Renormalization and Classical MEchanics

Renormalization (group): cancellations fixing apparently singular problems can be (easily) exhibited by decomposing the singularity into “scales”, i.e. as sums of regular terms.

An interesting illustration is the analysis of the KAM problem. First consider a trivial case (“1–scale problem)

$$h_j(\underline{\psi}) = \varepsilon\partial_j f(\underline{\psi} + \underline{h}(\underline{\psi})) \qquad \underline{\psi} = (\psi_1,\psi_2) \in T^2$$
$$f(\underline{\alpha}) = \sum_{|\underline{\nu}|\le N} e^{i\underline{\nu}\cdot\underline{\alpha}} f_{\underline{\nu}}, \qquad f_{\underline{\nu}} = f_{-\underline{\nu}}$$

Then cnsider the KAM problem

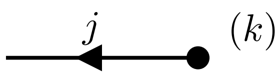
$$(\underline{\omega}\cdot\underline{\partial})^2 h_j(\underline{\psi}) = \varepsilon\partial_j f(\underline{\psi} + \underline{h}(\underline{\psi})) \qquad j = 1,2$$
$$\underline{\omega} = (\omega_1,\omega_2) \in R^2, \qquad |\underline{\omega}\cdot\underline{\nu}| \ge \frac{1}{C|\underline{\nu}|^\tau}, \quad C,\tau > 0$$

(example: $\underline{\omega} = (1,\sqrt{2})$).

Perturbative solution: $\underline{h}(\underline{\psi}) = \varepsilon\,\underline{h}^{(1)}(\underline{\psi}) + \varepsilon^2\,\underline{h}^{(2)}(\underline{\psi}) + \ldots \Longrightarrow$

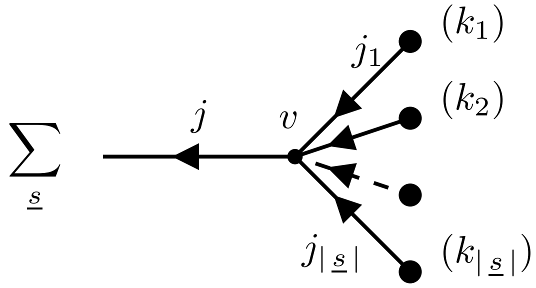
$$\underline{h}^{(k)}(\underline{\psi}) = \Big[\sum_{\underline{s}\ge 0} \frac{1}{\underline{s}!}\,\partial_{\underline{\psi}}\,\partial^{\underline{s}}_{\underline{\psi}} f(\underline{\psi})\,\underline{h}^{\underline{s}}\Big]^{(k-1)} =$$
$$= \sum_{\underline{s}\ge 0} \frac{1}{\underline{s}!} \sum_{\sum k_{ij}=k-1} \partial_{\underline{\psi}}\,\partial^{\underline{s}}_{\underline{\psi}} f(\underline{\psi}) \prod_{i=1}^{\ell}\prod_{j=1}^{s_i} \underline{h}^{(k_{ij})}$$

This is rather involved as an algebraic relation! A characteristic feature of the method: graph symbols (rather than algebraic ones). We can represent $h_j^{(k)}$ as



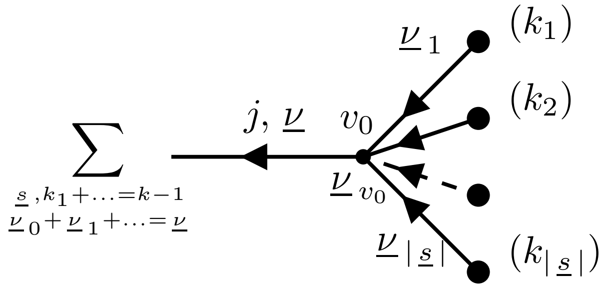
1Represent. of $\underline{h}^{(k)}$: the label $j = 1,\ldots,\ell$ indicates the j-th component of $\underline{h}^{(k)}$.

Rewrite $h_j^{(k)}$, for $k-1 = k_1 + \ldots + k_{|\underline{s}\,|}$, as



2:The *root* is marked j. *Node* $v \to \partial_{\underline{\psi}} \partial^{\underline{s}}_{\underline{\psi}}$. The dummy labels j_i wil be suppressed.

Since $\underline{h}\,(\underline{\psi}\,)$ is periodic we compute its FT. $\underline{h}_{\,\underline{\nu}}$: as a graph



Iteration:
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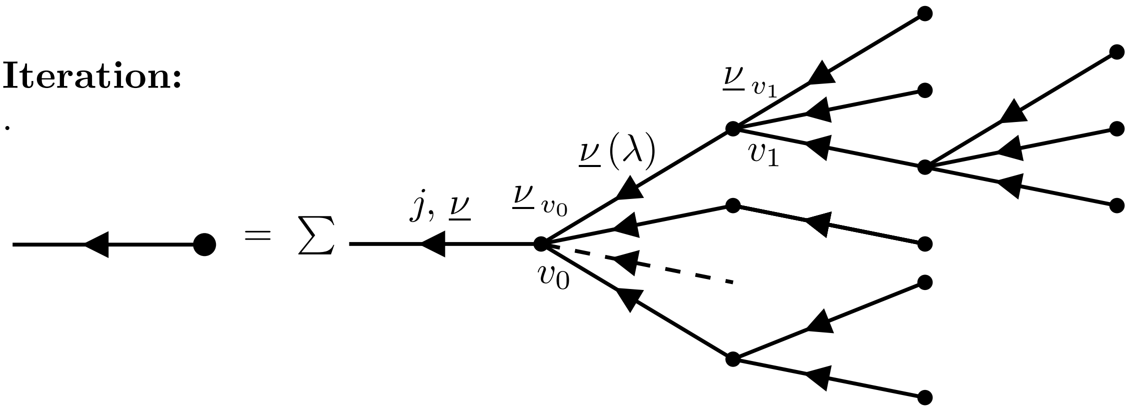

Incoming momenta $\underline{\nu}_{\,n}$ into each node.
Current through a generic line $\lambda = v_0 v_1$: $\underline{\nu}\,(\lambda) = \sum_{w \le v_1} \underline{\nu}_{\,v}$ (i.e. current is conserved at each node).

Value:

$$\mathrm{Val}(\vartheta) = \frac{1}{k!} \prod_{\lambda=(v'v)} (i\,\underline{\nu}_{\,v'} \cdot i\,\underline{\nu}_{\,v}) \prod_v f_{\underline{\nu}_{\,v}}$$

Number of trees $\le k!2^{2k}$. Each value is trivially bounded by

$$|\mathrm{Val}(\vartheta)| \le \frac{1}{k!} N^{2k-1} F^k \quad \Longrightarrow \text{ convergence if } |\varepsilon| < \varepsilon_0 < \tfrac{1}{2^2 N^2 F}$$

Cancellations

(1) If $\underline{\nu} = \underline{0}$ then the *sum* of the values is $\underline{0}$! In fact if we sum the values of the trees obtained by detaching the root line from the node v_0 and we reattach it to the other nodes w the value does not change other than by the factor "*propagator*" of the root line which equals $\underline{\nu}_{wj}$ hence the sum of all values is proportional to $\sum_w \underline{\nu}_{wj} \equiv \underline{\nu} = 0$

(2) The same argument justifies discarding the (otherwise undefined) values of the trees which contain an internal line with $\underline{0}$ current: this is the cancellation found by Lindstedt, Newcomb (in special) and by Poincaré (in general).

(3) If there are two lines that are comparabe (in the tree partial order) and which have the *same current* $\underline{\nu}$

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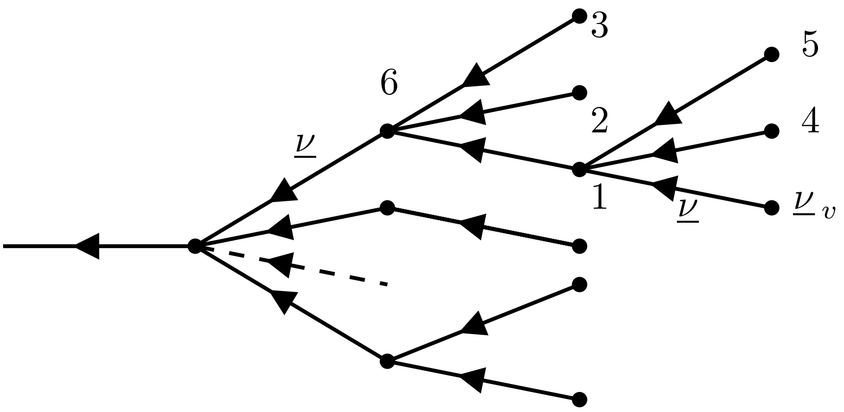


this means that $\sum_{i=1}^{6} \underline{\nu}_i = \underline{0}$: and if we detach the line starting in v and we attach it to the nodes $j = 1, \ldots, 6$ we only change the propagator of the line jv which assumes one after the other all values $\underline{\nu}_j \cdot \underline{\nu}_v$: their sum vanishes ($\underline{0}$). There we could have also excluded the graphs in which two comparable lines have the same momentum? NO *overlapping divergences.*

Same representation! But new propagator. *Value*: it is obtained by discarding graphs in which at least oneline carries a 0 current (Poincaré),

$$\mathrm{Val}(\vartheta) = \frac{1}{k!} \prod_{\lambda\equiv(v'v)\in\vartheta} \frac{-\underline{\nu}_v \cdot \underline{\nu}'_v}{(\underline{\omega}_0 \cdot \underline{\nu}(\lambda))^2} \prod_{v\in\vartheta} f_{\underline{\nu}_v}$$

where the root should be interpreted as: $\frac{i(\underline{\nu}_{v_0})_j}{(\underline{\omega}_0 \cdot \underline{\nu})^2}$.

The momentum of every node $\underline{\nu}_v$ is bounded, *but the currents* $\underline{\nu}(\lambda)$ *can be as large as* $k\,N$: $\Longrightarrow$ *large propagators* or *small divisors* (not zero, though) will be possible.

Is this a real problem? yes: because the value of the single graph ϑ_0

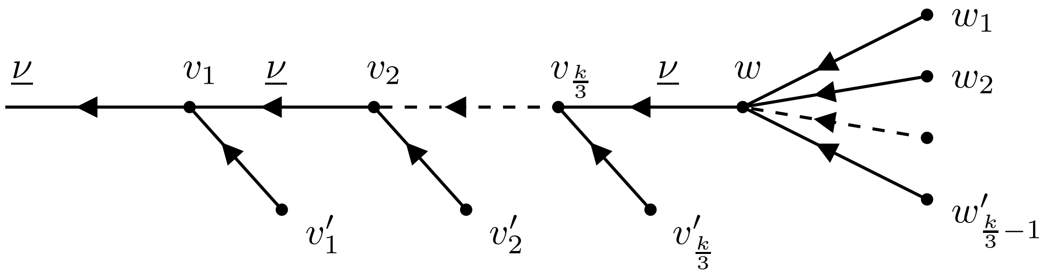


Fig.4 A graph (*comb*) resonating if $\underline{\nu}_{v_j} = -\underline{\nu}_{v'_j} = \underline{\nu}_0$: it has et trop gande valeur.

is easily computed and yields a value $k!\mathrm{Val}(\vartheta_0 \ge const(k!)^a$ $(a = \tau/3)$.

One must make use of the cancellations seen in the trivial case. Indeed if *we disregard certain graphs* then $\Longrightarrow$ easy bound. Indeed if N_n = number of propagators *of scale* n *i.e*

$$2^{n-1} < C|\underline{\omega} \cdot \underline{\nu}| \le 2^n \qquad n = 0, -1, -2, \dots$$

the value is trivially bounded by

$$\frac{1}{k!} N^{2k-1} F^k C^{2k} \prod_{n=-\infty}^{1} 2^{-2nN_n}$$

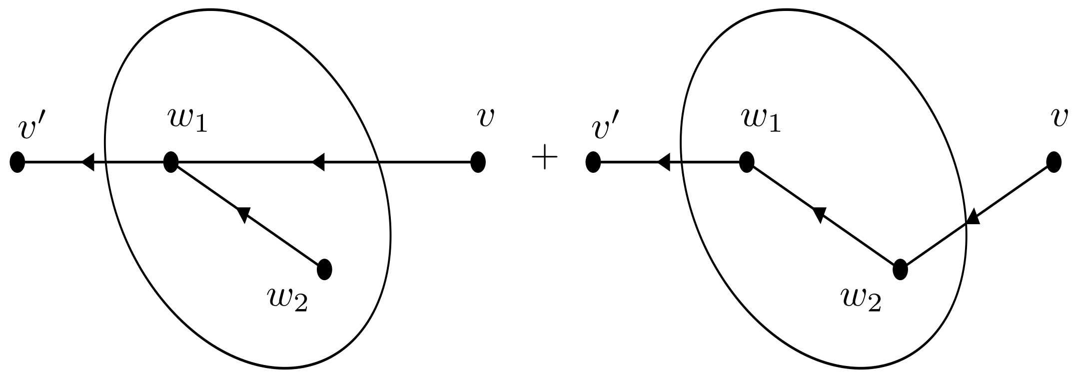
On the other hand to generate a propagator of scale n it is necessary that $|\underline{\nu}| > 2^{-n/\tau}$ and therefore the line must be preceded by at least $2^{-n/\tau}/N$ graph lines.

Once such *resonance* is created and $C\,\underline{\omega}\cdot\underline{\nu}\,(\lambda)\simeq 2^n$ one might think that one should "*wait*" as many new lines $(2^{-n/\tau}/N)$ in order of being able to recondtruct a propagator of the same size)The conlusion would be that if we only dispose of k lines (Siegel,Pöschel) then

$$N_n < const\frac{k}{2^{-n/\tau}} \qquad \Longrightarrow \qquad \prod_{n=-\infty}^{1} 2^{-2n\,c\,2^{n/\tau}\ k} = M^k$$

The "comb" graph exposes the faulty argument: between two equal propagators there might be *just a single line*: one should thereofore show that if bewteen two lines with equal propagator the are not many "intermediate" lines then we can collect several graphs whose sum can be bounded by counting the repeated propagators just once

The mechanism is simple



The simple cancellation

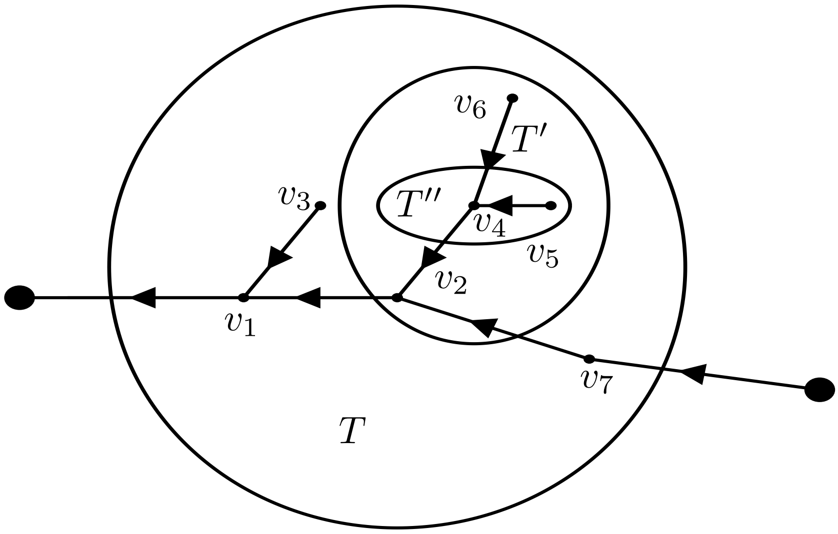
If in a graph one gets the situation of the figure with $\underline{\nu}_{w_1}=-\underline{\nu}_{w_2}$.
We detach from the node w_1 the entering line and atach it to w_2: this amounts to a change in sign (because of the vector part of the propagator $\underline{\nu}_{w_1}\cdot\underline{\nu}_{v}$, which becomes). However the current on the line $w_w w_2$ changes from $\underline{\nu}_{w_2}$ to $\underline{\nu}_{w_2}+\underline{\nu}$ and $(\underline{\omega}\cdot\underline{\nu}_{w_2})$ changes into $(\underline{\omega}\cdot\underline{\nu}_{w_2}+\underline{\omega}\cdot\underline{\nu})\stackrel{def}{=}(\underline{\omega}\cdot\underline{\nu}_{w_2}+\delta)$: hence by summing the two terms and those obtained by exchanging the sign of $\underline{\nu}_{w}$ one gets the 4 terms

$$\Big(\frac{1}{(\underline{\omega}\cdot\underline{\nu}_{w_2})^2}-\frac{1}{(\underline{\omega}\cdot\underline{\nu}_{w_2}+\delta)^2}+\frac{1}{(-\underline{\omega}\cdot\underline{\nu}_{w_2})^2}-\frac{1}{(-\underline{\omega}\cdot\underline{\nu}_{w_2}+\delta)^2}\Big)$$

which vanishes to order 2 if $\delta=0$. If $\delta\ll\underline{\omega}\cdot\underline{\nu}_{w_2}$ (the only interesting case) then we bound by $const\,\delta^2$ which "*removes one of the two small divosors.*

If we have a chain of m resonating propagators we repeat the operations and we sum over 4^m graphs obtaining $\Longrightarrow$ bound prop. to δ^{2m}; which compensates the product of the ($\delta^{2(m+1)}$) propagators: only one of the m denominators remains!

But one must avoid superposing the cancellations. Given a graph we define the "clusters" of propagators of scale n:



Example of 3 clusters (encercled as visual help).

A cluster is formed by lines, forming a connected set, of scale $\geq n$ with at least one line of scale $n \Longrightarrow$: which leads to a hierarchical structure typical of multiscale problems.

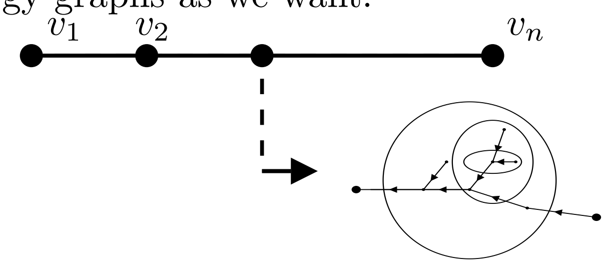
We declare a cluster a *self–energy* cluster if

• the sum of the moments inside a cluster vanished et
• a single line enters the cluster
• the number of internal lines is not large of the order $2^{-n/\tau}$ (*i.e*if the presence of the cluster would not permit to apply the mentioned argument of Siegel–P̈oschel)

There will not be conflicts between the cancellations (*i.e*we shall not need the same graph to exhibit the cancellations needed in two unrelated graphs) if in the operation of detaching attaching of the lines that enter a self energy graph *the scales of the graph lines do not change* (as one might fera they could): this is verified.

Then with each self energy graph one shall generate via the resummation just indicated a factor as small as the size of the two (equal) divisors relative to the entering and exiting line. *We can proceed as if the self energy subgraph did not produce an extra small divisor and the Siegel-Pöschel bound remains valid.*

The technique can be pushed further: since inside every line we can insert as many self energy graphs as we want.



Inserting self energy graphs on a line

Since the value of a graph in which we insert a self e. graph on a line with current $\underline{\nu}$ is changed by a factor depending only on $\underline{\nu}$ We can imagine defining $M(\underline{\nu})/(\underline{\omega}\cdot\underline{\nu})^2$ to be the sum of the factors that are obtained by adding together the values of all possible self-e. graphs and then inserting them in all lines of a graph *without self-e. graphs.*

Summing over the insertions of the factors M generates a geometric series. One sees that formally this means replacing $\frac{1}{(\underline{\omega}\cdot\underline{\nu})^2}$ by

$$\frac{1}{(\underline{\omega}\cdot\underline{\nu})^2}\sum_{k=0}^{\infty}\Big(\frac{M(\underline{\nu})}{(\underline{\omega}\cdot\underline{\nu})^2}\Big)^k=\frac{1}{(\underline{\omega}\cdot\underline{\nu})^2-M(\underline{\nu})}$$

Therefore if one can show that the series defining $M(\underline{\nu})$ is convergent and smaller than $(\underline{\omega}\cdot\underline{\nu})^2 O(\varepsilon)$ one achieves a proof of the KAM and an expression of the solution which involves only summations over graphs values of graphs without self-energy graphs (which can be bounded trivially). Difficult; and requires a proof: but possible.The technique can be extended to lower dimensional invariant tori of hyperbolic type. The self-e. graphs do not necessarily verify $M(\underline{\nu})$ proportional to $(\underline{\omega}\cdot\underline{\nu})^2$: however one can check that $M(\underline{\nu})<0$ at least formally (*i.e*diregarding convergence problems). The renormalized graphs are "even better" at least formally: this can be used to prove that $M(\underline{\nu})$ is well defined and >0 so that one achieves an (alternative) proof of the existence of hyperbolic lower dimesional tori under suitable nondegeneracy conditions.