On the 2D NS-equation

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I. "EQUATIONS".

The velocity $\mathbf{u}(\mathbf{x})$ of an incompressible viscous fluid in a 2-dimensional (2D) periodic container $[-L,L]^2$ can be written via its Fourier's components $u_\mathbf{k}=\overline{u}_{-\mathbf{k}}$, $\mathbf{k}\in Z^2$:

$$\mathbf{u}(\mathbf{x})=\sum_{\mathbf{0}\neq\mathbf{k}\in Z^2} i u_\mathbf{k}\frac{\mathbf{k}^\perp}{|\mathbf{k}|}e^{-i\mathbf{k}\cdot\mathbf{x}} \tag{1.1}$$

and the Navier-Stokes (NS) equation for its evolution with kinematic viscosity ν is:

$$\begin{aligned}\dot u_\mathbf{k}=&\sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}} T_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}}u_{\mathbf{k}_1}u_{\mathbf{k}_2}-\nu\mathbf{k}^2u_\mathbf{k}+f_\mathbf{k}\\ T_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}}=&-\frac{(\mathbf{k}_1^\perp\cdot\mathbf{k}_2)(\mathbf{k}_2\cdot\mathbf{k})}{\|\mathbf{k}_1\|\,\|\mathbf{k}_2\|\,\|\mathbf{k}\|},\quad \mathbf{k}=\mathbf{k}_1+\mathbf{k}_2\end{aligned} \tag{1.2}$$

with $\mathbf{k}=(k_1,k_2),\mathbf{k}_1=(k_{1,1},k_{1,2}),\mathbf{k}_2=(k_{2,1},k_{2,2})\in Z^2$, $\mathbf{k}^2=k_1^2+k_2^2$, $\|\mathbf{k}\|=(\mathbf{k}^2)^{\frac12}$, and the forcing $f_\mathbf{k}$ is supposed fixed once and for all and to act only on 'large scale':

$$f_\mathbf{k}=0\ \text{ unless }\ |\mathbf{k}|\stackrel{def}{=}\max_{i=1,\ldots,d}|k_i|\le k_f<\infty \tag{1.3}$$

The equation obtained by restricting in Eq.(**??**) to have components with $|\mathbf{k}|,|\mathbf{k}_1|,|\mathbf{k}_2|\le N$, where $N\ge k_f$ is an integer: the resulting equation will be called the 'regularized' NS with 'ultraviolet' (UV) cut off N and denoted NS^N: and the velocity fields are checked to be in a space of dimension $4N(N+1)$.

Consider also the equation , regularized at N, in which the viscosity ν is replaced by

$$\alpha(\mathbf{u})=\frac{\sum_\mathbf{k}\mathbf{k}^2 f_\mathbf{k}u_{-\mathbf{k}}}{\sum_\mathbf{k}\mathbf{k}^4|u_\mathbf{k}|^2} \tag{1.4}$$

The new equation will be denoted NS_R^N and it has been considered (also in the corresponding ones in 3D) in several works: for a review see [**?**].

The multiplier $\alpha(\mathbf{u})$ is so defined that the NS_R^N admits $En=\sum_\mathbf{k}\mathbf{k}^2|u_\mathbf{k}|^2$ as an exact integral of motion. Hence it is possible to consider its time average $\overline{\alpha}$.

Given N imagine to run the NS^N with a viscosity ν and that in this evolution the enstrophy has an average $\overline{E}n$ (typically depending on N): assume first that for N large and there is no intermittency and the average is independent on the initial data (with probability 1 with respect to the volume). The intermittent cases will be addressed later.

The basic idea is the equations NS^N and NS_R^N are *equivalent* in the limit $N\to\infty$ if

the average $\overline{E}n$ of $En=\sum_\mathbf{k}\mathbf{k}^2|u_\mathbf{k}|^2$ in NS^N and of $a(u)$ in NS_R^N exist for N large

Assume that the NS_R^N is run with enstrophy $\equiv\overline{E}n$.

a) If averages of observables $O(\mathbf{u})$ which depend only on finitely many Fourier's components of $\mathbf{u}$ (in number N-indpendent) are considered, then the averages of such observables are converge to the same limit as $N\to\infty$: such observables will be called "local observables" or "large scale observables".

In other words the equations are equivalent.

b) It is also possible to fix a value of the average enstrophy $\overline{E}n$ and run the NS_R^N to compute the average $\overline{\alpha}$ (typically N dependent) and then run the NS^N equation with viscosity $\nu=\overline{\alpha}$. Then again in the limit $N\to\infty$ the averages of the local observables converge to the same limit.

Again the equations are equivalent.

In case a) it is expected that average of $a(\mathbf{u})$ in the NS_R^N run with enstrophy $\overline{E}n$ average in NS^N converges to ν.

In case b) it is expected that the average of En in the NS^N evolution run with viscosity $\overline{a}$ average of a in NS_R^N converges to $\overline{E}n$.

Here we shall test the above two points at various choices of N, at fixed forcing $\mathbf{f}$ with only two modes $\mathbf{k}=\pm(2,-1)$ or ten modes $\mathbf{k}=\pm(2,-1)$ and $\mathbf{k}=(k1,k2),|k_i|\le1$. We also fix the $\sum_\mathbf{k}|f_\mathbf{k}|^2=1$ so that the equations have just one parameter (the viscosity for NS^N and the enstrophy En for NS_R^N) the components of $\mathbf{f}$ are chosen randomly and fixed once and for all.

We shall consider various values of N and for each determine average of $En(\mathbf{u})$ as a function of ν, N in NS^N and average of $\alpha(\mathbf{u})$ as a function of En, N in NS_{En}^N. Th expectation is that the results will become functions N-independent for large N.

Attention will also be dedicated to test the equivalence for selected local observables: namely $O_1(\mathbf{u}) = |u_{1,1}|^2$, $O_{2,-2} = |u_{2,-2}|^2$.

II. "ONE MODE FORCING".

2 armoniche
-2 +1 +0.504344589531564 -i 0.495617327189272

N	N0	En	t	rR	h	a/nu I	a/nu R	dove
3	16	155.4160	1.e05	11	13	1.0227	1.02900	rm2/00/SLF16/F13_13
7	32	143.3187	3.e04	11	13	1.0607	1.03541	rm2/00/E/e32-13 solo En
7	32	143.8409	5.e04	11	13	1.0410	1.00839	rm2/00/SLF32/F13-7
7	32	144.5381	5.e04	11	13	?	?	rm2/00/SLF32/F13_13.2
15	64	357.5826	1.5e04	11	13	1.0257	?	rm2/00/SLF64/F13_13.2
20	128	147.78 ?	1.e04	11	13	1.0292	?	rm2/00/SLF128/F-13_13 ?trans
20	128	3.601445	*1.e04	11	13	1.0318	?	rm2/00/SLF128/F20
3	16	153.8429	3.5e06	11	13	1.0217	1.03245	rm1 NS2/SLF16/F13-13 F161
3	16	153.8429	1.e03	11	13	1.0217		rm1 0/SLF16/F13-13 n16- *
7	32	140.2072	9.5e05	11	13	1.0464	1.05028	rm1 NS2ns/SLF32/g20_16Mdt1
3	64	158.2390	1.1e04	11	13	?	?	C NS/0/SLF64/f13_13
15	64	360.2179	1.5e04	11	13	?	?	C NS/0/SLF64/F13_13.2 Est
3	16	152.1929	9.e04	10.95	13	1.0244	0.9918	ipp NS/0/SLF16/F13_13 0-16 trans*
15	256	345.9115	2.7e03	12.83	13	1.0123	1.000	ipp NS2NS/SLF256/F63 MDT1 *
63	512	360.1549	2.e03	11.5342	13		1.000	ipp NS2NS/SLF512/F63 MDT1 *
63	512	360.1549		11.5342	13		1.000	ipp 0/SLF256/F63 MDT1 *
31	128							
63	256							
127	512							

The * means that average En is found at the end of a run at fixed enstrophy with the alfa average taken as the viscosity in an irreversible run.

The ?trans in the rm2 N=20 N0=128 line is doubtful: it is on purpose obtained by starting with a u with very high enstrophy = 5.e07 to check that it starts decreasing. But it reaches a low Enstrophy comparable with the average enstrophy of the cases with low N ¡= 7. Two attractors??

III. "THREE MODES FORCING".

6 armoniche
-2 -1 +0.028055631427720 +2 +1
-2 +1 +0.665353907422766 +2 -1
-1 -1 -0.237733168536597 +1 +1

N	En	t	rR	N0
3				
7				
15	+3.54734732657739e+03	282000	11	64
31				
63				
127				

IV. "FOUR MODES FORCING".

8 armoniche
-1 -1 +3.01944356155454e-01 -i 8.69087515006728e-02 -1 -1
-1 +0 +2.45249290707804e-01 -i 2.95419917152201e-01 -1 +0
-1 +1 +2.89975095588292e-01 +i 2.84957318488153e-01 -1 +1
+0 -1 +2.88348938869411e-01 -i 7.36545496697665e-02 +0 -1

N	En	t	rR	N0	a/nu I	a/nu R	dove
7	1673663.6	3.6e04	11	32	1.003	?	rm1/NS2ns/SLF32/F20-16 F320
15							
31							
63							
127							

V. "FIVE MODES FORCING".

10 armoniche
-2 +1 +1.39175656498932e-01 -i 0.145766596967491 +2 -1
-1 -1 +2.89420315586534e-01 -i 0.0833039524461469 +1 +1
-1 +0 +2.35076846667346e-01 -i 0.283166497103586 +1 +0
-1 +1 +2.77947515714421e-01 +i 0.273137865849274 +1 -1
+0 -1 +2.76388808683880e-01 -i 0.705995080723843 -0 +1

N	En	t	rR	N0	a/nu I	a/nu R	dove
3							
7	547073.1	57800	11	32	1.0019	?	NS2ns/SLF32/f20_16/MDT0
15							
31							
63							
127							