Resummations of self energy graphs and KAM theory

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Hamiltonian for a rotator system

$$\begin{aligned} H &= \frac{1}{2}(\underline{A}^{\,2} + \underline{B}^{\,2}) + \varepsilon f(\underline{\alpha}, \underline{\beta}) \\ \underline{A} &= (A_1, \ldots, A_r), \qquad \underline{B} = (B_1, \ldots, B_{n-r}) \\ \underline{\alpha} &= (a_1, \ldots, a_r), \qquad \underline{\beta} = (\beta_1, \ldots, \beta_{n-r}) \end{aligned}$$

with f an even trigonometric polynomial of degree N.

Unperturbed motions

$$\underline{A} = \underline{\omega}, \qquad \underline{B} = \underline{0}, \qquad |\underline{\omega} \cdot \underline{\nu}| > \frac{1}{C|\underline{\nu}|^{\tau}}$$

$$\underline{\alpha} = \underline{\alpha}_0 + \underline{\omega}\, t, \qquad \underline{\beta} = \underline{\beta}_0$$

$r = n \;\longleftrightarrow\;$ *maximal tori* or *KAM tori*

Are there motions of the "same type" in the perturbed system ?? *i.e.*

$$\begin{aligned} &\underline{A} = \underline{\omega} + \underline{H}(\underline{\psi}), \qquad \underline{B} = \underline{K}(\underline{\psi}) \\ &\underline{\alpha} = \underline{\psi} + \underline{h}(\underline{\psi}), \qquad \underline{\beta} = \underline{\beta}_0 + \underline{k}(\underline{\psi}) \qquad \text{with} \\ &\underline{\psi} \Rightarrow \underline{\psi} + \underline{\omega}\, t \qquad \text{is a solution?} \end{aligned}$$

It must be (*iff*):

$$\begin{aligned} (\underline{\omega} \cdot \underline{\partial}_{\,\psi})^2\, \underline{h}(\underline{\psi}) &= -\varepsilon\, \underline{\partial}_{\,\alpha} f(\underline{\psi} + \underline{h}(\underline{\psi}), \underline{\beta}_0 + \underline{k}(\underline{\psi})) \\ (\underline{\omega} \cdot \underline{\partial}_{\,\psi})^2\, \underline{k}(\underline{\psi}) &= -\varepsilon\, \underline{\partial}_{\,\beta} f(\underline{\psi} + \underline{h}(\underline{\psi}), \underline{\beta}_0 + \underline{k}(\underline{\psi})) \end{aligned}$$

and $\underline{\beta}_0$ must be an extremum for the $\underline{\alpha}$–average $\overline{f}(\underline{\beta}) = \int f(\underline{\alpha}, \underline{\beta})\frac{d\,\underline{\alpha}}{(2\pi)^r}$ of f:

$$(\underline{\omega} \cdot \underline{\partial}_{\,\psi})^2\, \underline{k}^{\,1}(\underline{\psi}) = -\,\underline{\partial}_{\,\beta} f(\underline{\psi}, \underline{\beta}_0) \;\Rightarrow\; \underline{0} = \underline{\partial}_{\,\beta} \overline{f}(\underline{\beta}_0)$$

Case of maximal tori: r=n

Given a tree ϑ with p nodes $v_0, \ldots, v_{p-1}$ and a root r
Associate with each node v a "*node momentum*" $\underline{\nu}_v \in \mathcal{Z}^r$ for $f_{\underline{\nu}_v} \neq 0$.
The root momentum will be a unit vector $\underline{n}$; and define a *line current*

$$\underline{\nu}(v) \stackrel{def}{=} \sum_{w<v} \underline{\nu}_w \qquad \text{and } \textbf{assume} \neq \underline{0}$$

Note that $0 < |\underline{\nu}_v| \leq N \Rightarrow |\underline{\omega} \cdot \underline{\nu}_v| > c > 0$ (Diophantine $\underline{\omega}$)

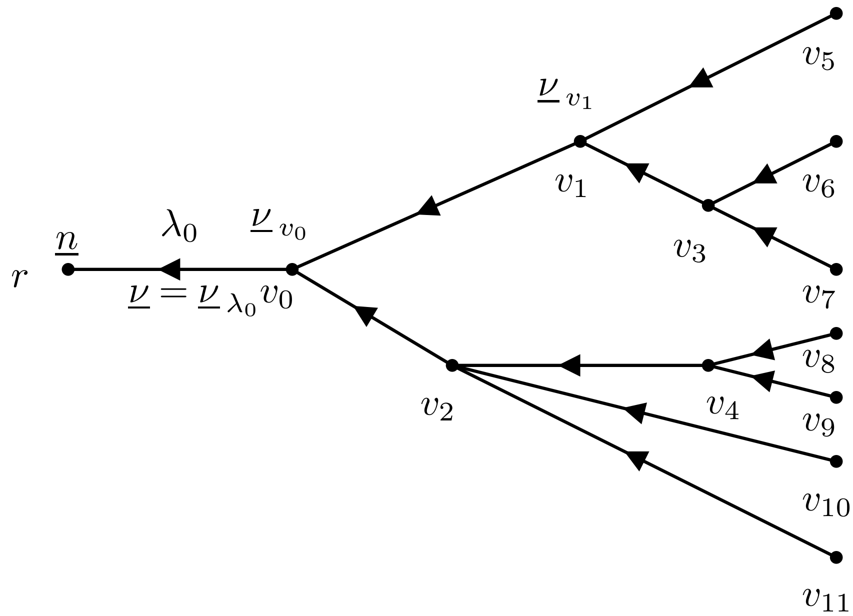


Fig. 1: A tree ϑ with $m_{v_0} = 2, m_{v_1} = 2, m_{v_2} = 3, m_{v_3} = 2, m_{v_4} = 2$ and $k = 12$, and some decorations. Only two mode label and one momentum are explicitly written; the number labels, distinguishing the branches, are not shown. The arrows represent the partial ordering on the tree.

Define the **value** of a tree ϑ with distinguishable branches by

$$\mathrm{Val}(\vartheta) = \frac{1}{p!} \Big(\prod_{\text{lines}\lambda=(v'v)\in\vartheta} \frac{\underline{\nu}_{v'} \cdot \underline{\nu}_v}{(\underline{\omega} \cdot \underline{\nu}(v))^2} \Big) \Big(\prod_{v\in\vartheta} f_{\underline{\nu}_v} \Big)$$

counting tree up to *pivoting equivalence.* Linstedt, Newcomb, Poincaré:

$$\underline{h}^{(p)}_{\underline{\nu}} \cdot \underline{n} = \sum_{\substack{\vartheta \\ \text{root current}=\underline{\nu}}} \mathrm{Val}(\vartheta)$$

Siegel–Bryuno–Pöschel bound

Define **scale** of $\lambda = (v'v)$ to be $n = 0, -1, -2, \ldots$ if $2^{n-1} < |\underline{\omega} \cdot \underline{\nu}(v)| \le 2^n$

$$|\frac{\underline{\nu}_{v'} \cdot \underline{\nu}_v}{(\underline{\omega} \cdot \underline{\nu}(v))^2}| \le N^2 2^{-2n} \qquad \text{if} \qquad 2^{n-1} < |\underline{\omega} \cdot \underline{\nu}(v)| \le 2^n$$

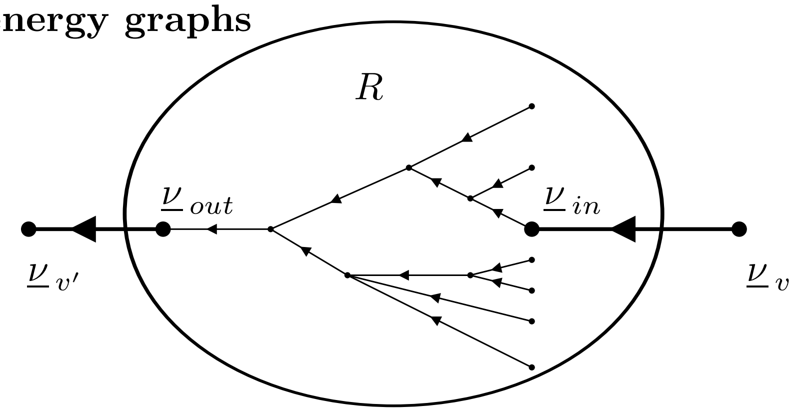
$$|\mathrm{Val}(\vartheta)| \le \frac{1}{p!} N^{2p} F^p \prod_{n=-\infty}^{0} 2^{-2n\mathcal{N}_n}$$

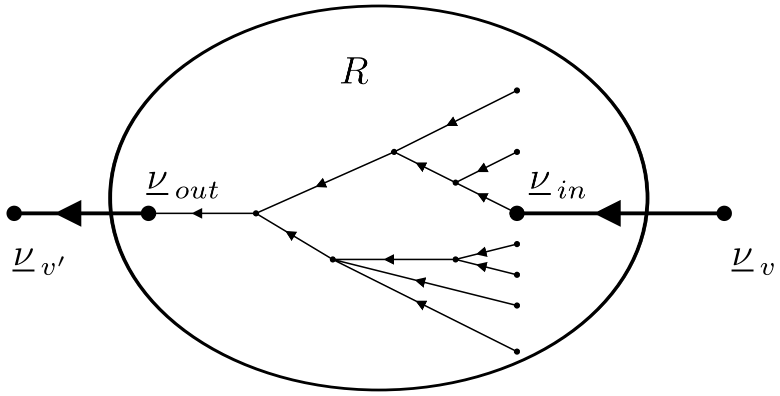
$$\mathcal{N}_n \stackrel{def}{=} \text{nombre des lignes d'échelle}\, n$$

Si $\underline{\nu}(v) \neq \underline{\nu}(w)$ pour tout $v > w$ alors $\mathcal{N}_n$ est "petit" $\mathcal{N}_n \le aN2^{n/\tau}p$ for some $a > 0$.
Il faut consommer $2^{-n/\tau}N^{-1}$ noeuds avec moment $\le N$ pour atteindre une ligne $v'v$ telle que $\underline{\omega} \cdot \underline{\nu}(v) \sim 2^{-n}$, *i.e.* $\underline{\nu}(v) = O(2^{-n/\tau})$. Pour en trouver une autre de la meme echelle il en faut encore autant: donc $\mathcal{N}_n = O(p2^{n/\tau}N)$.

$$\sum_{\vartheta\ avec\ p\ noeds} |\mathrm{Val}(\vartheta)| \le \frac{1}{p!} p! 4^p N^{2p} F^p (\prod_{n=-\infty}^{0} 2^{-2naN2^{n/\tau}p}) =$$
$$= \frac{1}{p!} p! 4^p N^{2p} F^p (2^{-2aN\sum_{n=-\infty}^{0} n2^{n/\tau}})^p = B^p$$

Simple self energy graphs





This is a *self-energy subgraph* if the entering line and the exiting line have the same current $\underline{\nu}$ with scale n and all internal lines have scale $m \geq n+3$ and their number is $< a2^{-n/\tau}$, *i.e.* not too large, **and** $\sum_{w\in R} \underline{\nu}_w = \underline{0}$ and all lines in the subgraph have different currents (*i.e.* no self energy sub-sub graphs! $\to$ "simple").

Resummation of simple self energy subgraphs

The contribution to the value of a self-energy subgraph R inserted in the line $v'v$ on which is

$$\frac{\underline{\nu}_{v'} \cdot \underline{\nu}_{out}}{(\underline{\omega} \cdot \underline{\nu})^2} \Big(\prod_{\lambda=(w'w)\in R} \frac{\underline{\nu}_{w'} \cdot \underline{\nu}_{w}}{(\underline{\omega} \cdot \underline{\nu}(\lambda))^2} \Big) \frac{\underline{\nu}_{in} \cdot \underline{\nu}_{v}}{(\underline{\omega} \cdot \underline{\nu})^2} \equiv$$

$$\equiv \frac{1}{(\underline{\omega} \cdot \underline{\nu})^2} \underline{\nu}_{v'} \cdot \frac{M_R(\underline{\nu})}{(\underline{\omega} \cdot \underline{\nu})^2} \underline{\nu}_{v}$$

We define $M(\underline{\nu}) = \sum_R \varepsilon^{|R|} M_R(\underline{\nu})$. One can insert $m = 0, 1, 2, \ldots$ self–energy subgraphs inside each line of a tree which has no such subgraphs

$$\sum_{m=0}^{\infty} \frac{1}{(\underline{\omega} \cdot \underline{\nu})^2} \underline{\nu}_{v'} \cdot \Big(\frac{M(\underline{\nu})}{(\underline{\omega} \cdot \underline{\nu})^2} \Big)^m \cdot \underline{\nu}_{v} =$$

$$= \underline{\nu}_{v'} \cdot \frac{1}{(\underline{\omega} \cdot \underline{\nu})^2 - M(\underline{\nu})} \cdot \underline{\nu}_{v}$$

convergent by the Siegel–Bryuno–Pöschel boun..

Cancellations

Not useful because $(\underline{\omega}\cdot\underline{\nu})^2-M(\underline{\nu})$ may vanish!! However (theorem)

$$M(\underline{\nu})=(\underline{\omega}\cdot\underline{\nu})^2\, m^1_\varepsilon(\underline{\nu})$$

and the propagator is $(\underline{\omega}\cdot\underline{\nu})^{-2}\,(1+m^1_\varepsilon(\underline{\nu}))^{-1}\stackrel{def}{=}(\underline{\omega}\cdot\underline{\nu})^{-2}\,\underline{\nu}_{v'}G^{(1)}(\underline{\nu})\,\underline{\nu}_v$ *i.e.* we have **eliminated** the self–energy subgraphs without inner self energy subgraphs.

Elimination of overlapping self energy graphs

Define $m^2_\varepsilon(\underline{\nu})$ in the "same way": consider all trees with at most simple self energy graphs and define their values as in the previous case but using the new propagators. Iterate: one checks that $G^{(k)}_\varepsilon(\underline{\nu})$ converges to a limit $G^{(\infty)}_\varepsilon(\underline{\nu})$.

The equation of the invariant torus is then obtained by considering all tree graphs without self energies and evaluating them with the propagator

$$(\underline{\omega}\cdot\underline{\nu})^{-2}\,\underline{\nu}_{v'}\cdot(1+m^\infty_\varepsilon(\underline{\nu}))^{-1}\cdot\underline{\nu}_v\stackrel{def}{=}(\underline{\omega}\cdot\underline{\nu})^{-2}\,\underline{\nu}_{v'}G^{(\infty)}(\underline{\nu})\,\underline{\nu}_v$$

which by the Siegel–Bryuno–Pöschel bound has no convergence problems and in fact it yields an effective computational algorithm to evaluate the LNP–series.

Low dimensional tori

If $f(\underline{\alpha},\underline{\beta})=\sum_{\underline{\nu},\underline{\mu}}e^{i\underline{\nu}\cdot\underline{\alpha}+i\underline{\mu}\cdot\underline{\beta}}f_{\underline{\nu},\underline{\mu}}$ the Feynman rules change in a minor way. Namely after resummation of the self energy graphs (defined in the same way) the propagator is the a matrix ($n\times n$ as before) which has the form

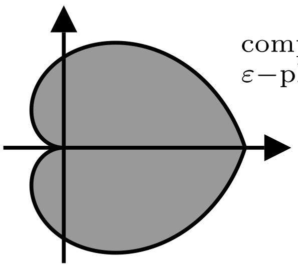
$$
\begin{array}{ccc}
 & \alpha & \beta \\
\begin{array}{c}\alpha \\ \beta\end{array} & \multicolumn{2}{c}{\left(\begin{array}{cc} (r\times r) & (r\times(n-r)) \\ (r\times r) & ((n-r)\times(n-r)) \end{array}\right) =}
\end{array}
$$

$$
= \left(\begin{pmatrix} (\underline{\omega} \cdot \underline{\nu})^2(1+O(\varepsilon^2)) & i(\underline{\omega} \cdot \underline{\nu})b\varepsilon + O(\varepsilon^2) \\ -i(\underline{\omega} \cdot \underline{\nu})b\varepsilon + O(\varepsilon^2) & (\underline{\omega} \cdot \underline{\nu})^2 - \varepsilon \underline{\partial}_{\underline{\beta}} \overline{f}(\underline{\beta}_0) + O(\varepsilon^2) \end{pmatrix} \right)^{-1}
$$

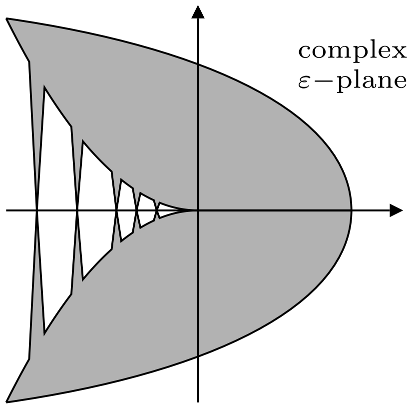
where the $\alpha \times \alpha$ elements take into account the cancellations.

However the $\beta \times \beta$ elements can vanish on or close to the infinite set $(\underline{\omega} \cdot \underline{\nu})^2 - \varepsilon \underline{\partial}_{\underline{\beta}} \overline{f}(\underline{\beta}_0) = 0$.

If $\varepsilon > 0$ and $\underline{\beta}_0$ is a *maximum* there is no 0 eigenvalue and in fact the eigenvalues are bounded below by $(\underline{\omega} \cdot \underline{\nu})^2$. Hence we are back to the maximal case. The convergence occurs in the domain D_γ $(\gamma > 0)$ of complex ε where $(\underline{\omega} \cdot \underline{\nu})^2 - \varepsilon \underline{\partial}_{\underline{\beta}} \overline{f}(\underline{\beta}_0) \geq \gamma\, \underline{\omega} \cdot \underline{\nu}^{\,2}$. This has the form:



Analyticity domain D_0 for the lower dimension invariant torus. The cusp at the origin is a second order cusp. The figure corresponds to the hyperbolic case.



The domain D_0 of Figure 1 can be further extended? the conjecture above asks whether the extended analyticity domain could possibly be represented (close to the origin) as here: the domain reaches the real axis at cusp points which are in I_{ε_0} and correspond, in the complex ε-plane, to the elliptic tori which are the analytic continuations of the hyperbolic tori. The analytic continuation would be continuous thorough the real axis at the points of I_{ε_0}. The cusps would be at least quadratic.

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