of size n.

are approximations, still provide an improvement in MSE over the median.

Example 3.3. Let $X_1, \ldots, X_n$ and an iid sample from a normal distribution having mean θ and variance $v^2\theta^2$. Khan (1968) and Gleser and Healy (1976) examined properties of the estimator $T = c_n S_n$, where

$$S_n^2 = \frac{1}{n}\sum(X_i - \overline{X})^2 \quad \text{and} \quad c_n = \frac{n^{1/2}\Gamma\left(\frac{n-1}{n}\right)}{(2v)^{1/2}\Gamma\left(\frac{n}{2}\right)}.$$

Khan (1968) has shown that $T = c_n S_n$ is an unbiased estimator of θ and $\sigma_T^2 = d_n\sigma_X^2/v^2$, where $d_n = (n-1/n)v^2c_n^2 - 1$. It follows from (2) that

$$\delta_k^* = \frac{c_n}{d_n + 1}S_n$$

minimizes MSE over the class C_T. The minimum MSE is given by

$$\text{MSE}(\delta_k^*) = \frac{d_n\sigma_x^2}{v^2(d_n + 1)}$$

and the relative efficiency with respect to the Khan (1968) estimator is simply RE $= d_n + 1$, which is clearly greater than 1.

[*Received November 1994. Revised April 1995.*]

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Graphical Demonstration of an Optimality Property of the Median

Yoong-Sin Lee

The property that the mean absolute deviation is a minimum when measured from a median is demonstrated with graphs. Deficiencies of the usual proof of the property are pointed out, and a new, general proof is suggested.

KEY WORD: Mean absolute deviation.

1. INTRODUCTION

It is well known that the mean absolute deviation attains a minimum when the deviation is measured from the median. Whereas the corresponding optimality property for the mean, that the mean squared deviation attains a minimum when measured from it, receives a good deal of attention in textbooks and is often proved in the class, this property of the median has been given less than a cursory treatment. Some textbooks relegate the proof to exercises for the students. The reason for this may well be in the proof itself. Students attempting to prove the theorem often find themselves getting into some unpleasant, although not necessarily difficult mathematics. We shall discuss the proof in Section 3. The purpose of this note is to present a graphical demonstration of this optimality property of the median. This approach is not only simple and convincing, but also covers any form of distribution and any configuration of the median. It can be shown to classes at a very elementary level.

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2. THE GRAPHICAL DEMONSTRATION

Consider the graph of a distribution function $F(x)$ and a point b on the x axis, as shown in Figure 1a. The infinitesimal quantity $(x - b)dF(x)$ can be represented as the horizontal strip in the picture. Summing up such infinitesimal areas, one can easily see that the integrals $\int_b^c |x - b|dF(x)$ and $\int_a^b |x - b|dF(x)$ are representable by the shaded area shown in Figure 1b.

Now, let m be the median of the distribution, as shown in Figure 2a. Consider the area formed by the mean absolute deviation about m:

$$I = E[|X - m|] = \int_{-\infty}^{\infty} |x - m|\, dF(x). \qquad (1)$$

Suppose the point m is changed to another point $m' < m$ and the mean absolute deviation now measured from m'. Consider the effect of this change on the mean absolute deviation. The area representing the value of the mean absolute deviation will be increased by an amount equal to the area of region A, and simultaneously decreased by

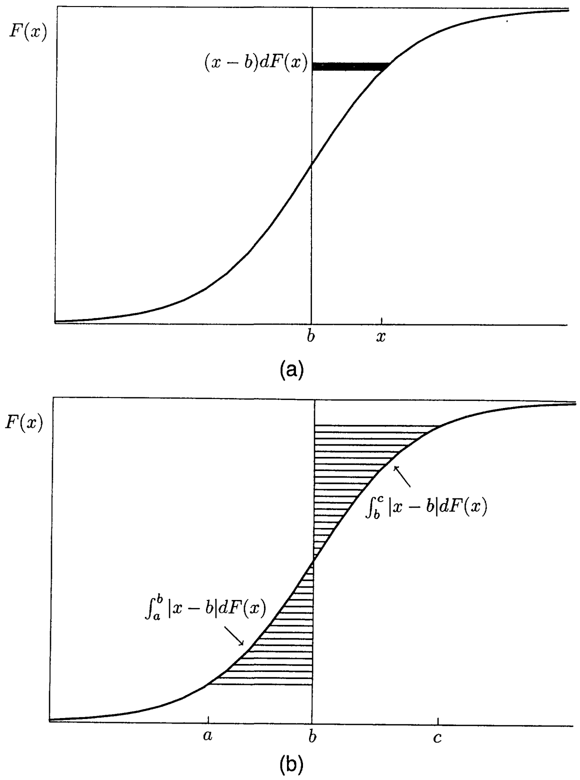

Figure 1. Representation of Integrals as Areas.

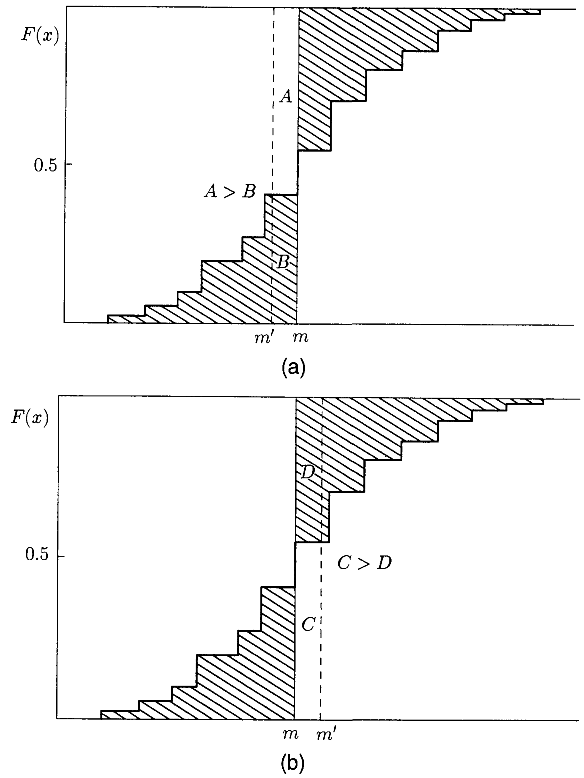

Figure 3. Change in Mean Absolute Deviation: Discrete Distributions.

an amount equal to the area of region B in Figure 2a. However, by definition of the median, region A has area greater than region B because the boundary between the two regions in this case is below the .5 horizontal line. A similar argument applies to the case of $m' > m$. In Figure 2b the resulting increase in the mean absolute deviation represented by the area of C would be greater than

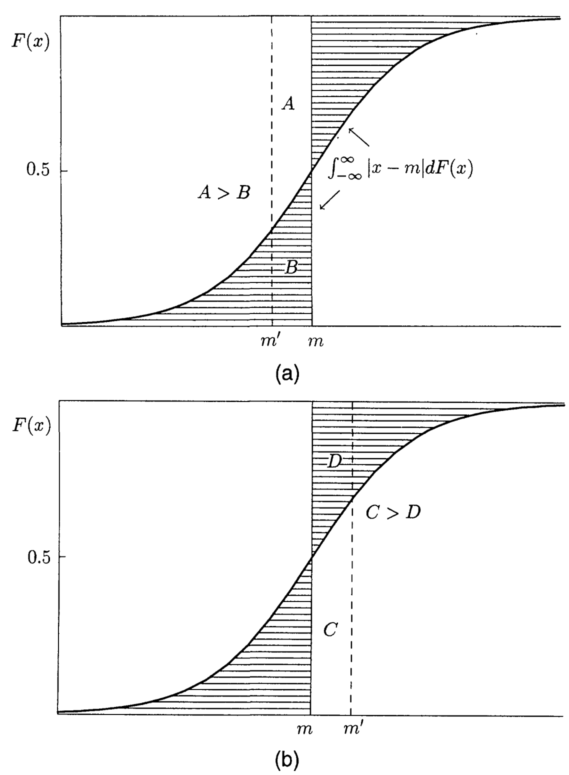

Figure 2. Change in Mean Absolute Deviation: Continuous Distributions.

the corresponding decrease represented by the area of D, again leading to an increase in the mean absolute deviation. This shows that the expected absolute deviation is a minimum when the deviation is measured from the median.

The graph demonstrates the result for discrete distributions just as well. In Figure 3 we can apply exactly the same argument as we do for the continuous case, showing that the mean absolute deviation is a minimum when measured from the median. In Figure 4a the median is indeterminate within the interval (s, t). The graph shows that the expected absolute deviation measured from any point within this interval is a minimum because the area representing the integral I in (1) remains the same when measured from any point within this interval, as is clear from the resulting change in the mean absolute deviation shown in Figure 4a.

In Figure 4b the result for a mixed discrete and continuous distribution is demonstrated. The median happens to be at a point between an interval where the distribution is discrete, and an interval where the distribution is continuous. The same argument as before shows that changing the point from which the mean absolute deviation is measured from the median to anywhere else would lead to an increase in value for the mean absolute deviation.

Finally, the graphs can also be used to demonstrate the optimality property of the sample median. The form of an empirical distribution function for a given sample size would be a special case of the distribution function for a discrete random variable such as shown in Figure 3a and b, and the optimality result follows. The graph would also show the indeterminancy of the median value for minimizing the mean absolute deviation when the sample size is even.

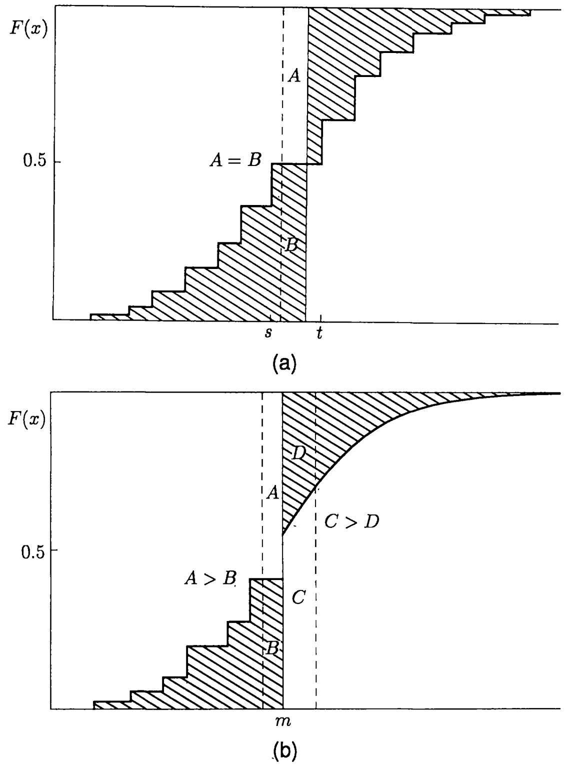

Figure 4. Change in Mean Absolute Deviation: Special Cases.

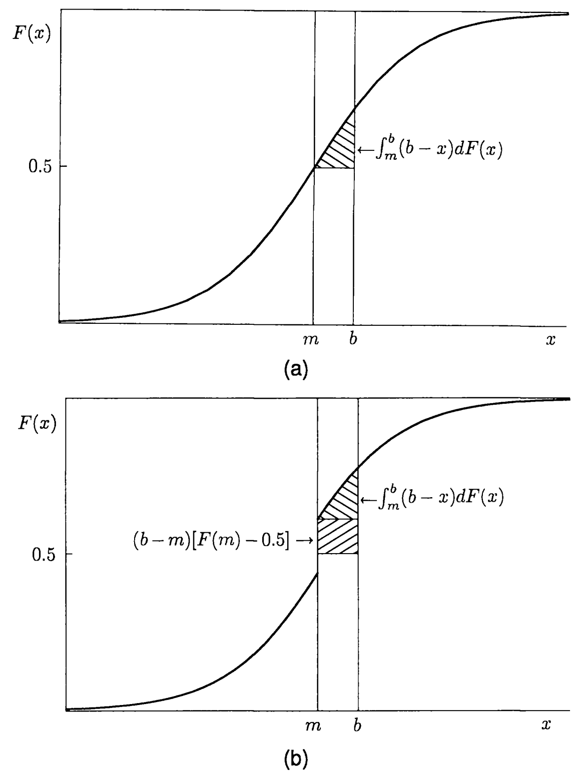

Figure 5. Area Representing the Integral.

3. PROOF OF OPTIMALITY

The usual proof of the optimality property is by showing that

$$E[|X-b|] = E[|X-m|] + 2\int_m^b (b-x)\, dF(x).$$

Most students would find obtaining the equality unpleasant and not particularly insightful. In fact, this equality is strictly true only when m is a continuity point of $F(\cdot)$. If m is a jump point, then the equality would be more precisely expressed as

$$E[|X-b|] = E[|X-m|] + 2\left\{ \int_m^b (b-x)\, dF(x) + (b-m)[F(m) - .5] \right\}.$$

Figure 5a illustrates the area representing $\int_m^b (b-x)\, dF(x)$ for the continuous case, and Figure 5b that of $\int_m^b (b-x)\, dF(x) + (b-m)[F(m) - .5]$ for the discontinuous case.

There is therefore a need for a more general and, perhaps, simpler proof. I propose the following approach. First, we note the identities

$$\int_{-\infty}^b (b-x)\, dF(x) = \int_{-\infty}^b F(x)\, dx$$

and

$$\int_b^\infty (x-b)\, dF(x) = \int_b^\infty [1 - F(x)]\, dx,$$

which are of interest in their own right. These can most conveniently be proved by expressing the integral on the right as a double integral and interchanging the order of integration. Thus, for example,

$$\begin{aligned}\int_{-\infty}^b (b-x)\, dF(x) &= \int_{-\infty}^b \int_x^b dt\, dF(x) \\ &= \int_{-\infty}^b \int_{-\infty}^t dF(x)\, dt = \int_{-\infty}^b F(x)\, dx.\end{aligned}$$

Figure 6 shows that the right-hand side integral is built up from the horizontal, and the left-hand side the vertical, strips, both being represented by the area of the same region. Then

$$\begin{aligned}E[|X-b|] &= \int_{-\infty}^b F(x)\, dx + \int_b^\infty [1 - F(x)]\, dx \\ &= \int_{-\infty}^m F(x)\, dx + \int_m^b F(x)\, dx \\ &\quad + \int_m^\infty [1 - F(x)]\, dx - \int_m^b [1 - F(x)]\, dx \\ &= E[|X-m|] + 2\int_m^b [F(x) - .5]\, dx. \qquad (2)\end{aligned}$$

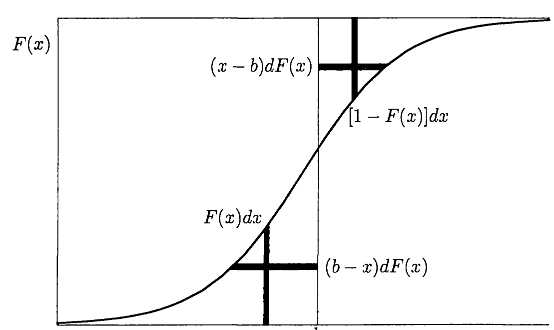

Figure 6. Two Ways of Evaluating the Same Area.

This equality is true whether m is a continuity or a jump point. The corresponding equality for the case of $b \leq m$ can be similarly established, and the proof of the optimality property follows directly. Alternatively, we can obtain from the right-hand side of (2) the left derivative with respect to b of $E[|X - b|]$ as $2[F(b-) - .5]$, which is nonpositive for $b \leq m$, and the right derivative as $2[F(b) - .5]$, which is nonnegative for $b \geq m$. The result follows remembering that the mean absolute deviation is continuous with respect to b.

[*Received August 1993. Revised May 1995.*]

Accent on Teaching Materials

Harry O. POSTEN, Section Editor

In this section *The American Statistician* publishes announcements and selected reviews of teaching materials of general use to the statistical field. These may include (but will not necessarily be restricted to) curriculum material, collections of teaching examples or case studies, modular instructional material, transparency sets, films, filmstrips, videotapes, probability devices, audiotapes, slides, and data deck sets (with complete documentation).

Authors, producers, or distributors wishing to have such materials announced or reviewed should submit a single, complete copy of the product (three copies of printed material double-spaced) to Section Editor Harry O. Posten, Statistics Department, University of Connecticut, Storrs, CT 06268. A statement of intention that the material will be available to all requesters for a minimum of a two-year period should be provided, along with information on the cost (including postage) and special features of the material. Information on classroom experience may also be included. All materials submitted must be of general use for teaching purposes in the area of probability and statistics.

The *Journal of Statistics Education* Information Service and Other Internet Resources for Statistics Teachers

The Internet has added a new dimension to the teaching of statistics. One of the most important resources for teachers is the *Journal of Statistics Education* Information Service, which includes the electronic *Journal of Statistics Education*. This article describes these and other Internet resources available to students and teachers of statistics.

KEY WORDS: Hypertext; Web sites; World Wide Web.

1. INTRODUCTION

The Internet provides a new dimension to learning, teaching, and doing statistics. Electronic mail has made it possible to communicate and transmit text materials worldwide with the greatest of ease. Now the World Wide Web (WWW) and Web browsers such as the popular Mosaic or Netscape allow us to obtain more general kinds of information from computers around the world by a simple click of the mouse. Documents are prepared using hypertext markup language (html) that can be read on your computer be it a Mac, PC, or Unix workstation. The hypertext format allows you to click on an item to bring more detailed information, show a picture, get a related document from another server on the Internet, listen to music, start a video clip, and even run a computer at another site and have it give you the result. The coming MBone software will extend this to provide electronic classrooms and conferences.

The expression "surfing the Internet" has been introduced to describe the process of moving through the Internet looking for interesting information. With this article we hope to encourage you to jump in and start surfing for materials to liven up your statistic classes. We will help you get started by describing a few of the resources available from the Internet for statistics teachers. The resources are available at Web sites and, at the end of the article, we will indicate how you can easily go to any of these sites using a Web browser. When you go to a particular Web site, you will find a "homepage" that will tell you what is available at this site.

2. THE *JOURNAL OF STATISTICS EDUCATION* INFORMATION SERVICE

The first Web site that we visit is the *Journal of Statistics Education* Information Service. This Information Service is maintained at North Carolina State University by Tim Arnold.

The centerpiece of this Information Service is the *Journal of Statistics Education* (*JSE*). The *JSE* is a refereed electronic journal that deals with postsecondary statistics education. The managing editor is Tim Arnold and the editor is E. Jacquelin Dietz. The first issue was published on July 1, 1993.

The inaugural issue emphasized current efforts to reform the teaching of statistics, particularly in the beginning course. This reform stresses more data and less theory, as well as active learning. An article by George Cobb reported on a meeting of principal investigators of NSF projects participating in these reform efforts. Cobb identified and discussed several common themes in these projects including: the use of real-life data sets, developing new software and laboratories for exploratory data

Preparation of this article was supported by the NSF under Grant DUE-9354592.