Lyapunov spectra and non equilibrium ensembles equivalence in 2D fluid mechanics

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NS and GNS equations ($d = 2$ *Geometry*):

1) Side of the periodic cell $= L$,

2) Forcing term is $F\mathbf{f}$; *i.e.* a dimensional parameter F times a fixed $\mathbf{f}$

3) Fourier comp. $F_{\mathbf{k}}$ have max. mod. 1; vanish but for one (or few) $\mathbf{k}$

4) Viscosity ν

5) $V = FL^2/\nu, C = \nu/L^2$ have dimension of velocity and inverse time. The velocity field is defined $V\mathbf{u}(\mathbf{x}/L, Ct)$ with $\mathbf{u}$ dimensionless.

6) *Reynolds number* R : $R^2 = FL^3\nu^{-2}$, or $R = \nu^{-1}\sqrt{FL^3}$

Equations (incompressible fluids):

(NS): $\dot{\mathbf{u}} + R^2(\mathbf{u} \cdot \partial)\mathbf{u} = \Delta\mathbf{u} + \mathbf{f} - \partial p \ , \quad \partial \cdot \mathbf{u} = 0$

(GNS): $\dot{\mathbf{u}} + R^2(\mathbf{u} \cdot \partial)\mathbf{u} = \alpha\,\Delta\mathbf{u} + \mathbf{f} - \partial p \ , \quad \partial \cdot \mathbf{u} = 0$

where α is constant in space but may depends on $\mathbf{u}$.

Energy, Enstrophy, etc.

Energy (kinetic): $Q_0 = \int \mathbf{u}^2 d\mathbf{x} = (2\pi)^2 \sum_{\mathbf{k}} |\mathbf{u}_{\mathbf{k}}|^2$,
Enstrophy: $Q_1 = \int (\partial\mathbf{u})^2 d\mathbf{x} = (2\pi)^2 \sum_{\mathbf{k}} |\mathbf{k}^2|\,|\mathbf{u}_{\mathbf{k}}|^2$,

Consider Euler equations ($\nu = 0$) which are Hamiltonian: *impose* the constraint that $Q_0 \equiv$ constant constant via *Gauss' least effort principle.* Of course the "effort" has to be defined.

Effort: $\mathbf{G}_1$ as $\qquad \mathbf{G}_1 \stackrel{def}{=} \big(\Delta^{-1}(\mathbf{a} - \mathbf{f} - \partial p), (\mathbf{a} - \mathbf{f} - \partial p)\big)$

Resulting equations have been studied: *are the above GNS equations* with $\alpha = \frac{\sum_{\mathbf{k}} f_{\mathbf{k}} \overline{u}_{\mathbf{k}}}{\sum_{\mathbf{k}} |\mathbf{k}|^2 |u_{\mathbf{k}}|^2}$. Or in Fourier transform:

$$\dot{u}_{\mathbf{k}} = -iR^2 \sum_{\mathbf{j}+\mathbf{h}=\mathbf{k}} \frac{(\mathbf{j}^{\perp} \cdot \mathbf{h})(\mathbf{h} \cdot \mathbf{k})}{|\mathbf{j}||\mathbf{h}||\mathbf{k}|} u_{\mathbf{j}} u_{\mathbf{h}} - \alpha \mathbf{k}^2 u_{\mathbf{k}} + f_{\mathbf{k}}$$

Effort definition is arbitrary. More general constraints on Euler eq.:

$$Q_m = (2\pi)^2 \sum_{\mathbf{k}} |\mathbf{k}|^{2m} |u_{\mathbf{k}}|^2 \qquad \textit{constant under}$$

$$\mathbf{G}_{\ell,m} = \big((\Delta)^{-(\ell-m)}(\mathbf{a} - \mathbf{f} - \partial p), (\mathbf{a} - \mathbf{f} - \partial p)\big) \qquad \textit{as effort}$$

The case $\ell = 1, m = 1$ ($\to \alpha = \frac{\sum_{\mathbf{k}} |\mathbf{k}|^2 f_{\mathbf{k}} \overline{u}_{\mathbf{k}}}{\sum_{\mathbf{k}} |\mathbf{k}|^4 |u_{\mathbf{k}}|^2}$) studied in [RS00] together other cases (G-hyperviscous equations NS). Here I fix $\ell = 1, m = 0$: energy conserved with an effort $\mathbf{G}_1$.

Mathematically GNS *is no easier* than NS. Truncation (*i.e.* $|k_i| \le N$) is necessary. Physically the cut–off on NS should be of the order of $|k_i| < O(R^2)$ (very large). Chaos starts around $R^2 \sim 70$ (experimentally).

Our interest is not motivated by any deep meaning attached to the Gauss principle: it is not used in any other way than in the above interpretation of the equations that are interesting only for the reasons that follow ("equivalemce conjecture").

In general G-hyperviscous equations are *reversible*: velocity reversal $Iu_{\mathbf{k}} = -u_{\mathbf{k}}$ anticommutes with time evolution S_t since $I \cdot S_t = S_{-t} \cdot I$. They are also *dissipative* and the phase space is contracted at rate (divergence of the equations)

$$\sigma^{GNS} = 2\Big(\sum_i |k_i|^2 - \frac{Q_2}{Q_1}\Big) \cdot \alpha + \frac{\sum_{|k_i|<N} \mathbf{k}^2 f_{k_i} \overline{u}_{k_i}}{Q_1}$$

related to the (*constant*) NS dissipation $\sigma^{NS} = 2\sum_{|k_i|<N} |k_i|^2 \stackrel{def}{=} M$. For instance $\langle\sigma^{GNS}\rangle = M \cdot \langle\alpha\rangle + o(M)$

Equivalence conjecture

The stationary probability distributions on the phase space of the NS equations and the GNS equations are equivalent in the limit of large Reynolds number, provided the value of Q_0 is chosen so that σ^{NS} and $\langle\sigma^{GNS}\rangle$ coincide or, equivalently, provided the value of Q_0 is chosen so that $\langle\alpha\rangle$ equals 1.

Equivalence takes place in the same sense as canonical and microcanonical ensembles are equivalent.

It han been studied by [RS00] and they concluded that there was evidence for its validity at least in the small truncations that they could study.

Why? Note that α is the sum of the local Lyapunov exponents: they are in number $O(R^2)$ and furthermore fluctuate on short time scales: therefore the value of α will have fluctuations in time which are less strong than those of the individual exponents whose sum is α. Then α can be regarded as a constant for the purpose of studying observable on large scale. Locality is in "momentum space" and $R \to \infty$ corresponds to the thermodynamic limit.

Here we attempt to study

1) whether the equivalence extends beyond the original formulation and one can identify even the Lyapunov spectrum of the two equations

2) given the equivalence we study the NS equation and try to see whether a part of the fluid behaves like the entire sample and we look at the *fluctuation relation* in a smaller portion of the fluid. This is necessary in order to compare with certain experiments that are being attempted or planned ([CL][G]). In macroscopic systems one can only hope to see important fluctuations if one examines small portions of the systems (anlogy with density fluctuations in equilibrium)

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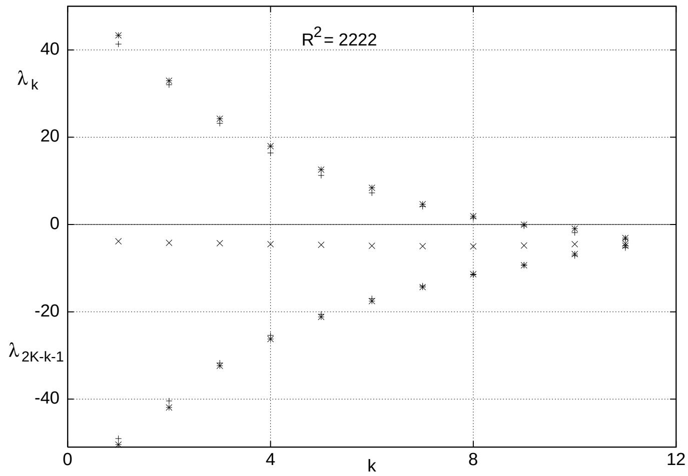

Lyapunov spectra for NS with normal viscosity ($N = 5$ truncation) at $R^2 = 2222$ (left) and $R^2 = 5000$ (right), and corresponding GNS runs with constrained energy Q_0. The $2K-2$ nontrivial exponents are drawn by associating each value of the abscissa $k = 1, 2, \ldots, K-1$ with the k–th largest exponent λ_k and the k–th smallest exponent $\lambda'_k = \lambda_{2K-k-1}$. Symbols
"$+$" $\to$ NS spectra,
"$*$" $\to$ GNS spectra,
"$\times$" to the sums $(\lambda_k + \lambda'_k)/2$ (NS case).
No "pairing" of exponents to a common average value, *unlike the cases of isokinetic Gaussian systems [DM96].*

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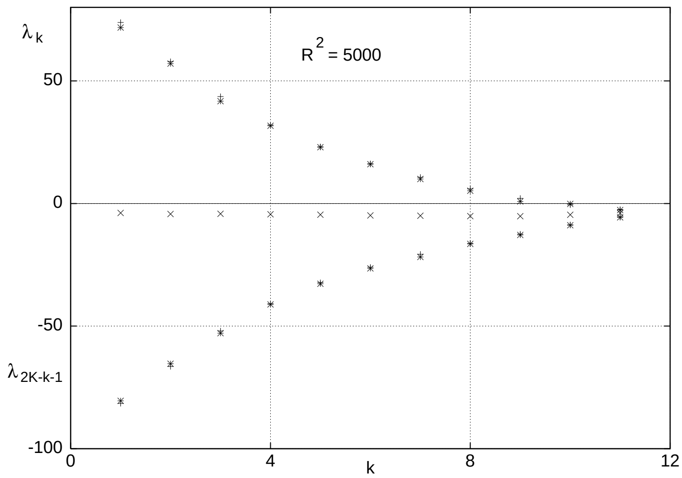

An attempt to check equivalence on systems with many more modes yields the following result

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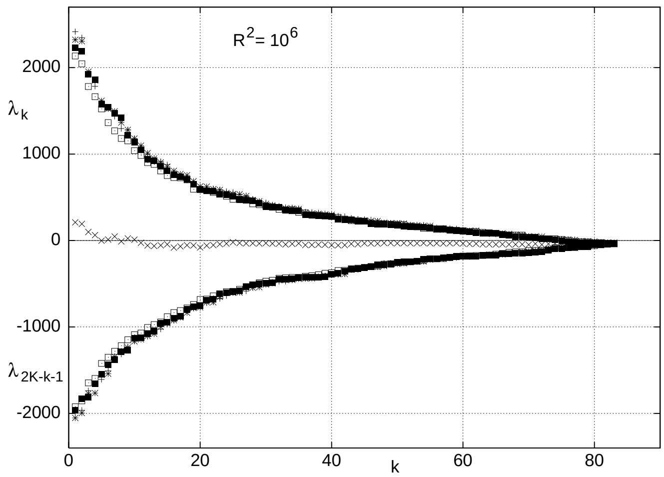

Lyapunov exponents [$N = 7$, $R^2 = 10^6$, and forcing modes $(4, -3), (3, -4)$].
All $2K - 2 = 164$ nontrivial exponents are drawn as in Fig.1.
$(+)$ NS exponents
$(\times) \to$ graph of $(\lambda_k + \lambda'_k)/2$ (NS only)
$(*) \to$ corresponding GNS runs with fixed energy $(*)$ *i.e.* $m = 0, \ell = 1$
("box") $\to$ fixed "enstrophy" *i.e.* $m = 1, \ell = 1$,
("square") $\to$ fixed "palinstrophy" *i.e.* $m = 2, \ell = 1$.
Error bars identified with the size of the symbols.
units of $1/\lambda_{max}$, λ_{max} being the largest Lyapunov exponent; runs of length $T \in [125, 250]$.
Overlap (best between NS and GNS because ...) reflects the validity of the extension of the EC to the whole spectrum and to different members of the hierarchy of equations.

R^2	$\delta Q_0 / \langle Q_0 \rangle_{NS}$	$\triangle\alpha$	$\triangle Q_1$	$o(M)/M$
800	0.005	0.030	0.053	0.068
1250	0.020	0.018	0.062	0.057
2222	0.002	0.039	0.058	0.077
4444	0.050	0.021	0.093	0.059
5000	0.010	0.008	0.058	0.033

Equivalence of NS and GNS dynamics, i.e. with $\ell = 1$ and $m = 0$, for different Reynolds numbers. The last column gives the relative difference of the computed sums of the NS and GNS Lyapunov exponents, cfr. [GNStoNS]).

Fluctuation relation (global)

The system is clearly chaotic (ie it has one, in fact many, positive Lyapunov exponents). We study the *asymmetry* of fluctuations of the phase space contraction rate. Define

$$\overline{\sigma^{GNS,\,\tau}}(i) = \frac{1}{\tau\,\langle\sigma^{GNS}\rangle_\infty}\int_{t_i}^{t_{i+1}} \sigma^{GNS}(S_t\mathbf{u})\;dt,$$

with t_i a sequence of time intervale of length τ spaced by a gap $\overline{t}$ ($i = 1, ..., \sim T/(\tau+\overline{t})$). An histogram yields probability density of the distribution (in the stationary state, "SRB") $\pi^\tau(p) \stackrel{def}{=} \exp T\zeta(p)$ of the values of $p = \overline{\sigma^{GNS,\,\tau}}(i)$. It was observed (cf. [RS99]) that the "*asymmetry*"

$$F(p;\tau) = \frac{1}{\tau\,\langle\sigma^{GNS}\rangle}\left[\log\left(\pi^\tau(p)\right) - \log\left(\pi^\tau(-p)\right)\right]$$

could be fitted linearly in p with slope $c(\tau)$, as (within error bars):

$$F(p;\tau) = c(\tau)p\ ,$$

Fluctuation relation (local)

How to define the phase space contraction in a small cubic ideal cell V_0 of side L_0 in the fluid? We propose (inspired by [CL], [GGK]) to define it *for GNS as well as for NS* via ($\omega \stackrel{def}{=} \nabla \times \mathbf{u}$)

$$\begin{aligned}
\hat{p}_{L_0}(t) &= \int \mathbf{f}(x)\cdot \mathbf{u}(\mathbf{x},t)\chi_{V_0}(\mathbf{x})\; d\mathbf{x} \\
Q_{1,L_0}(t) &= \int \omega^2(\mathbf{x},t)\chi_{V_0}(\mathbf{x})\; d\mathbf{x}\;, \\
Q_{2,L_0}(t) &= \int (\nabla\omega)^2(\mathbf{x},t)\;\chi_{V_0}(\mathbf{x})\; d\mathbf{x}\;, \quad \alpha_{L_0}(t) = \frac{\hat{p}_{L_0}(t)}{Q_{1,L_0}(t)}\;,
\end{aligned}$$

By the expression (reducing to σ^{GNS} in the case $L_0 = L$:

$$\hat{\sigma}_{L_0} = 2\left(\left(\frac{L_0}{2\pi}\right)^2 \sum_{|k_i|\le N} |k_i|^2 \;-\; \frac{Q_{2,L_0}}{Q_{1,L_0}}\right)\cdot \alpha_{L_0} + \frac{\int \nabla f\,\nabla\omega\;\chi_{V_0}\; d\mathbf{x}}{Q_{1,L_0}}\;.$$

or *simply* as proportional to

$$\alpha_{L_0}(t) \equiv \frac{\text{work per unit time on } V_0}{\text{vorticity}}$$

The two def. differ by “boundary terms” and yield (in our experiments!) the same results within errors.

Experimental results: global fluctuations

Only on GNS (in NS $\sigma^{GNS} = const$!): even there it is a difficult task because the attractor is sensibly smaller than phase space. Smallness implies s FR with a slope different from 1 (theoritical result in systems that are hyperbolic and with dense attractor) and equal to $1 - \frac{M}{N}$ where M is the number of exponents in "negative pairs" and M is the total number. Remarkably ≤ 1.

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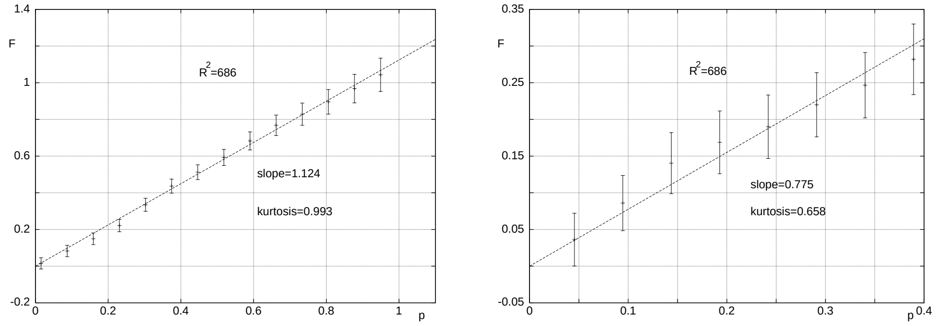

Computed values of $F(p;\tau)$, large deviation asymmetry, for GNS at $R^2 = 686$. The straight lines interpolate the data; error is statistical. Left: $\tau \approx 1.2$; Right $\tau \approx 2.5$, in $1/\lambda_{max}$ units. The fit below is over various τ. Data refer to a 24–real modes truncation ($2K = 24$). The kurtosis of π^τ would be 3 in Gaussian case.

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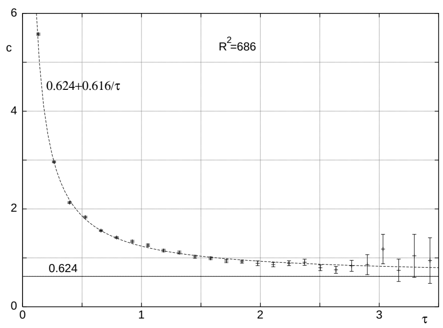
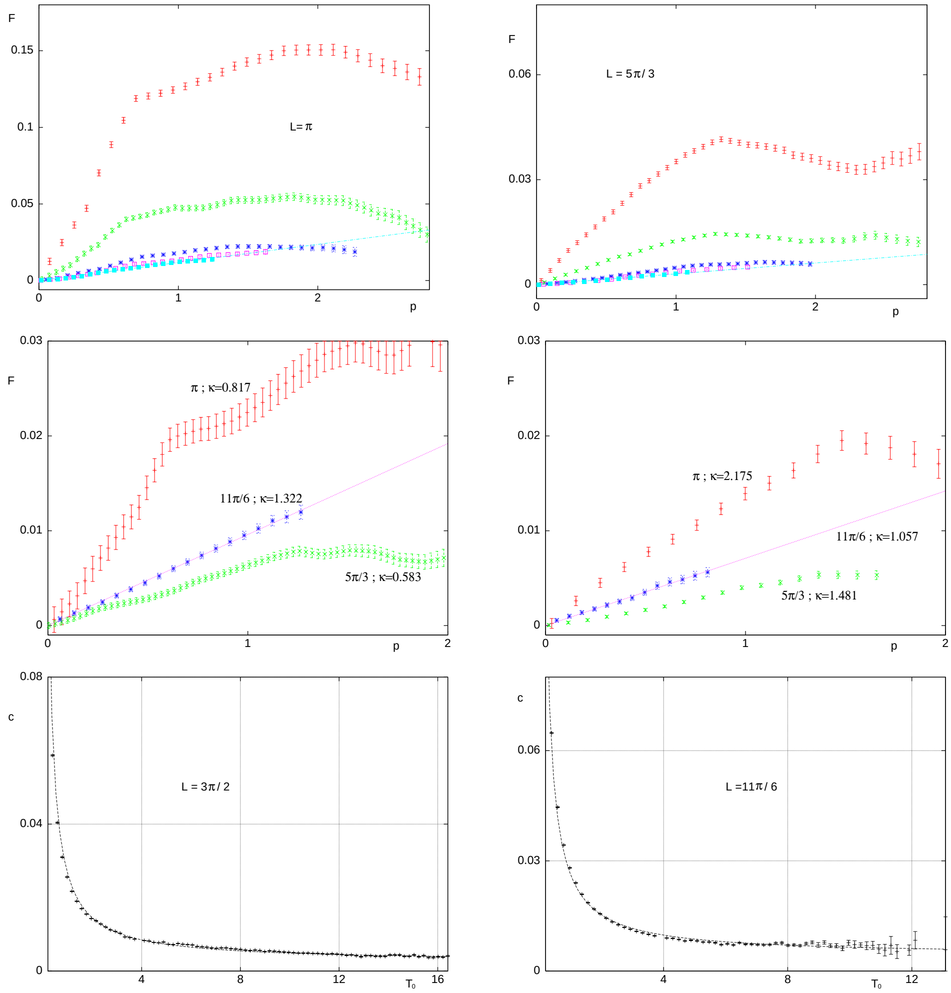


The value $c(\tau)$ (GNS at fixed energy and 24–real modes truncation) ($R^2 = 686$). Times τ, in the horizontal axis are given in $1/\lambda_{max}$ units. The values of τ corresponds to cases in which already $F(p;\tau)$ has linear graph. The datum with smallest τ has not been used in constructing of the fit (but it matches it, nevertheless).

The slope is related to the dimension of the attracting set ([BG]): hence it can be compared with the KY dimension: it appears that this dimension is, in all cases, below KY dimension.

Local fluctuations experiments (preliminary)

We define the "local contraction rate" of phase space by localizing the global expression for GNS (as in equilibrium one does for the energy fluctuations) as explained above



Theoretical interpretation (an attempt)

A FR will hold if several properties hold: among them at least

i) attracting set is smooth: this is not possible to check

ii) if there is a time reversal invariance *on the attracting set*

iii) if a kind of *pairing* holds: $(\lambda_k(x) + \lambda_{2N-k}(x))/2$ averages to its limiting value faster than the individual $\lambda_k(x)$ and $\lambda_{2N-k}(x)$ approach their asymptotic value on a faster time scale

The first assumption cannot be checked directly (attracting set is "out of reach"). The second property might be less difficult than it seems. Bonetto and G. proposed to link it to a geometric property called *Axiom C* described by the picture:

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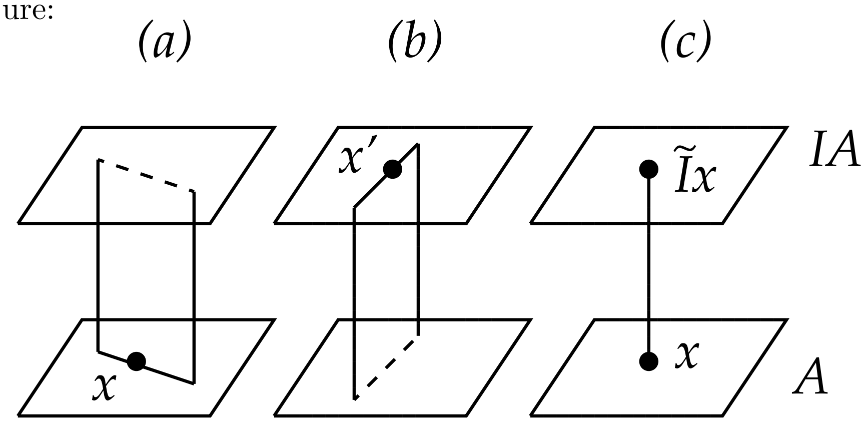

Illustration of axiom-C. Bottom squares are the attr. set A, top squares represent the repelling set IA. Left: with vertical surface depicts a point $x \in A$ with a piece of its stable manifold connecting A to IA. Center: a point $x' \in IA$ with a piece of its unstable manifold. Right: the corresponding intersection between stable manifold of x and unstable manifold of Ix. Thus one associates a point $\tilde{I}x$ in IA with a point x in A.

Local time reversal: $I^* = \tilde{I} \circ I$. In axiom C systems time reversal symmetry cannot be broken.

The third property can be tested experimentally: it seem verified in our case although the time scales involved are not really so different to provide a clear cut answer.

So test the consequences (predictions [BG], [G]): *linear asymmetry with slope* < 1 *and equal to the ratio of the dimension of the attracting set to the total phase space dimension.*

References

[BG] F. Bonetto, G. Gallavotti: *Reversibility, coarse graining and the chaoticity principle*, Communications in Mathematical Physics, 189:263–276, 1997.

[CG] G.Gallavotti, E.G.D. Cohen: *Dynamical ensembles in nonequilibrium statistical mechanics*, Physical Review Letters, **74**, 2694–2697, 1995

[CL] S. Ciliberto and C. Laroche: *An experimental test of the Gallavotti-Cohen fluctuation theorem*, Journal de Physique IV, 8(6):215, 1998.

[DM96] C. P. Dettmann and G. P. Morriss: *Proof of lyapunov exponent pairing for systems at constant kynetic energy*, Physical Review E, 53:R5541, 1996.

[G] G. Gallavotti: *Foundations of fluid mechanics*, Texts and Monographs in Physics. Springer–Verlag, Berlin, Springer–Verlag, Berlin, 2002. See also the papers by the same author quoted there.

[GGK]W. I. Goldburg, Y. Y. Goldschmidt, and H. Kellay: *Fluctuation and dissipation in liquid crystal electroconvection*, Physical Review Letters, December 2001.

[RS99] L. Rondoni and E. Segre: *Fluctuations in two-dimensional reversibly damped turbulence*, Nonlinearity, 12(6):1471–1487, 1999.