SRB distribution for Anosov maps

The lectures illustrate a method due to *F. Bonetto and P. Falco*. This was part of a program to obtain analyticity of the SRB distributions in weakly interactin chains of Anosov maps of "cat" type. The program has been completed by *F. Bonetto, P. Falco, A. Giuliani* and with some contributions by *G. Gentile and GG* it can be found in

Bonetto, F., Falco, P.L., Giuliani, A.: , *Analyticity and large deviations for the SRB measure of a lattice of coupled hyperbolic systems*, Preprint 2003.

as well as in

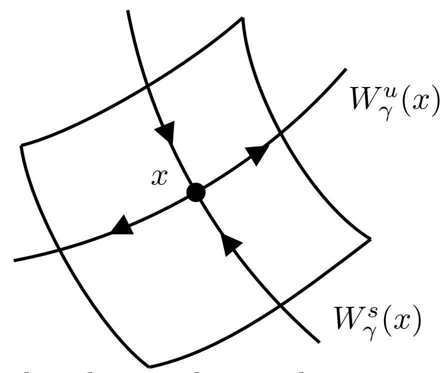
Gallavotti, G., Bonetto, F., Gentile, G.: *Aspects of the ergodic, qualitative and statistical theory of motion*, p. 1–517, to appear printed by Springer–Verlag.

Preprints in

`http://ipparco.roma1.infn.it`

Anosov maps: paradigm of systems with chaotic evolution.
Admit a system of curvilinear coordinates based on smooth surfaces W^s_x and W^u_x

(1) *covariant*: $S\partial W^i_x = \partial W^i_{Sx}$, $i = u, s$;
(2) *continuous*: ∂W^i_x depends continuously on x
(3) *hyperbolic*:



(4) *transitivity*: there is a point with a dense orbit in phase space M under S
A key example is the map of the torus $\mathcal{T}^2$ (Arnord's or cat map) defined by

$$S\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}$$

Problem: check the above four properties ("by Fourier analysis")

Key question is *which is the statistics* μ of motions? *i.e.* does it exist μ such that ($\forall f$ smooth)

$$\lim_{T\to\infty}\frac{1}{T}\sum_{i=0}^{T-1} f(S^i x)=\int_M \mu(dy)\, f(y)$$

for all but a set of zero volume of points x? if so μ is called *SRB distribution* for (M,S).

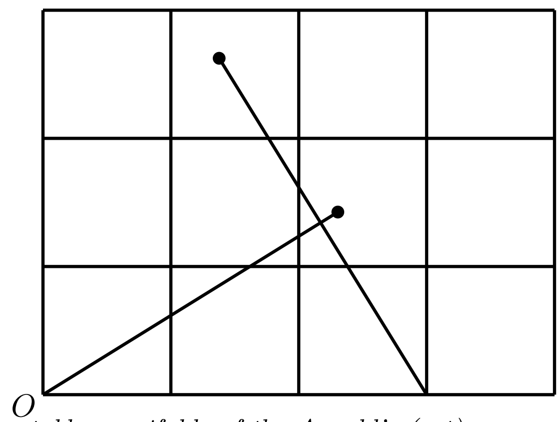
Anosov maps in general do not have an invariant distribution which admits a density with respect to the volumeμ_0 . Very simple examples are small perturbations of the above map, *e.g.* (*non trivial!*)

$$S_\varepsilon\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}=\begin{cases}\varphi_1+\varphi_2+\varepsilon\sin\varphi_1\\ \varphi_1\end{cases}$$

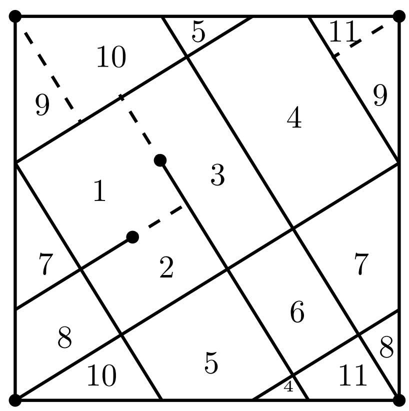
The lack of a density of μ with respect to μ_0 is at the same time a difficulty and a main point of interest. Question is more general: "*Chaotic hypothesis*".

Anosov maps admit special partitions of M, called *Markov partitions*, $\mathcal{P}=(P_1,\ldots,P_n)$. Histories $\underline{\sigma}$ on $\mathcal{P}$ of points x, *i.e.* sequences $\{\sigma_i\}$ s.t. $S^k x\in P_{\sigma_k}$, consist of sequences s.t. $SP_{\sigma_i}\cap P_{\sigma_{i+1}}$ hence defining $T_{\sigma\sigma'}=1$ if $SP_\sigma\cap P_{\sigma'}$ and 0 otherwise it is $T_{\sigma_i\sigma_{i+1}}\equiv 1$ (*compatibility matrix*).

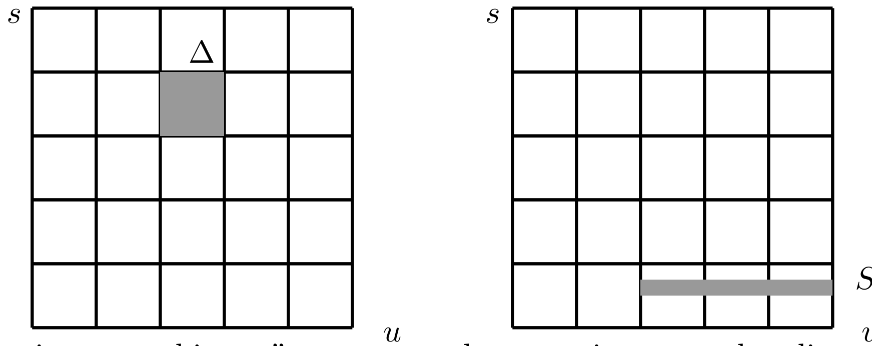
It is easy to construct $\mathcal{P}$ from a fixed point O of S. The stable and unstable lines through O cover densely phase space (a general property). Then



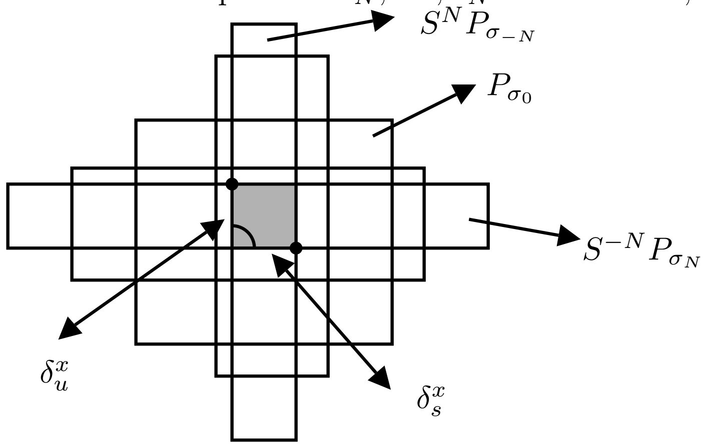
Portions of the stable and unstable manifolds of the Arnold's (cat) map.

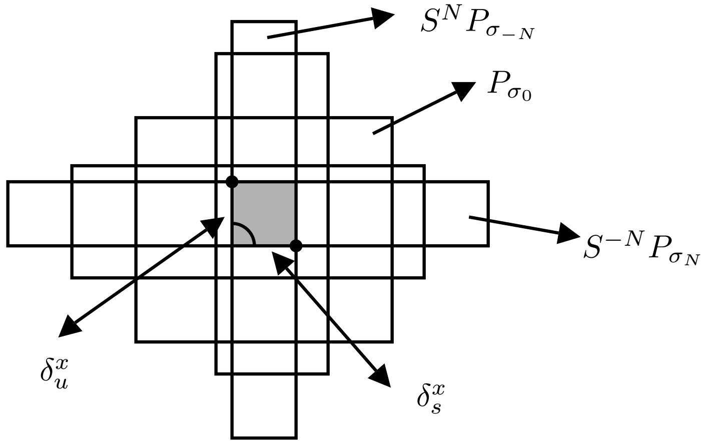


The main property of Markovian partitions is



The map "point $\longleftrightarrow$ history" converts volume μ_0 into a prob. dist. on T–compatible sequences. The points with histories equal to $\sigma_{-N},\ldots,\sigma_{N'}$ between $-N,N'$ form a rectangle





In formulae

$$\mu_0(\cap_{-N}^{N'} S^{-k} P_{\sigma_k}) = m_0(C^{-N\ \ldots 0\ \ldots\ N'}_{\sigma_{-N}\ldots\sigma_0\ldots\sigma_{N'}}) = \lambda^{N+N'} \delta^u_{\sigma_{N'}} \delta^s_{\sigma_{-N}} \prod_{j=-N}^{N'-1} T_{\sigma_j \sigma_{j+1}}$$

The volume distribution μ_0 becomes a **Gibbs dist.** *m_0 for a "spin system" interacting with a hard core (expressed by $T_{\sigma_i \sigma_{i+1}} \equiv 1$) and 0 potential.*

For $\varepsilon>0$ (small) the sides of the rectangles bedome slightly curved and the area of the "rectangle" $\cap_{-N}^{N'} S^{-k} P_{\sigma_k}$ is approximately

$$\sin\varphi(x)\prod_{i=1}^{N}\lambda_u^{-1}(S^{-i}x)\prod_{i=1}^{N'}\lambda_s(S^i x)\delta^u_{\sigma_{-N}}\delta^s_{\sigma_{N'}}$$

x is a point in $\cap_{-N}^{N'} S^{-k} P_{\sigma_k}$ and the sides are computed by composition rule of differentiations and $\lambda^u(x), \lambda^s(x)$ denote the expansion and the contraction of the length of the unstable/stable manifold under S.

The functions $\lambda^u(x), \lambda^s(x), \varphi(x)$ are no longer constant but can be shown to be Hölder continuous with exponent $\alpha<1$ ($i=u,s$):

$$|\lambda^i(x)-\lambda^i(x')|\le L\,d(x,x')^\alpha, \qquad |\sin\varphi(x)-\sin\varphi(x')|\le L\,d(x,x')^\alpha$$

if $d(x,x')$ is the distance between x and x' and $L>0$ is a constant. Since the history code determines points exponentially fast with the number of specified digits the latter functions can be expressed as functions of the histories $-\log\lambda(^i x)\equiv\Lambda^i(\underline{\sigma}),\ i=u,s,\ -\log\sin(x)=s(\underline{\sigma})$. and verify for $\ell,\delta>0$

$$|\Lambda^i(\underline{\sigma})-\Lambda^i(\underline{\sigma}')|\le \ell\,d(\underline{\sigma},\underline{\sigma}')^\delta, \qquad |s(\underline{\sigma})-s(\underline{\sigma}')|\le \ell\,d(\underline{\sigma},\underline{\sigma}')^\delta$$

The Hölder regularity allows us to represent Λ^i, s as exponentially convergent series, *e.g.*

$$\Lambda^u(\underline{\sigma}) = \Lambda^u(\underline{1}) + (\Lambda^u(\underline{1}\sigma_0\underline{1}) - \Lambda^u(\underline{1})) + \\ + (\Lambda^u(\underline{1}\sigma_{-1}\sigma_0\sigma_1\underline{1}) - \Lambda^u(\underline{1}\sigma_0\underline{1})) + \ldots = \sum_{k=0}^{\infty} \Phi^u_k(\sigma_{-k}\ldots\sigma_k)$$

Noting that the history of $S^k x$ is $\tau^k\underline{\sigma}$

$$\sin\varphi(x) \prod_{i=1}^{N} \lambda_u^{-1}(S^{-i}x) \prod_{i=1}^{N} \lambda_s(S^i x)\delta^u_{\sigma_{-N}}\delta^s_{\sigma_N} = e^{-s(\underline{\sigma}) - \sum_{i=1}^{N} \Lambda^u(\tau^{-i}\underline{\sigma}) - \sum_{i=1}^{N} \Lambda^s(\tau^i\underline{\sigma})} = \\ = e^{-\sum_{h\ge k\ge 1}^{N} \Phi^u_k(\tau^h\underline{\sigma}) - \sum_{h\ge k\ge 1}^{N} \Phi^s_k(\tau^{-h}\underline{\sigma}) + c(\underline{\sigma}) + \Delta_N(\underline{\sigma})}$$

$\Delta_N(\underline{\sigma})$ depends only on the σ_j with j close to $\pm N$ in the sense that $|\Delta_N(\underline{\sigma}) - \Delta_N(\underline{\sigma}')| < De^{-\kappa\ell}$ if $\underline{\sigma}$ and $\underline{\sigma}'$ differ only at distance $\ge \ell$ from $\pm N$; likewise $c(\underline{\sigma})$ depends only on the values of $\underline{\sigma}$ at sites near the origin.

Conclusion:

(1) The volume distribution μ_0 is **Gibbs state with potential Φ^u to the right of the origin and Φ^s to the left, locally perturbed near the origin**.
(2) μ_0-random initial data have the same future statistics μ and the same past statistics μ_-. They are Gibbs states for a 1-dim. spin system with hard core and *different potentials.*

Existence of statistics is solved. How to compute expectations? *i.e.* how to compute $\Phi^u(\underline{\sigma})$....

To compute Φ^u first construct the conjugacy H by the following

Let $\underline{f}(\underline{\varphi})$ be a real trigonometric polynomial, $\underline{f}(\underline{\varphi}) = \sum_{\underline{\nu}\in\mathbb{Z}^2,|\underline{\nu}|\le N} e^{i\underline{\nu}\cdot\underline{\varphi}} \underline{f}_{\underline{\nu}}$, defined on the two-dimensional torus $\mathbb{T}^2$ and let

$$S_\varepsilon\underline{\varphi} = S_0\underline{\varphi} - \varepsilon\underline{f}(\underline{\varphi}), \qquad \text{with} \qquad S_0 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

For $\beta \in (0,1)$ there exist $C(\beta) < \infty$ and $\varepsilon_0(\beta) > 0$ such that for $|\varepsilon| < \varepsilon_0(\beta)$ the equation

$$H \circ S_0 = S_\varepsilon \circ H$$

defines a unique homeomorphism $\underline{\varphi} \to H(\underline{\varphi})$ which is analytic in ε in the complex disk $|\varepsilon| < \varepsilon_0(\beta)$ and Hölder continuous with exponent at least as large as β and with Hölder continuity modulus bounded by $C(\beta)$.

Once H is constructed we shall have also consytructed a Markov partition for S_ε as H–image of the one for S. The $W^u(x), W^s(x)$ are in fact given by parametric equations of the form

$$\underline{\varphi}(t) = H(\underline{\psi} + t\underline{v}_\alpha) \qquad t \in \mathbb{R}, \alpha = \pm,$$

not immediately useful because H is not differentiable.

Need the derivative $\lambda^u(x)$ of S_ε in the direction of the tangent plane to $W^u(x)$. In principle, from the parametric rep., it is not even clear its existence.

Calling $\widehat{\Omega}$ the (*non compact*) space $\mathbb{T}^2 \times \mathbb{R}^2$ we define

$$\widehat{S}_0(\underline{\varphi}, \underline{v}) \stackrel{def}{=} (S_0\underline{\varphi}, S_0\underline{v}), \qquad \widehat{S}_\varepsilon(\underline{\varphi}, \underline{v}) \stackrel{def}{=} (S_0\underline{\varphi} + \varepsilon \underline{f}(\underline{\varphi}), S_0\underline{v} + \varepsilon(\underline{v} \cdot \partial_{\underline{\varphi}})\underline{f}(\underline{\varphi})),$$

and we find an isomorphism between $\widehat{S}_\varepsilon$ and $\widehat{S}_0$ or, since this turns out to be in general impossible, between $\widehat{S}_\varepsilon$ and $\widehat{S}_{0,\varepsilon}$ defined by

$$\widehat{S}_{0,\varepsilon}(\underline{\varphi}, \underline{v}) = (S_0\underline{\varphi}, (S_0 + \Gamma_\varepsilon(\underline{\varphi}))\underline{v}).$$

with $\Gamma_\varepsilon(\underline{\varphi})$ a matrix *diagonal* on the basis $\underline{v}_\pm$ (on which S_0 is diagonal too). Therefore we look for a map $\widehat{H}$ of a simple form and such that $\widehat{S}_\varepsilon \circ \widehat{H} = \widehat{H} \circ \widehat{S}_{0,\varepsilon}$, *i.e.* (setting $H(\underline{\psi}) = \underline{h}(\underline{\psi})$)

$$\widehat{H} \,:\, (\underline{\psi}, \underline{w}) \longleftrightarrow (\underline{\varphi}, \underline{v}) = (\underline{\psi} + \underline{h}(\underline{\psi}), \underline{w} + K(\underline{\psi})\underline{w}),$$

with

$$\Gamma(\underline{\psi}) = \begin{pmatrix} \gamma_+(\underline{\psi}) & 0 \\ 0 & \gamma_-(\underline{\psi}) \end{pmatrix}, \qquad K(\underline{\psi}) = \begin{pmatrix} 0 & k_+(\underline{\psi}) \\ k_-(\underline{\psi}) & 0 \end{pmatrix}.$$

The above is useful because the diagonal form of Γ implies that a tangent vector at $\underline{\psi}$ parallel to $\underline{v}_+$ is mapped by $\widehat{S}_{0,\varepsilon}$ into $(\lambda_+ + \gamma_+(\underline{\psi}))\underline{v}_+$, tangent at $S_0\underline{\psi}$ so that in the coordinates $(\underline{\psi}, \underline{v})$ is a tangent vector to the unstable manifold.

The unstable tangent vector corresponding to $(\underline{\psi}, \underline{v}_+)$ at the point $\underline{\varphi} = H(\underline{\psi})$ is simply $\underline{v}_+ + K(\underline{\psi})\underline{v}_+$ *by construction* $\underline{w}^u(\underline{\psi}) = \underline{v}_+ + k_+(\underline{\psi})\underline{v}_-$; so that if we know how to compute K, Γ and if K, Γ are Hölder continuous in $\underline{\psi}$ we get that the image of $\underline{w}^u(\underline{\psi})$ is $(\lambda_+ + \gamma_+(\underline{\psi}))(\underline{v}_+ + k_+(S_0\underline{\psi})\underline{v}_-)$ so that the expansion is

$$\lambda^u(\underline{\psi}) = (\lambda_+ + \gamma_+(\underline{\psi}))\frac{|\underline{v}_+ + k_+(S_0\underline{\psi})\underline{v}_-|}{|\underline{v}_+ + k_+(\underline{\psi})\underline{v}_-|}$$

Therefore *all we need is* $\underline{h}(\underline{\psi}), \gamma_+(\underline{\psi})$**: we need the** γ_+ **function to compute the potential** $\Phi^u(\underline{\psi})$ **and** $\underline{h}(\underline{\psi})$ **to express the results in the original coordinates.** Note that $k_+(\underline{\psi})$ is *not* necessary (although it will be computed).

The methods that we use to solve the equations $S_\varepsilon \circ H = H \circ S_0$ and $\widehat{S}_{0,\varepsilon}(\underline{\varphi}, \underline{v}) = (S_0\underline{\varphi}, (S_0 + \Gamma_\varepsilon(\underline{\varphi}))\underline{v})$. are the same and for simplicity I illustrate the first.

Write $\underline{\varphi} = H(\underline{\psi}) = \underline{\psi} + \underline{h}(\underline{\psi})$, $\underline{\psi} \in \mathbb{T}^2$; then $S_\varepsilon \circ H = H \circ S_0$ becomes

$$S_0 \underline{h}(\underline{\psi}) - \underline{h}(S_0 \underline{\psi}) = \varepsilon \underline{f}(\underline{\psi} + \underline{h}(\underline{\psi}))$$

Hence we look for $\underline{h}(\underline{\psi}) = \varepsilon \underline{h}^{(1)}(\underline{\psi}) + \varepsilon^2 \underline{h}^{(2)}(\underline{\psi}) + \cdots$. For instance to first order

$$S_0 \underline{h}^{(1)}(\underline{\psi}) - \underline{h}^{(1)}(S_0 \underline{\psi}) = \underline{f}(\underline{\psi}).$$

Let $\underline{v}_+, \underline{v}_-$ the two eigenvectors of S_0; call $\lambda < 1$ the inverse of the largest one ($\lambda = (\sqrt{5}-1)/2$): so that $\lambda_+ = \lambda^{-1}, \lambda_- = -\lambda$. Split $\underline{f}, \underline{h}$ into components on $\underline{v}_\pm$:

$$\underline{f}(\underline{\psi}) = f_+(\underline{\psi})\underline{v}_+ + f_-(\underline{\psi})\underline{v}_-, \qquad \underline{h}(\underline{\psi}) = h_+(\underline{\psi})\underline{v}_+ + h_-(\underline{\psi})\underline{v}_-,$$

and the equations for $h_\pm^{(1)}$ are

$$\begin{aligned} \lambda_+ h_+^{(1)}(\underline{\psi}) - h_+^{(1)}(S_0\underline{\psi}) &= f_+(\underline{\psi}), \\ \lambda_- h_-^{(1)}(\underline{\psi}) - h_-^{(1)}(S_0\underline{\psi}) &= f_-(\underline{\psi}). \end{aligned}$$

$$\begin{aligned}\lambda_+ h_+^{(1)}(\underline{\psi}) - h_+^{(1)}(S_0\underline{\psi}) &= f_+(\underline{\psi}),\\ \lambda_- h_-^{(1)}(\underline{\psi}) - h_-^{(1)}(S_0\underline{\psi}) &= f_-(\underline{\psi}).\end{aligned}$$

The equations can be solved by simply setting

$$h_\alpha^{(1)}(\underline{\psi}) = -\sum_{p\in\mathbb{Z}_\alpha} \alpha\lambda_\alpha^{-|p+1|\alpha} f_\alpha(S_0^p\underline{\psi}), \qquad \alpha=\pm,$$

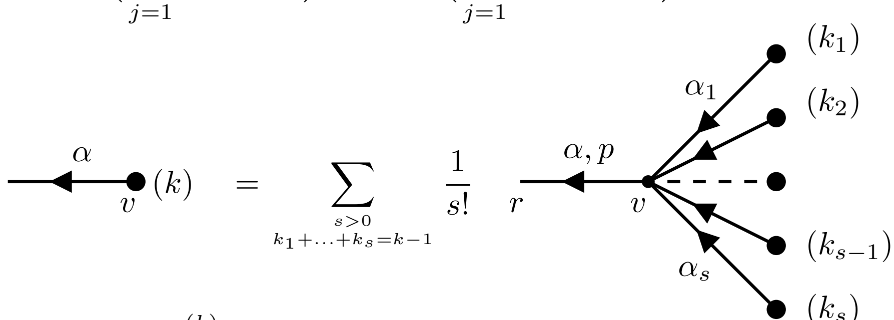
where $\mathbb{Z}_+ = [0,\infty)\cap\mathbb{Z}$ and $\mathbb{Z}_- = (-\infty,0)\cap\mathbb{Z}$. Therefore the equations for $h_\pm^{(k)}$ become

$$\begin{aligned}h_\alpha^{(k)}(\underline{\psi}) = \sum_{s=0}^{\infty}\frac{1}{s!}\sum_{\substack{k_1+\cdots+k_s=k-1,\,k_i\ge0\\ \alpha_1,\ldots,\alpha_s=\pm}}\sum_{p\in\mathbb{Z}_\alpha}\alpha\lambda_\alpha^{-|p+1|\alpha}\cdot\\ \cdot\Big(\prod_{j=1}^{s}(\underline{v}_{\alpha_j}\cdot\partial_{\underline{\varphi}})\Big) f_\alpha(S_0^p\underline{\psi})\Big(\prod_{j=1}^{s}h_{\alpha_j}^{(k_j)}(S_0^p\underline{\psi})\Big),\end{aligned}$$

Proceeding as usual in perturbation theory we shall use a graphical representation.

$$h_\alpha^{(k)}(\underline{\psi}) = \sum_{s=0}^{\infty} \frac{1}{s!} \sum_{\substack{k_1+\cdots+k_s=k-1,\ k_i\ge 0\\ \alpha_1,\ldots,\alpha_s=\pm}} \sum_{p\in\mathbb{Z}_\alpha} \alpha\lambda_\alpha^{-|p+1|\alpha}\cdot$$

$$\cdot \Big(\prod_{j=1}^{s} (\underline{v}_{\alpha_j}\cdot\partial_{\underline{\varphi}})\Big) f_\alpha(S_0^p\underline{\psi}) \Big(\prod_{j=1}^{s} h_{\alpha_j}^{(k_j)}(S_0^p\underline{\psi})\Big),$$



where the l.h.s. represents $h_\alpha^{(k)}(\underline{\psi})$. Representing again the graph elements that appear on the r.h.s. one obtains an expression for $h_\alpha^{(k)}(\underline{\psi})$ in terms of *trees*, oriented toward the root.

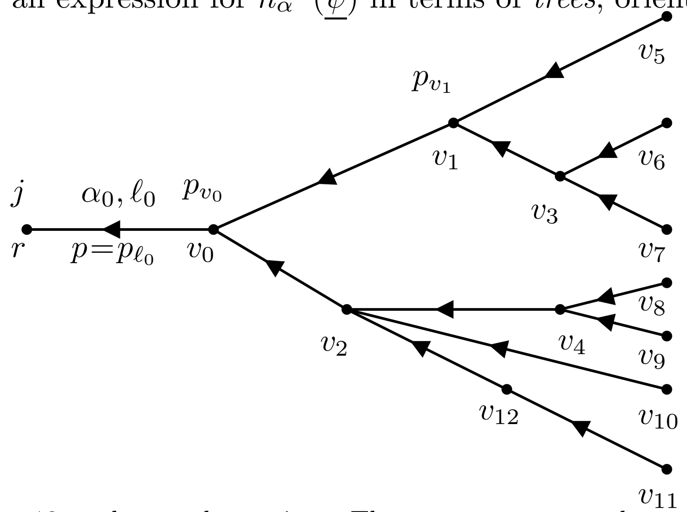


Fig. 3: A tree ϑ with $k = 13$, and some decorations. The arrows represent the partial ordering on the tree.

A tree ϑ with k nodes carries on the branches ℓ a pair of labels α_ℓ, p_ℓ, with $p_\ell \in \mathbb{Z}$ and $\alpha_\ell \in \{-,+\}$, and on the nodes v a pair of labels α_v, p_v, with $\alpha_v = \alpha_{\ell_v}$ and $p_v \in \mathbb{Z}_{\alpha_v}$ with

$$p(v) \equiv p_{\ell_v} = \sum_{w\succeq v} p_v,$$

where the sum is over the nodes following v (*i.e.* over the nodes along the path connecting v to the root), ℓ_v denotes the branch $v'v$ exiting from the node v.

$$
\begin{aligned}
h_\alpha^{(k)}(\underline{\psi}) = \sum_{s=0}^{\infty} \frac{1}{s!} \sum_{\substack{k_1+\cdots+k_s=k-1,\, k_i\ge 0\\ \alpha_1,\ldots,\alpha_s=\pm}} \sum_{p\in\mathbb{Z}_\alpha} \alpha\lambda_\alpha^{-|p+1|\alpha} \cdot \\
\cdot \Big(\prod_{j=1}^{s} (\underline{v}_{\alpha_j} \cdot \partial_{\underline{\varphi}}) \Big) f_\alpha(S_0^p\underline{\psi}) \Big(\prod_{j=1}^{s} h_{\alpha_j}^{(k_j)}(S_0^p\underline{\psi}) \Big),
\end{aligned}
$$

hence to each tree we shall assign a *value* given by

$$
\mathrm{Val}(\vartheta) = \prod_{v\in V(\vartheta)} \frac{\alpha_v}{s_v!} \lambda_{\alpha_v}^{-|p_v+1|\alpha_v} \Big(\prod_{j=1}^{s_v} \partial_{\alpha_{v_j}} \Big) f_{\alpha_v}(S_0^{p(v)}\underline{\psi}),
$$

If $\Theta_{k,\alpha}$ denotes the set of all trees with k nodes and with label α associated to the root line, then one has

$$
h_\alpha(\underline{\psi}) = \sum_{k=1}^{\infty} \varepsilon^k \sum_{\vartheta\in\Theta_{k,\alpha}} \mathrm{Val}(\vartheta),
$$

and the "only" problem left is to estimate the radius of convergence of the above formal power series.

For this purpose it is convenient to study the Fourier transform of the function $h_\alpha(\underline{\psi})$. This is easily done graphically because it is enough to attach a label $\underline{\nu}_v \in \mathbb{Z}^2$ to each node and define the momentum that flows on the tree branch $v'v$, *i.e.* $\underline{\nu}_{\ell_v} \stackrel{def}{=} \sum_{w \preceq v} \underline{\nu}_w$. Then

$$h_\alpha(\underline{\psi}) = \sum_{k=1}^{\infty} \varepsilon^k \sum_{\underline{\nu} \in \mathbb{Z}^2} e^{i\underline{\nu}\cdot\underline{\psi}}\, h^{(k)}_{\alpha,\underline{\nu}},$$

with

$$\begin{aligned} h^{(k)}_{\alpha,\underline{\nu}} = & \sum_{\vartheta \in \Theta_{k,\underline{\nu},\alpha}} \sum_{p_v \in \mathbb{Z}_{\alpha_v}} \Big(\prod_{v \in V(\vartheta)} \frac{\alpha_v}{s_v!} \lambda_{\alpha_v}^{-|p_v+1|\alpha_v} f_{\alpha_v, S_0^{-p(v)}\underline{\nu}_v} \Big) \cdot \\ & \cdot \prod_{\substack{v \in V(\vartheta) \\ v' \neq v_0}} \big(- S_0^{-p(v')} \underline{\nu}_{v'} \cdot \underline{v}_{\alpha_v} \big), \end{aligned}$$

where $\Theta_{k,\underline{\nu},\alpha}$ denotes the set of all trees with k nodes and with labels $\underline{\nu}$ and α associated with the root line.

Calling $F = \max_{\underline{\nu}} |f_{\underline{\nu}}|$ we can estimate $\sum_{\underline{\nu}} |\underline{\nu}|^\beta |h^{(k)}_{\alpha,\underline{\nu}}|$. Consider first the case $\beta = 0$: there are only $(2N+1)^2$ possible choices for each $\underline{\nu}_v$, given p_v, such that $|S_0^{-p_v} \underline{\nu}_v| \le N$. Hence fixed ϑ, $\{\alpha_v\}_{v\in V(\vartheta)}$ and $\{p_v\}_{v\in V(\vartheta)}$ the remaining sum of products in is bounded by

$$(3N)^{2k} N^k F^k \prod_{v\in V(\vartheta)} \frac{\lambda^{|p_v|}}{s_v!}.$$

The sum over the p_v is a geometric series bounded by $(2/(1-\lambda))^k$.

The combinatorial problem is well known: the factor $\prod_v (1/s_v!)$ becomes, after summing over all the trees, simply bounded by 2^{3k}, (2^{2k} due to the number of trees for fixed labels and 2^k due to the sum of labels α_v).

Therefore for $\beta = 0$ we have that the conjugating function H exists and that inside the complex domain $|\varepsilon| < \varepsilon_0(0) \stackrel{def}{=} (3N)^{-3} F^{-1} 2^{-4} (1-\lambda)$ it is uniformly continuous and uniformly bounded with a uniformly summable Fourier transform.

Taking $\beta > 0$ requires estimating $|\underline{\nu}|^\beta$: we bound it by $\sum_v |\underline{\nu}_v|^\beta$. Then we can make use of the fact that $|S_0^{-p(v)}\underline{\nu}_v| \le N$ to infer that $|\underline{\nu}_v| \le \lambda^{-|p(v)|}BN$, where $B \ge 1$ is a suitable constant (because the eigenvectors of S_0 have components with ratio a quadratic irrational, hence Diophantine).

The sum $\sum_v |\underline{\nu}_v|^\beta$ is over k terms which can be estimated separately so that we can write $\sum_v |\underline{\nu}_v|^\beta \le k|\underline{\nu}_{\overline{v}}|^\beta$ where $|\underline{\nu}_{\overline{v}}| = \max_v |\underline{\nu}_v|$. This can be taken into account by an extra factor $(BN)^\beta \lambda^{-\beta|p(\overline{v})|} \le BN\lambda^{-\beta\sum_v |p_v|}$.

Therefore if $\beta < 1$ the bound that we found for $\beta = 0$ is modified into

$$\varepsilon_0(\beta) = (3N)^{-3}F^{-1}(1-\lambda^{1-\beta})2^{-5}.$$

This shows that $H(\underline{\psi})$ analytic in ε in the disk with radius $\varepsilon_0(\beta)$. Note that $\varepsilon_0(\beta) \to 0$ for $\beta \to 1$.