On the Cassandro–Olivieri renormalization method and the foundations of nonequilibrium statistical mechanics.

Marzio Cassandro's 65-th birthday meeting: he is among the few who appreciate the power of the cluster expansion and who applied it to various problems in Statistical Mechanics and Field Theory devising ways of combining the technique with the renormalization group with the aim of converting heuristic ideas into rigorous results and methods.

$$H_\Lambda = -\sum_i h\sigma_i - \sum_{i,j\in\Lambda} J_{ij}\sigma_i\sigma_j, \qquad i,j\in\Lambda \stackrel{def}{=} [-\frac12 L, \frac12 L] \subset \mathbb{Z}$$

Ising model with $|J_{ij}| \le C|i-j|^{-2-\varepsilon}$ does not have phase transitions for $\varepsilon > 0$ (Ruelle) and does with $J_{ij} = -C|i-j|^{-2-\varepsilon}$ for $\varepsilon < 0$ (Dyson).

Surprisingly (for me) if $|J_{ij}| \le C|i-j|^{-2-\varepsilon}$, $\varepsilon > 0$, the free energy and correlations are analytic in temperature and h (Dobrushin).

At the same time Renormalization group was developed and the simplest problem was its application to the 1–dim. Ising model.

RG = map of potential into a new "equivalent" potential: $\Phi' = \mathcal{R}\Phi$ apparently far more complicated but in fact (hopefully) simpler.

Example of $\mathcal{R}$ is "decimation" (corresponding to the potential generating the distribution of every other spin).

The work [CO81] first developed a rigorous and complete theory of decimation transformation for **long range** 1-dim. systems.

Strategy: apply decimation renormalization enough many times and check that the new potential (for the spins left after the decimation) is Gibbsian with potential Φ_R which, although very complicated, fulfills the requirements for an application of the general cluster expansion technique which, whenever working, yields analyticity and clustering.

This achieved a new, conceptually very clear, proof of the quoted Dobrushin's result.

Remarkable: the RG map is *not* applied infinitely many times but **only** finitely many times. Delicate questions in checking that $\mathcal{R}^n\Phi \xrightarrow[n\to\infty]{} 0$ and courageously dealing with the corresponding exchange of limits problems.

Given Φ the prob. dist. of blocks of spins β of size τ spaced by blocks η of size h remains a Gibbs distribution with potential smaller and smaller as h grows provided τ is *fixed* large enough.

Once the interaction is small one **stops and applies the cluster expansion** thus describing analytically and in "arbitrary detail" the properties of the corresponding Gibbs state.

After the Dobrushin's analyticity result, justly appreciated for mathematical depth, came the novelty and beauty of the technique of [CO81].

Anosov maps: *paradigm of chaotic motions.* Existence of statistics μ (SRB): for *all* smooth F

$$\lim_{T\to+\infty} T^{-1}\sum_{j=0}^{T-1} F(S^j x) = \int_\Omega F(y)\,\mu(dy) \qquad L-a.e.$$

Natural dynamical coordinates for the points $x\in\Omega$. There is a partition $\mathcal{P}=(P_0,P_1,\ldots,P_n)$ s.t. history $\underline{\sigma}=\{\ldots,\sigma_{-1},\sigma_0,\sigma_2,\ldots\}$ is 1-to-1 between x and "compatible" spin config.

This means that if $T_{\sigma\sigma'}=1$ if

$$\mathrm{int}(SP_\sigma)\cap\mathrm{int}(P_{\sigma'})\neq\emptyset$$

and $T_{\sigma\sigma'}=0$ otherwise then a sequence $\underline{\sigma}$ is compatible if $T_{\sigma_i,\sigma_{i+1}}\equiv 1$ for all $i\in\mathbb{Z}$: one says that configuration space is the *space of spin–*$\frac{n}{2}$ *configurations with nearest neighbor hard core.* The matrix T is *mixing*, *i.e.* there exists $a_0>0$ such that $T^{a_0}_{\sigma\sigma'}>0$ for all σ,σ'.

SRB distribution $\mu(d\underline{\sigma}\,)$ is simply a Gibbs distribution with potential Φ_+. The potential $\Phi_+=\{\Phi_{+,I}(\underline{\sigma}_{\,I})\}$ is

(a) translation invariant
(b) short range: there exist $\varphi>0,\kappa>0$ such that

$$\max_{\underline{\sigma}_{\,I}}|\Phi_{+,I}(\underline{\sigma}_{\,I})|\le\varphi\, e^{-\kappa\mathrm{diam}(I)},\qquad \text{if } |I|>1$$

(c) $\Phi_I\ne0$ only if I is an interval ("*Fisher potential*", or "*extended nearest neighbor potential*"), with hard core T.

In [Co81] it is shown that if a potential is a pair potential verifying (b) and if $T_{\sigma\sigma'}\equiv1$, *i.e.* "no hard core", then by decimation it becomes a potential verifying (b) with φ,κ^{-1} prefixed arbitrarily small (much more: as (b) is replaced by a decay $|i-j|^{-(2+\varepsilon)}$, $\varepsilon>0$).

Cluster expansion:

On $\mathbb{Z}^d$ and no hard core. Let Φ be not necessarily translation invariant and set, for $\kappa > 0$,

$$\|\overline{\Phi}\|_\kappa \stackrel{def}{=} \sup_\xi \sum_{X \ni \xi, |X|>1} \|\Phi_X\| \, e^{\kappa\delta(X)},$$

where $\delta(X)$ denotes the tree diameter of X.

(i) The partition function $Z_\Phi(\Lambda)$ can be expressed as

$$Z_\Phi(\Lambda) = \big(\prod_{\xi\in\Lambda} \Xi_1(\xi)\big) \, \exp\Big(\sum_{X\subset\Lambda} \varphi^T(X)\zeta(X)\Big),$$

where $\Xi_1(\xi) = \sum_\sigma e^{-\Phi_\xi(\sigma)}$, $X = \{\gamma_1, \ldots, \gamma_n\}$ and $\varphi^T(X)$ are combinatorial coeff., $\zeta(X) = \prod_{\gamma\in X} \zeta(\gamma)$ for suitably defined $\zeta(\gamma)$, which verify

$$Z_\Phi(\Lambda) = \big(\prod_{\xi\in\Lambda} \Xi_1(\xi)\big)\, \exp\Big(\sum_{X\subset\Lambda} \varphi^T(X)\zeta(X)\Big),$$

—————————————————————————————————-

$$\zeta(X) = \prod_{\gamma\in X} \zeta(\gamma), \qquad |\zeta(\gamma)| \le (e^{2\|\overline{\Phi}\|_\kappa+1})^{|\gamma|}(\|\overline{\Phi}\|_\kappa)^{\nu(\gamma)} e^{-\kappa\delta(\gamma)}$$

where $\nu(\gamma) =$ *min. number of sets* $Y, |Y|\ge 2, \Phi_Y \ne 0$, *to cover* γ.
(ii) If $\nu(\gamma) > c|\gamma|$ *for* $|\gamma| > 1$ *and a suitable* $0 < c \le 1$ *there exist* $B, \varepsilon > 0$, **independent of the size** n **of the spins**, *such that if* $\|\overline{\Phi}\|_\kappa < \varepsilon$

$$\sup_{\xi\in\mathbb{Z}^d} \sum_{\widetilde{X}\ni\xi, \delta(X)\ge r} |\varphi^T(X)||\zeta(X)| < B\,\varepsilon\, e^{-\kappa r}.$$

(iii) Furthermore "analyticity" and "clustering".

—————————————————————————————————-

Extension of $(1+z) = \exp\log(1+z) = \exp\sum_{k=1}^\infty \frac{(-1)^{k-1}}{k} z^k$!!!

Space time chaotic evolutions

Recently nonequilibrium statistical mechanics started to develop with attention paid to rigorous foundations and questions of principle.

Simplest systems may play the role of Ising m.: *lattices ($d \ge 1$) of Anosov maps*? at each point is a pair of angles (think of watch arms). Configurations: $\underline{\varphi} = \{\underline{\varphi}_{-L/2}, \ldots, \underline{\varphi}_0, \underline{\varphi}_1, \ldots, \underline{\varphi}_{L/2}\}$

$$\bullet_0^{\underline{\varphi}_0} \qquad \bullet_1^{\underline{\varphi}_1} \qquad \bullet_2^{\underline{\varphi}_2} \ldots \qquad \ldots \bullet_{V-1}^{\underline{\varphi}_{V-1}}$$

and the "nearest neighbor evolution" is $\underline{\varphi}' = \mathcal{S}\,\underline{\varphi}$ with

$$\underline{\varphi}'_j = \mathcal{S}\,\underline{\varphi} \stackrel{def}{=} \begin{pmatrix} \varphi'_{j1} \\ \varphi'_{j2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varphi_{j1} \\ \varphi_{j2} \end{pmatrix} + \varepsilon \begin{pmatrix} \delta_1(\underline{\varphi}_j) + f_1(\underline{\varphi}_{nn(j)}) \\ \delta_2(\underline{\varphi}_j) + f_2(\underline{\varphi}_{nn(j)}) \end{pmatrix}$$

e.g.

$$\begin{aligned} &f_1 = \sin(\varphi_{j+1,1} - 2\varphi_{j,1} + \varphi_{j-1,1}), && f_2 = 0 \\ &\delta_1(\varphi) = \sin\varphi, && \delta_2 = 0 \end{aligned}$$

(1) if $\varepsilon = 0$ maps are independent and each $\underline{\varphi}_j$ is a symbolic sequence $\underline{\sigma}_{j,t}$. A state $\underline{\varphi} = (d+1)$–dim. array $\underline{\sigma}_{j,t}$ each with $n+1$–values. Time evolution: shift $\uparrow$; space transl. shift $\rightarrow$. "Thermodynamic formalism".

(2) Pertubation keeps the representation as a spin–$\frac{n}{2}$ valid at ε small enough. How small is **independent** on the spacial extension L.

Key: (thesis of Falco, and Bonetto, Falco, Giuliani))
Given $\beta \in (0,1)$, *there is* $\varepsilon_0(\beta) > 0, C(\beta) < \infty$ *s.t. for all* L *and* $|\varepsilon| < \varepsilon_0(\beta)$ *there is a* $H : \Omega_L \longleftrightarrow \Omega_L$ *and* κ *s.t.*

$$H \circ \mathcal{S}_0 = \mathcal{S}_\varepsilon \circ H,$$

analytic *in* $|\varepsilon| < \varepsilon_0(\beta)$ *and Hölder continuous in* $\underline{\varphi}$ *with*

$$|(H(\underline{\varphi}))_\xi - (H(\underline{\varphi}'))_\xi| \le C(\beta) e^{-\kappa|\xi-\xi'|} |\underline{\varphi}_{\xi'} - \underline{\varphi}'_{\xi'}|^\beta$$

for $|\varepsilon| < \varepsilon_0(\beta)$ *and* $\kappa = -const \log(|\varepsilon|/\varepsilon_0(\beta)) \xrightarrow[\varepsilon\to 0]{} \infty$.

The above result leads to identifying the SRB distribution with the Gibbs state with a potential on $\mathbb{Z}^{d+1}$

$$|\Phi_X(\underline{\sigma}_X)| < C e^{-(\kappa\delta_\perp(X)+\kappa_0\delta_\|(X)+\kappa n(X))}$$

here $n(X)$ is the number of timelike intervals whose union is X and $\kappa \xrightarrow[\varepsilon\to 0]{} \infty$. *I.e.* the tree diameter decay in space directions is as fast as wished while in the time direction it is fast but with a fixed range.

SRB distribution μ_ε is Gibbs on $(d+1)$-dimensional lattice with *n.n. hard core* in time direction and a *longer range exponentially decreasing* "many-body" potential Φ with $||\Phi||_\kappa$ small with ε and analytic.

First problem: potential Φ_+ is *not small* having a hard core. *But* extension of [CO81] to the hard core case easy, at least if the potential decays fast at infinity as in our cases. After all in our cases [CO81] holds no matter how large the potential: hence also for infinite potentials.

The second problem is that the system is *not* one dimensional.

Proposition 5 ([BFG03], [GBG03])
Consider a $\frac{n}{2}$-spin system subjected to a oriented hard core potential s.t. there are $\kappa, \kappa_0 > 0$ for which $\|\overline{\Phi}\|_{\kappa,\kappa_0} < \infty$. Then given κ_0 there is a constant $\overline{\kappa}$ such that if $\kappa > \overline{\kappa}$ the correlation functions $\mu_\Phi(C^V_{\underline{\sigma}_V})$ are analytic functions of Φ in the region $\|\overline{\Phi}\|_{\kappa,\kappa_0} < 1$. Furthermore the Gibbs distribution μ_Φ is exp. mixing.

The extension proceeds by considering a box Λ with sides L of $\mathbb{Z}^{d+1}$.

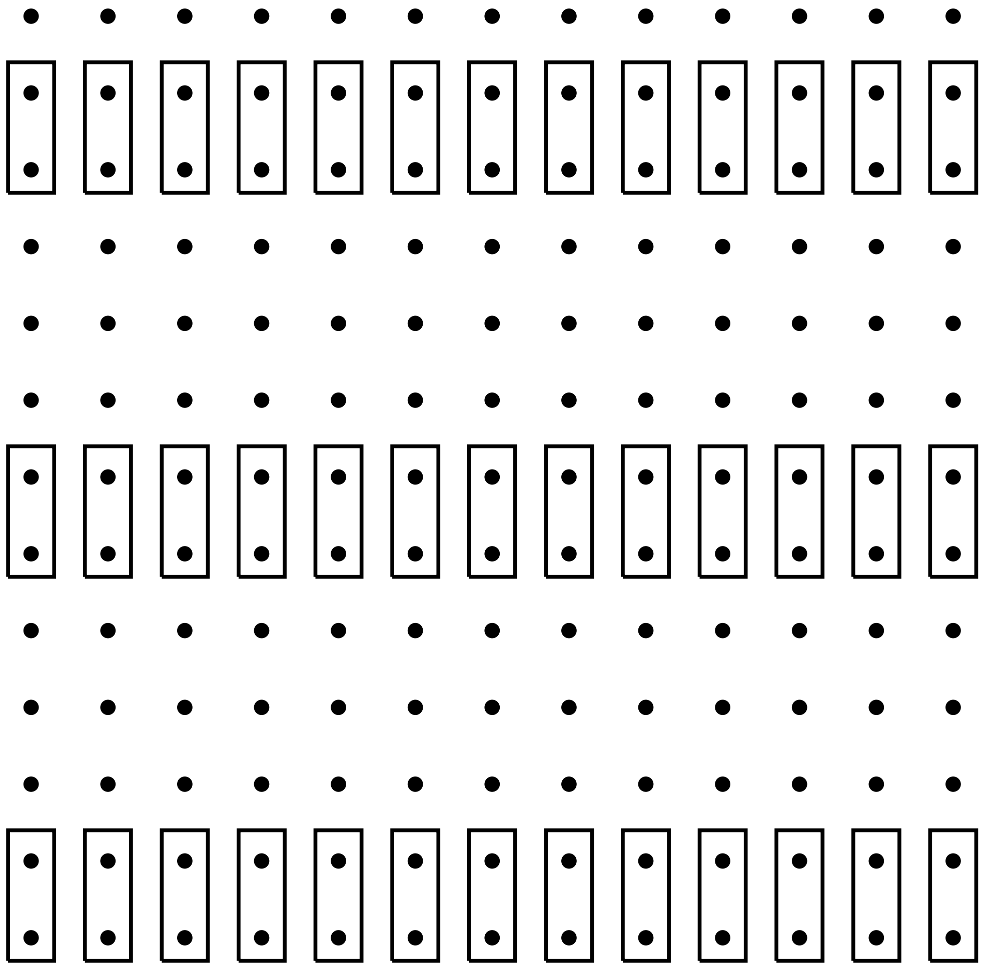

Fig. 1: The lattice points surrounded by a contour represent the β–blocks of length $\tau = 2$ in the figure; while the vertical intervals outside the blocks represent the η blocks of length $h = 3$ int he figure. In applications, however, $\tau \gg 2$ and $h \gg \tau \gg 3$.

Fix all spins of B-type (which can be arbitrarily assigned among the T–compatible strings β of length τ) and to sum over the spins of H–type.

If $h > \tau > a_0$ where a_0 is the mixing time for the symbols **no compatibility** in fixing configurations of distinct β-blocks; nor there will be any compatibility to respect the configurations of distinct η-blocks: *of course* there will be compatibility conditions between β and η blocks.

Distribution of η–blocks is $(d+1)$–dim system of weakly coupled spins.

In fact the part of the energy depending on $\underline{\eta}$ consists of a large "single block" energy plus a small energy $O(e^{-\kappa\tau})$: indeed it comes from potentials relative to sets containing points at distance at least τ or on different vertical "time" lines (which are small of order ε at least).

More mathematically the partition function in the space time region, say, $[-\frac{L}{2},\frac{L}{2}]\times[0,\ell(\tau+h)+\tau]$ can be expressed as

$$\sum_{\beta_{i,t}}\Big(\prod_{i=-L/2}^{L/2}\prod_{t=0}^{\ell-1}e^{-U^B_\tau(\beta_{i,t})}\Big)\Big(\prod_{i=-L/2}^{L/2}\prod_{t=0}^{\ell-1}Z_h(\beta_{i,t},\beta_{i+1,t})\Big)\cdot$$
$$\cdot\,\frac{\sum_{\eta_{i,t}}\prod_{i=-L/2}^{L/2}\prod_{t=0}^{\ell-1}e^{-U^H(\eta_{i,t})-W_\tau(\beta_{i,t},\eta_{i,t},\beta_{i+1,t})}e^{-\sum_X^*\Phi_X(\underline{\sigma}_X)}}{\prod_{i=-L/2}^{L/2}\prod_{t=0}^{\ell-1}Z_h(\beta_{i,t},\beta_{i+1,t})},$$

The $Z_h(\beta_{i,t},\beta_{i+1,t})$ are normalization functions which are partition functions of a single $\underline{\eta}$ block with boundary conditions $\beta_{i,t},\beta_{i+1,t})$:

$$Z_h(\beta_i,\beta_{i+1})=\sum_\eta e^{-U^H(\eta)}e^{-W_\tau(\beta_i,\eta,\beta_{i+1})}$$

The $\underline{\eta}$ blocks are “high spin” (spin of order $O((n+1)^h)$) and we can apply the proposition 2 to evaluate the partition function: we lack translation invariance but this is not required in the above cluster expansion.

Again $\underline{\beta}$ spins form a weakly coupled Gibbs state *without hard core.*

Result of summation over $\underline{\eta}$ blocks yields an *effective* short range potential btwn. $\underline{\beta}$ blocks: range $O(\kappa\tau)$ can be made as short as wished increasing τ. Also small *except possibly* the n.n. $\log Z(\beta_{i,t}, \beta_{i,t+1})$.

At this point we can fix h so large that

$$Z_h(\beta, \beta') = e^{-z(\beta)-z(\beta')-e^{-\kappa_0 h}\Delta(\beta,\beta')}$$

for a suitable $z(\beta)$ and with $\Delta(\beta, \beta') \le D$ for some $D, \kappa_0 > 0$. This is simply the property (1–dim. short range spins): *partition function on an interval of size h converges exp. to product of two factors resp. depending on the right (i.e. β') and left (i.e. β) b.c.*

Therefore also the n.n. potential is as small as liked by increasing the η–corridors size h. The potential acting on the β–blocks falls into the conditions of proposition 2 as well (again the size of the β–blocks is not important) and analyticity follows.