Introduction to Probabilistic Reasoning

– inference, forecasting, decision –

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– Part 2 –

"Probability is good sense reduced to a calculus" (Laplace)

Probability

What is probability?

Standard textbook definitions

$$p = \frac{\#\,\text{favorable cases}}{\#\,\text{possible equiprobable cases}}$$

$$p = \frac{\#\,\text{times the event has occurred}}{\#\,\text{independent trials under same conditions}}$$

Standard textbook definitions

It is easy to check that 'scientific' definitions suffer of circularity

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Standard textbook definitions

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$$p = \frac{\#\text{ favorable cases}}{\#\text{ possible equally possible cases}}$$

$$p = \frac{\#\text{ times the event has occurred}}{\#\text{ independent trials under same conditions}}$$

aplace: *"lorsque rien ne porte à croire que l'un de ces cas doit arriver plutot que les autres"*

Pretending that replacing 'equi-probable' by 'equi-possible' is just cheating students (as I did in my first lecture on the subject. . .).

Standard textbook definitions

It is easy to check that 'scientific' definitions suffer of circularity, plus other problems

$$p = \frac{\#\ \text{favorable cases}}{\#\ \text{possible equiprobable cases}}$$

$$p = \lim_{n\to\infty} \frac{\#\ \text{times the event has occurred}}{\#\ \text{independent trials under same condition}}$$

Future $\Leftrightarrow$ Past (believed so)

$n \to \infty$: $\rightarrow$ *"usque tandem?"*

$\rightarrow$ *"in the long run we are all dead"*

$\rightarrow$ It limits the range of applications

Definitions → evaluation rules

Very useful evaluation rules

$$A)\quad p = \frac{\#\,\text{favorable cases}}{\#\,\text{possible equiprobable cases}}$$

$$B)\quad p = \frac{\#\,\text{times the event has occurred}}{\#\text{independent trials under same condition}}$$

If the implicit beliefs are well suited for each case of application.

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BUT they cannot define the concept of probability!

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In the probabilistic approach we are going to see

- Rule A will be recovered immediately (under the assumption of equiprobability, when it applies).
- Rule B will result from a theorem (under well defined assumptions).

Probability

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$\rightarrow$ how much we are confident that something is true

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What <u>is</u> probability?

It is what everybody knows what it is before going at school

→ how much we are confident that something is true

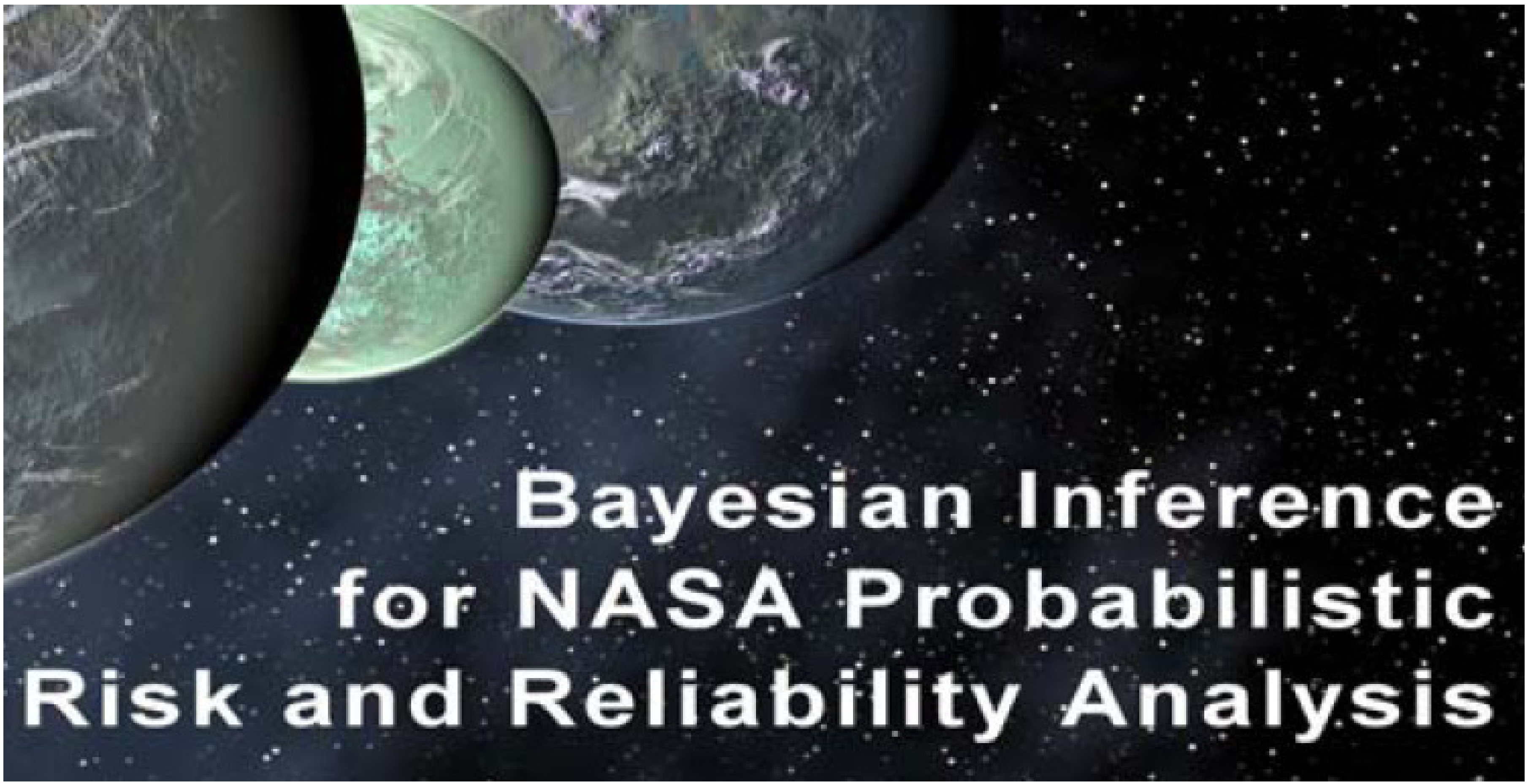
→ how much we believe something

→ "A measure of the degree of belief that an event *will* occur"

[Remark: 'will' does not imply future, but only uncertainty.]

An helpful diagram

The previous diagram seems to help the understanding of the concept of probability



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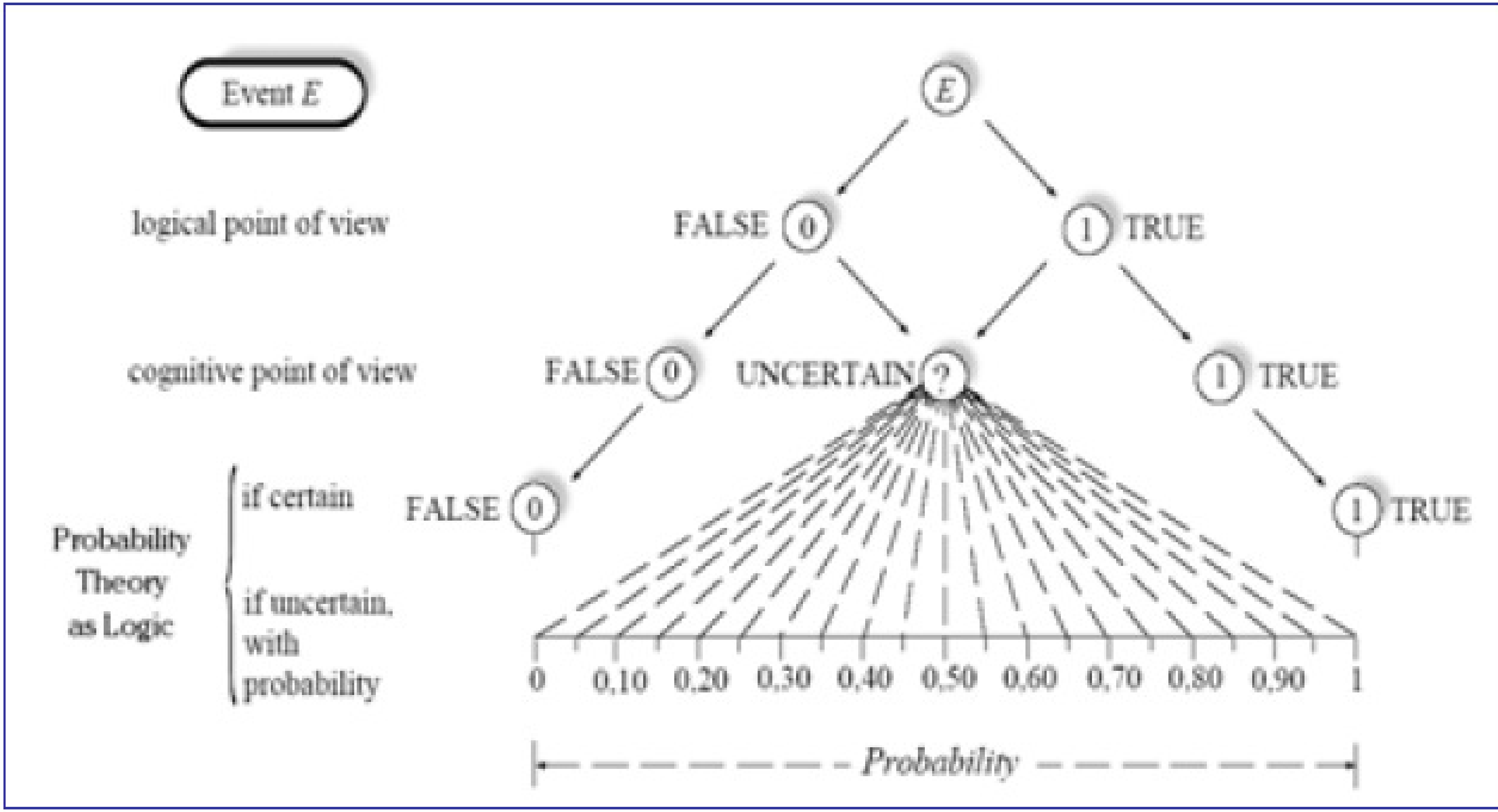

• Figure 2-1. Graphical abstraction of probability as a measure of information (adapted from "Probability and Measurement Uncertainty in Physics" by D'Agostini, [1995]).

(. . . but NASA guys are afraid of 'subjective', or 'psychological')

Beliefs and ‘coherent’ bets

Remarks:

- Subjective <u>does not mean</u> arbitrary!

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  "The usual touchstone, whether that which someone asserts is merely his persuasion – or at least his subjective conviction, that is, his firm belief – is betting. It often happens that someone propounds his views with such positive and uncompromising assurance that he seems to have entirely set aside all thought of possible error. A bet disconcerts him. Sometimes it turns out that he has a conviction which can be estimated at a value of one ducat, but not of ten. For he is very willing to venture one ducat, but when it is a question of ten he becomes aware, as he had not previously been, that it may very well be that he is in error." (Kant)

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	1	X	2				
11/07 20:30							
VOJVODINA - HIBERNIANS	1,05	10,00	25,00	3,10	1,30	2,55	1,42
GLENTORAN - KR REYKJAV	4,75	3,50	1,65	1,90	1,75	1,75	1,90
HONV BUDAP. - CELIK NIKS.	1,15	7,00	12,00	2,80	1,35	2,00	1,70
GERMANIA - OLANDA	1,15	6,50	13,00	2,50	1,45	2,20	1,57
11/07 20:45							
S PATRICKS - ZALGIRIS	1,90	3,40	3,50	1,75	1,90	1,73	1,95
11/07 21:00							
LIBERTAS - SARAJEVO	22,00	8,00	1,08	3,20	1,28	2,25	1,55
11/07 22:00							
STJARNAN - HAFNARFJOR	2,65	3,40	2,35	2,15	1,60	1,50	2,35

Beliefs and 'coherent' bets

Remarks:

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  - Coherent bet:
  - you state the odds according on your beliefs;
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"His [Bouvard] calculations give him the mass of Saturn as 3,512th part of that of the sun. Applying my probabilistic formulae to these observations, I find that the odds are 11,000 to 1 that the error in this result is not a hundredth of its value." (Laplace)

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$\rightarrow P(3477 \leq M_{Sun}/M_{Sat} \leq 3547 \,|\, I(\text{Laplace})) = 99.99\%$

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For more on the subject:

http://arxiv.org/abs/1112.3620

http://www.roma1.infn.it/~dagos/badmath/#added

Mathematics of beliefs

The good news:

The basic laws of degrees of belief are the same we get from the inventory of favorable and possible cases, or from events occurred in the past.

[Details skipped. . .]

Basic rules of probability

1. $0 \le P(A \,|\, I) \le 1$
2. $P(\Omega \,|\, I) = 1$
3. $P(A \cup B \,|\, I) = P(A \,|\, I) + P(B \,|\, I) \quad$ [if $P(A \cap B \,|\, I) = \emptyset$]

4. $P(A \cap B \,|\, I) = P(A \,|\, B, I) \cdot P(B \,|\, I) = P(B \,|\, A, I) \cdot P(A \,|\, I)$

Remember that probability is always conditional probability!

I is the background condition (related to information 'I'_s')

$\rightarrow$ usually implicit (we only care on 're-conditioning')

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Note: 4. does not define conditional probability.
(Probability is always conditional probability!)

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An even better news:

The fourth basic rule
can be fully exploided!

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(Liberated by a curious ideology that forbits its use)

A simple, powerful formula

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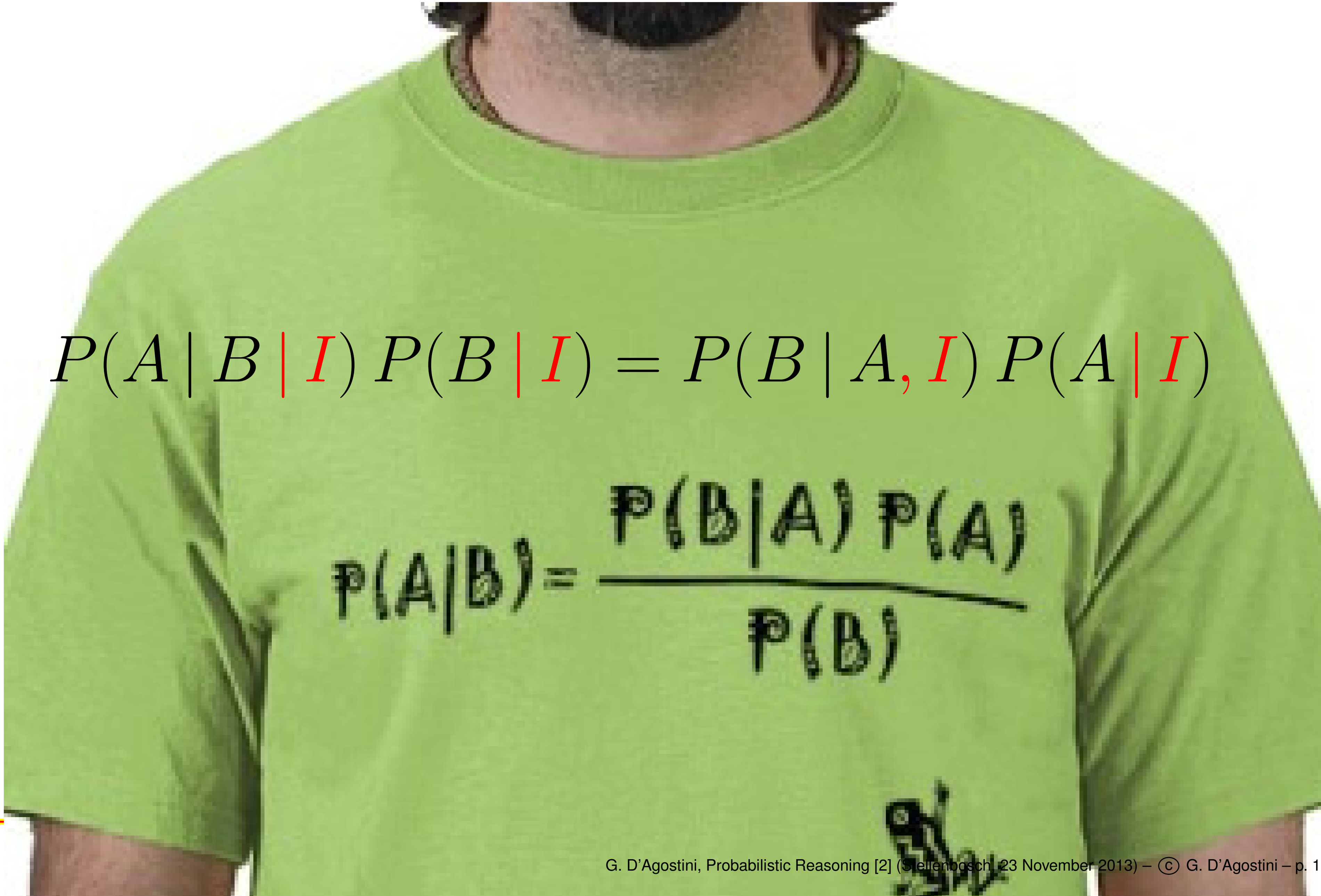
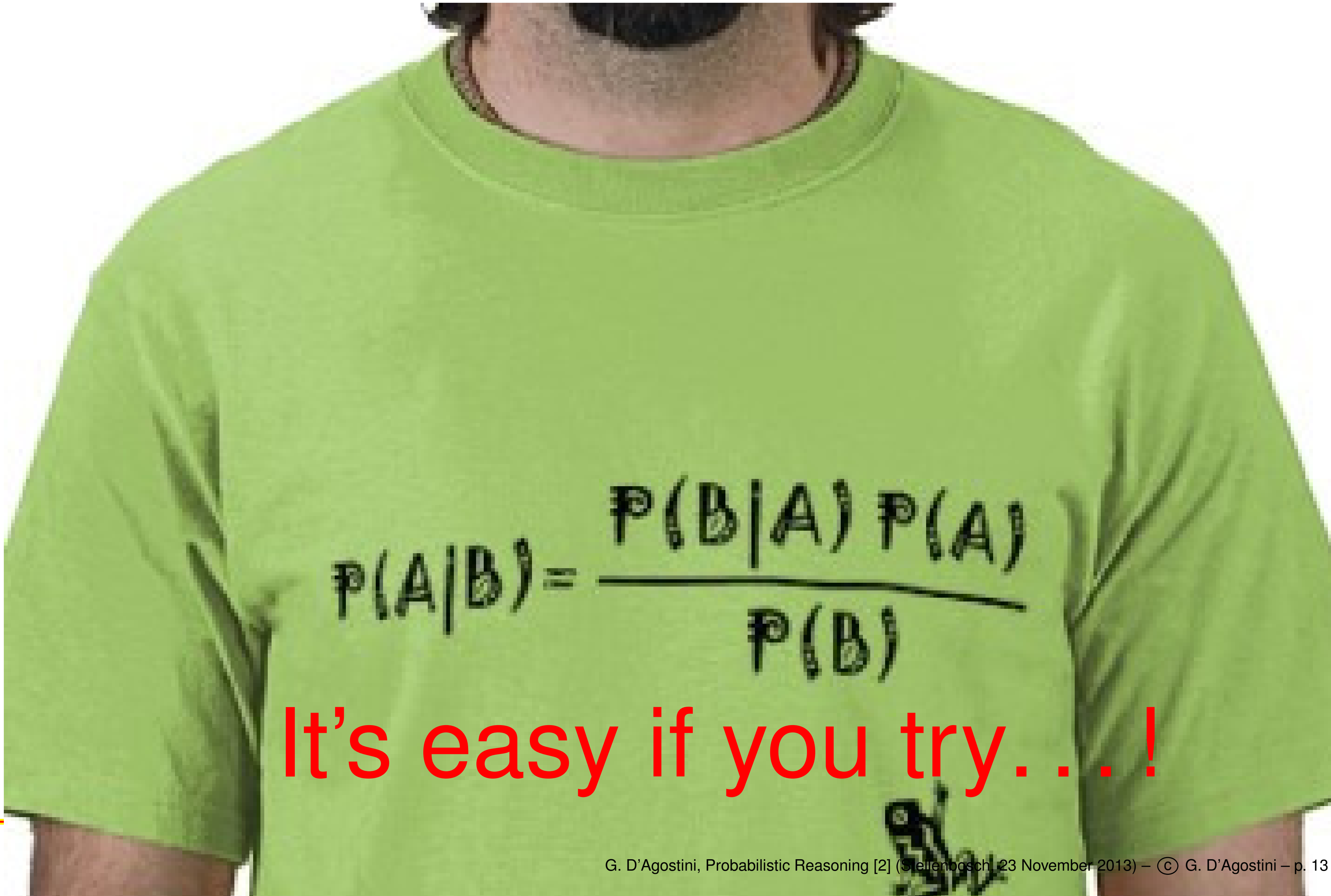


$$P(A\,|\,B\,|\,I)\,P(B\,|\,I) = P(B\,|\,A, I)\,P(A\,|\,I)$$

A simple, powerful formula

Take the courage to use it!

A simple, powerful formula



It's easy if you try. . . !

Telling it with Gauss' words

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"post illa observationes" *"ante illa observationes"*

(Gauss)

Bayes formulae

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$$P(C_i\,|\,E,I) \propto P(E\,|\,C_i,I)\cdot P(C_i\,|\,I)$$

or even (my preferred form to grasp its meaning):

$$\frac{P(C_i\,|\,E\,|\,I)}{P(C_j\,|\,E\,|\,I)} = \frac{P(E\,|\,C_i\,|\,I)}{P(E\,|\,C_j\,|\,I)}\cdot\frac{P(C_i\,|\,I)}{P(C_j\,|\,I)}$$

Bayesian parametric inference

If we want to infer a continuous parameter p from a set of data

$\rightarrow$ straghtforwad extension to probability density functions (pdf)

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If we want to infer a continuous parameter *p* from a set of data

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$$f(p \,|\, \text{data}, I) \;\propto\; f(\text{data} \,|\, p, I) \cdot f(p \,|\, I)$$

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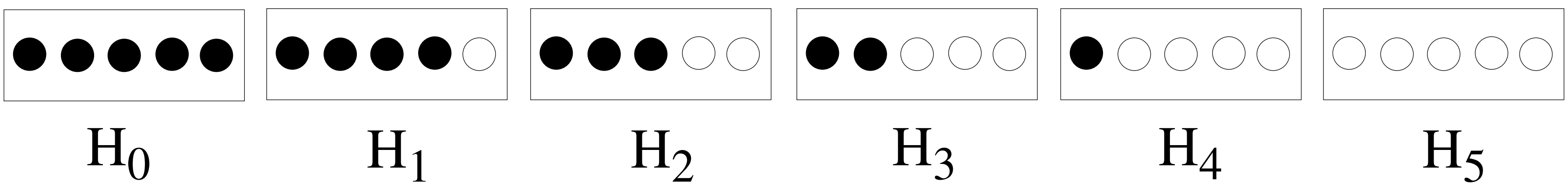
$$f(p\,|\,\text{data}, I) = \frac{f(\text{data}\,|\,p, I)\cdot f(p\,|\,I)}{\int_p f(\text{data}\,|\,p, I)\cdot f(p\,|\,I)\,dp}$$

Application to the six box problem



Remind:

- $E_1 =$ White
- $E_2 =$ Black

Collecting the pieces of information we need

Our tool:

$$P(H_j \,|\, E_i, I) = \frac{P(E_i \,|\, H_j, I)}{P(E_i \,|\, I)} \, P(H_j \,|\, I)$$

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- $P(E_i \,|\, I) = 1/2$
- $P(E_i \,|\, H_j, I)$:

$$\begin{aligned} P(E_1 \,|\, H_j, I) &= j/5 \\ P(E_2 \,|\, H_j, I) &= (5-j)/5 \end{aligned}$$

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Our prior belief about H_j

Collecting the pieces of information we need

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Probability of E_i under a well defined hypothesis H_j
It corresponds to the 'response of the apparatus in measurements.
$\rightarrow$ likelihood (traditional, rather confusing name!)

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Probability of E_i taking account all possible H_j
→ How much we are confident that E_i will occur.

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Probability of E_i taking account all possible H_j

$\rightarrow$ How much we are confident that E_i will occur.

We can rewrite it as

$P(E_i\,|\,I) = \sum_j P(E_i\,|\,H_j,\,I)\cdot P(H_j\,|\,I)$

We are ready

Now that we have set up our formalism, let's play a little

- analyse real data
- some simulations

Then

- $H_j \longleftrightarrow j \longleftrightarrow p_j$
- extending p to a continuum:

  $\Rightarrow$ Bayes' billiard

  (prototype for all questions related to efficiencies, branching ratios)
- On the meaning of p

Which box? Which ball?

Inferential/forecasting history:

1. $k = 0$
   $P_0(H_j) = P(H_j \,|\, I_0)$ (priors)
2. `begin loop`:
   $k = k + 1$
   $\Rightarrow E^{(k)}$ (k-th extraction)
3. $P_k(H_j \,|\, I_k) \propto P(E^{(k)} \,|\, H_j) \times P_{k-1}(H_j \,|\, I_k)$

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4. $\rightarrow$ `go to 2`

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Let's play!

Bayes' billiard

This is the original problem in the theory of chances solved by Thomas Bayes in late '700:

- imagine you roll a ball at random on a billiard;
- you mark the relative position of the ball along the billiard's length (l/L) and remove the ball
- then you roll at random other balls
  - write down if it stopped left or right of the first ball;
  - remove it and go on with n balls.
- Somebody has to guess the position of the first ball knowing only how mane balls stopped left and how many stoppe right

Bayes' billiard and Bernoulli trials

It is easy to recongnize the analogy:

- Left/Right $\rightarrow$ Success/Failure
- if Left $\leftrightarrow$ Success:
  - $l/L \leftrightarrow p$ of binomial (Bernoulli trials)

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Solution with modern notation:
Imagine a sequence $\{S, S, F, S, \ldots\}$ [f_0 is uniform]:

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\ldots & \quad \ldots \\
f(p\,|\,\#S,\#F) &\propto p^{\#S}(1-p)^{\#F} = p^{\#S}(1-p)^{(1-\#s)}
\end{aligned}$$

Bayes' billiard and Bernoulli trials

It is easy to recongnize the analogy:

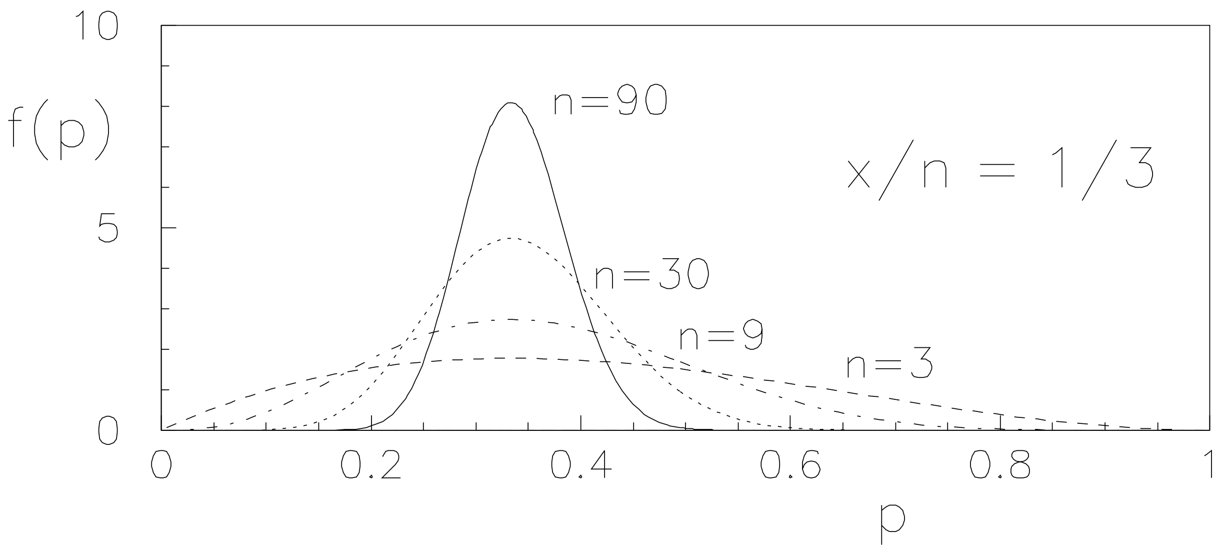
- Left/Right $\rightarrow$ Success/Failure
- if Left $\leftrightarrow$ Success:
  - $l/L \leftrightarrow p$ of binomial (Bernoulli trials)

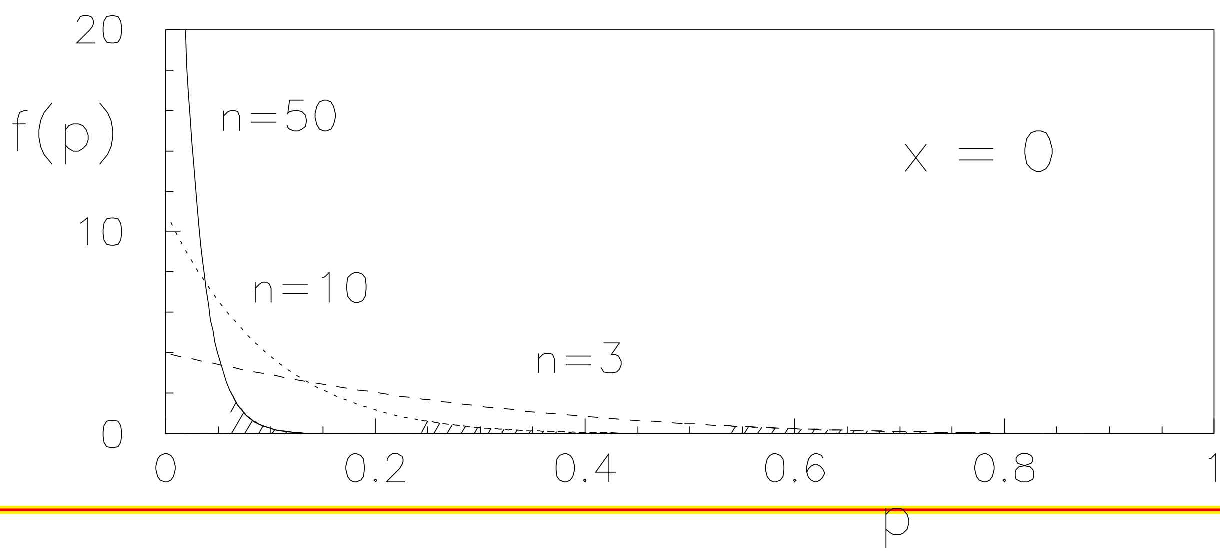
Solution with modern notation:
Imagine a sequence $\{S, S, F, S, \ldots\}$ [f_0 is uniform]:

$$
\begin{aligned}
f(p\,|\,S) &\propto f(S\,|\,p) = p \\
f(p\,|\,S,S) &\propto f(S\,|\,p)\cdot f(p\,|\,S) = p^2 \\
f(p\,|\,S,S,F) &\propto f(F\,|\,p)\cdot f(p\,|\,S,S) = p^2(1-p) \\
\ldots & \quad\quad \ldots \\
f(p\,|\,\#S,\#F) &\propto p^{\#S}(1-p)^{\#F} = p^{\#S}(1-p)^{(1-\#s)} \\
f(p\,|\,x,n) &\propto p^{x}(1-p)^{(n-x)} \qquad\qquad [x = \#S]
\end{aligned}
$$

Inferring the Binomial p

$$f(p\,|\,x,n,\mathcal{B}) = \frac{(n+1)!}{x!\,(n-x)!}\,p^x\,(1-p)^{n-x}\,,$$





Inferring the Binomial p

$$f(p \,|\, x, n, \mathcal{B}) = \frac{(n+1)!}{x!\,(n-x)!}\, p^x\,(1-p)^{n-x}\,,$$

$$\mathsf{E}(p) = \frac{x+1}{n+2}$$

Laplace's rule of successions

$$\begin{aligned} \mathsf{Var}(p) &= \frac{(x+1)(n-x+1)}{(n+3)(n+2)^2} \\ &= \mathsf{E}(p)\;(1-\mathsf{E}(p))\;\frac{1}{n+3}. \end{aligned}$$

Interpretation of E(p)

Think at any future event $E_{i>n}$
$\Rightarrow$ if we were sure of p, then our confidence on $E_{i>n}$ will be exactly p, i.e.

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How much should we believe $E_{i>n}$?.

$$\begin{aligned} P(E_{i>n}\,|\,x,n,\mathcal{B}) &= \int_0^1 P(E_i\,|\,p)\,f(p\,|\,x,n,\mathcal{B})\,\mathrm{d}p \\ &= \int_0^1 p\,f(p\,|\,x,n,\mathcal{B})\,\mathrm{d}p \\ &= \mathrm{E}(p) \\ &= \frac{x+1}{n+2} \quad \text{(for uniform prior)}\,. \end{aligned}$$

From frequencies to probabilities

$$\text{E}(p) = \frac{x+1}{n+2}$$

Laplace's rule of successions

$$\text{Var}(p) = \text{E}(p)\,(1-\text{E}(p))\,\frac{1}{n+3}.$$

For 'large' n, x and $n-x$: asymptotic behaviors of $f(p)$:

$$\text{E}(p) \approx p_m = \frac{x}{n} \quad \text{[with } p_m \text{ mode of } f(p)\text{]}$$

$$\sigma_p \approx \sqrt{\frac{p_m\,(1-p_m)}{n}} \xrightarrow[n\to\infty]{} 0$$

$$p \sim \mathcal{N}(p_m, \sigma_p)\,.$$

Under these conditions the frequentistic "definition" (evaluation rule!) of probability (x/n) is recovered.

Special case with $x = 0$

$$
\begin{aligned}
f(p \,|\, 0, n, \mathcal{B}) &= (n+1)\,(1-p)^n \\
F(p \,|\, 0, n, \mathcal{B}) &= 1 - (1-p)^{n+1} \\
p_m &= 0 \\
\mathsf{E}(p) &= \frac{1}{n+2} \longrightarrow \frac{1}{n} \\
\sigma(p) &= \sqrt{\frac{(n+1)}{(n+3)(n+2)^2}} \longrightarrow \frac{1}{n}
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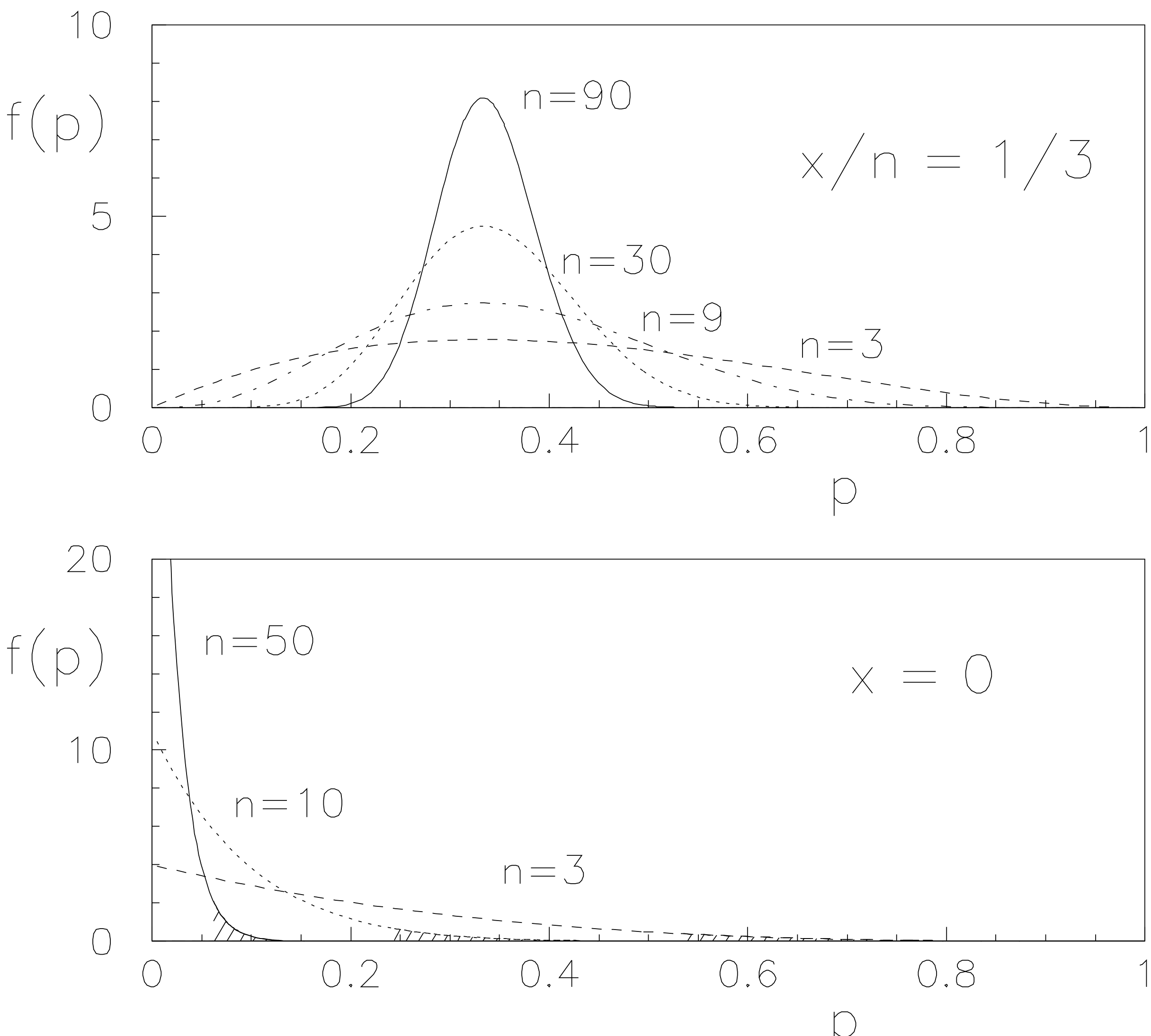
$$\begin{aligned} P(p \le p_u\,|\,0,n,\mathcal{B}) &= 95\% \\ &\Rightarrow \quad p_u = 1 - \sqrt[n+1]{0.05}\ : \end{aligned}$$

Probabilistic upper bound

Special case with $x = 0$



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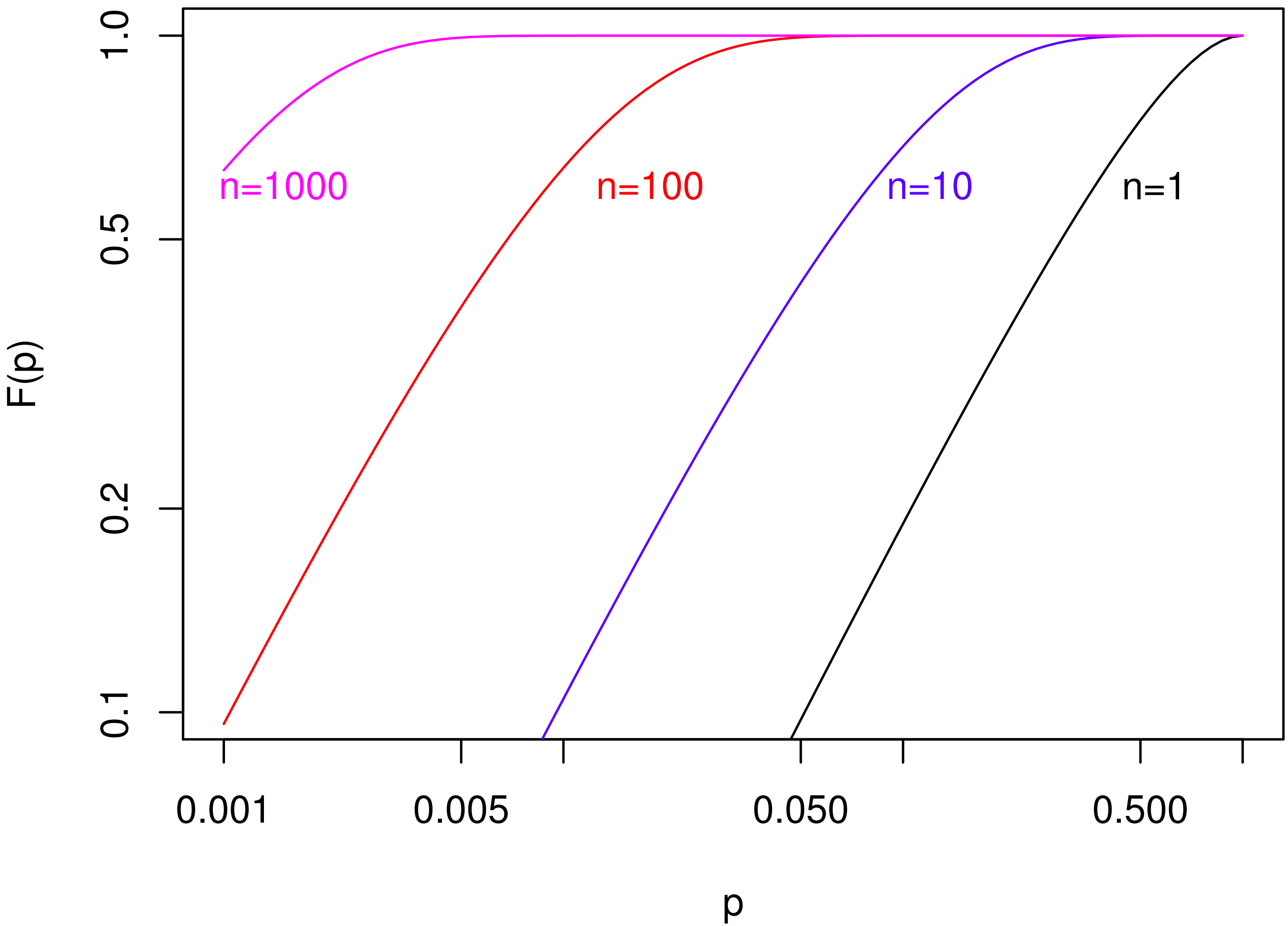

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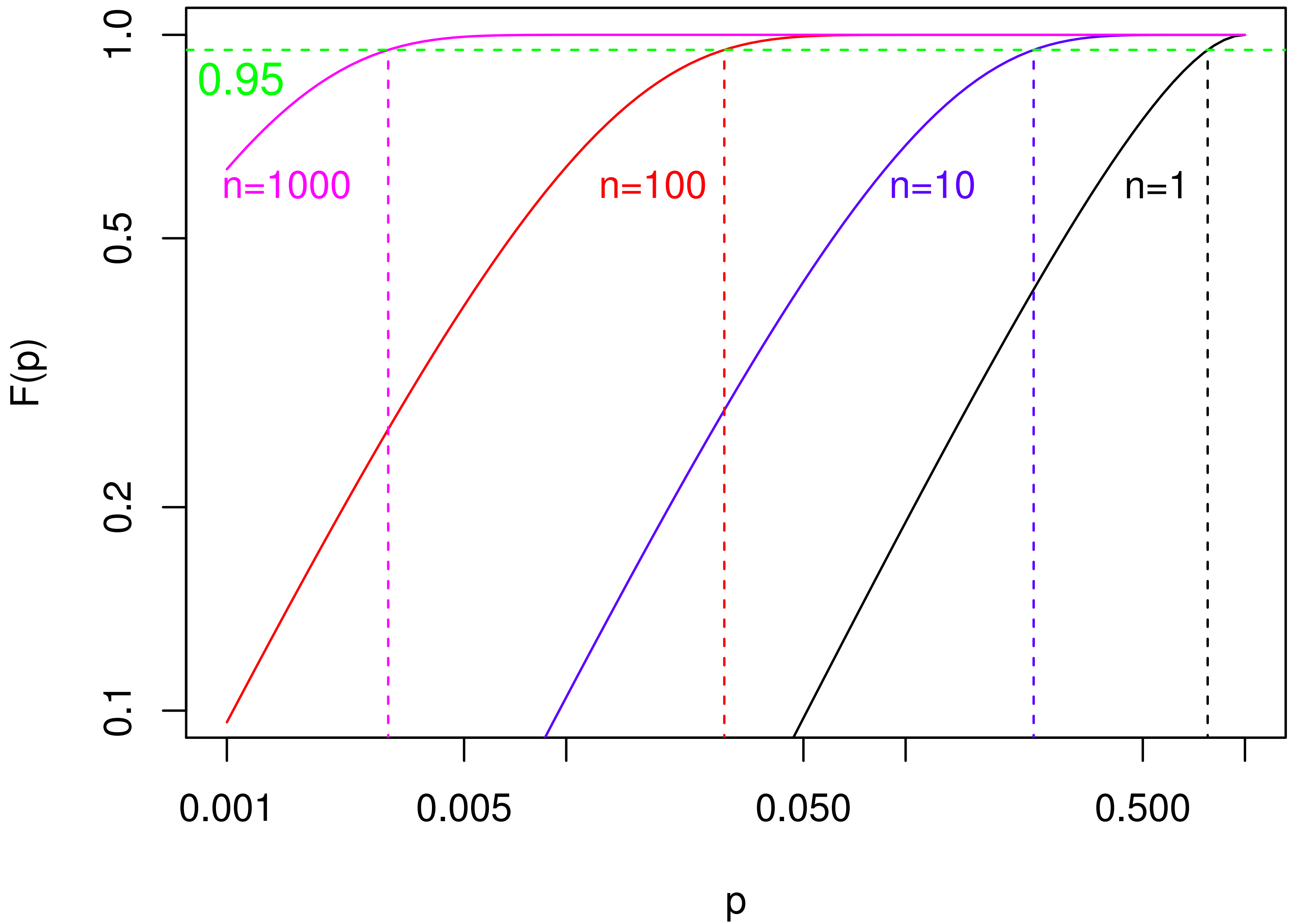

Special case with $x = 0$

For the case $x = n$

(like 'observing' a 100% efficiency):

$\rightarrow$ just reason on the complementary parameter

$$q = 1 - p$$

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We have no longer excuses!!

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  (Diffidate chi vi promette di far germogliar zecchini nel Campo dei Miracoli! – Pinocchio docet)

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  ⇒ Maximum Likelihood ...often a good approximated solution <u>under</u> some assumptions usually unkown to practitioners.

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- Mistrust all results that sound as ‘confidence’, ’probability’ etc about physics quantities, if they are obtained by methods that do not contemplate ’beliefs’.