Resummations of self-energy graphs and KAM theory

G. Gentile (Roma3), G.G. (The.H.E.S.)

Communications in Mathematical Physics, **227**, 421–460, 2002.

Hamiltonian for a rotators system

$$H=\frac{1}{2}(\,\underline{A}^{\,2}+\underline{B}^{\,2})+\varepsilon f(\,\underline{\alpha}\,,\,\underline{\beta}\,)$$
$$\underline{A}\,=(A_1,\dots,A_r),\qquad \underline{B}\,=(B_1,\dots,B_{n-r})$$
$$\underline{\alpha}\,=(a_1,\dots,a_r),\qquad \underline{\beta}\,=(\beta_1,\dots,\beta_{n-r})$$

where f is an even trigonometric polynomial of degree N.

Special unperturbed motions

$$\underline{\omega}\,=(\omega_1,\dots,\omega_r)\qquad\qquad \underline{\nu}\,=(\nu_1,\dots,\nu_r)\in\mathcal{Z}^r$$
$$\underline{A}\,=\,\underline{\omega}\,,\qquad \underline{B}\,=\,\underline{0}\,,\qquad |\,\underline{\omega}\,\cdot\,\underline{\nu}\,|>\frac{1}{C|\,\underline{\nu}\,|^{\tau}}$$
$$\underline{\alpha}\,=\,\underline{\alpha}_{\,0}+\underline{\omega}\,t,\qquad \underline{\beta}\,=\,\underline{\beta}_{\,0}$$

$r=n$ ⟷ *maximal tori* or *KAM tori*

Are there motions of the "same type" in presence of interaction ?? *i.e.*

$$\underline{A}\,=\,\underline{\omega}\,+\,\underline{H}\,(\,\underline{\psi}\,),\qquad \underline{B}\,=\,\underline{K}\,(\,\underline{\psi}\,)$$
$$\underline{\alpha}\,=\,\underline{\psi}\,+\,\underline{h}\,(\,\underline{\psi}\,),\qquad \underline{\beta}\,=\,\underline{\beta}_{\,0}+\,\underline{k}\,(\,\underline{\psi}\,)\qquad \text{with}$$
$$\underline{\psi}\Rightarrow\underline{\psi}\,+\,\underline{\omega}\,t\qquad \text{is a solution?}$$

It must be (*necessarily* if $\underline{h}\,=\varepsilon\,\underline{h}^{\,(1)}+\varepsilon^2\,\underline{h}^{\,(2)}+\dots,\ \ \underline{k}\,=\dots$):

$$(\,\underline{\omega}\,\cdot\,\underline{\partial}_{\,\underline{\psi}})^2\,\underline{h}\,(\,\underline{\psi}\,)=-\varepsilon\,\underline{\partial}_{\,\underline{\alpha}}\,f(\,\underline{\psi}\,+\,\underline{h}\,(\,\underline{\psi}\,),\,\underline{\beta}_{\,0}+\,\underline{k}\,(\,\underline{\psi}\,))$$
$$(\,\underline{\omega}\,\cdot\,\underline{\partial}_{\,\underline{\psi}})^2\,\underline{k}\,(\,\underline{\psi}\,)=-\varepsilon\,\underline{\partial}_{\,\underline{\beta}}\,f(\,\underline{\psi}\,+\,\underline{h}\,(\,\underline{\psi}\,),\,\underline{\beta}_{\,0}+\,\underline{k}\,(\,\underline{\psi}\,))$$

and $\underline{\beta}_{\,0}$ must be an extremum for the average over $\underline{\alpha}$: $\overline{f}(\,\underline{\beta}\,)=\int f(\,\underline{\alpha}\,,\,\underline{\beta}\,)\frac{d\,\alpha}{(2\pi)^r}$:

$$(\,\underline{\omega}\,\cdot\,\underline{\partial}_{\,\underline{\psi}})^2\,\underline{k}^{\,1}(\,\underline{\psi}\,)=-\,\underline{\partial}_{\,\underline{\beta}}\,f(\,\underline{\psi}\,,\,\underline{\beta}_{\,0})\ \Rightarrow\ \underline{0}\,=\,\underline{\partial}_{\,\underline{\beta}}\,\overline{f}(\,\underline{\beta}_{\,0})$$

Maximal tori: r=n. Series of Lindstedt, Newcomb, Poincaré:

Let ϑ be a tree with p nodes $v_0,\ldots,v_{p-1}$ and root r.
We attach to every node v a "*momentum*" $\underline{\nu}_v \in \mathcal{Z}^n$ with $f_{\underline{\nu}_v} \neq 0$. The root momentum will be a unit vector $\underline{n}$; and we define a *current* flowing on a line outgoing the node v

$$\underline{\nu}\,(v) \overset{def}{=} \sum_{w<v} \underline{\nu}_w \qquad \text{which we } \textbf{suppose} \neq \underline{0}$$

Note that $0 < |\underline{\nu}_v| \le N \Rightarrow |\underline{\omega}\cdot\underline{\nu}_v| > c > 0$ (if $\underline{\omega}$ is "Diophantine")

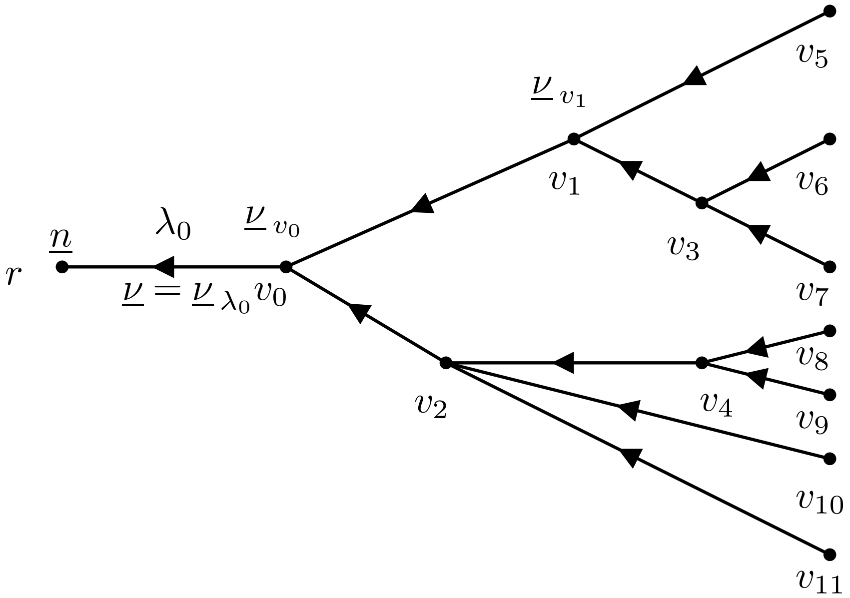


Fig. 1: A tree ϑ with $m_{v_0}=2, m_{v_1}=2, m_{v_2}=3, m_{v_3}=2, m_{v_4}=2$ and $k=12$, and a few decorations. Only two momentum labels and one current label are explicitly written down; the indices enumerating the lines (because they are distinct) are not marked. Arrows represnt the partial ordering of the nodes defined by the tree.

Define the **value** of a tree ϑ with distinct (*i.e.* labeled) branches

$$\mathrm{Val}(\vartheta) = \frac{1}{p!}\Big(\prod_{\text{lines}\lambda=(v'v)\in\vartheta} \frac{\underline{\nu}_{v'}\cdot\underline{\nu}_v}{(\underline{\omega}\cdot\underline{\nu}\,(v))^2}\Big)\,\Big(\prod_{v\in\vartheta} f_{\underline{\nu}_v}\Big)$$

counting trees up to *pivot* equivalence. Then

$$\underline{h}^{(p)}_{\underline{\nu}}\cdot\underline{n} = \sum_{\substack{\vartheta\\ \text{root current}=\underline{\nu}}} \mathrm{Val}(\vartheta)$$

Siegel–Bryuno–Pöschel bound

We say that $\lambda = (v'v)$ has **scale** $n = 0, -1, -2, \ldots$ bf if

$$2^{n-1} < |\underline{\omega} \cdot \underline{\nu}(v)| \le 2^n$$

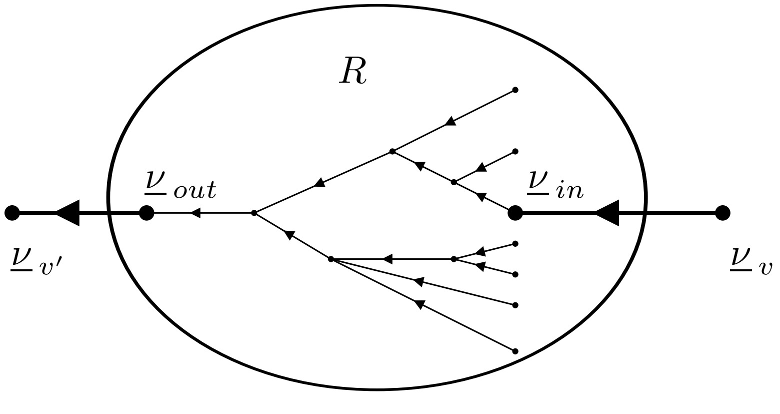
Then

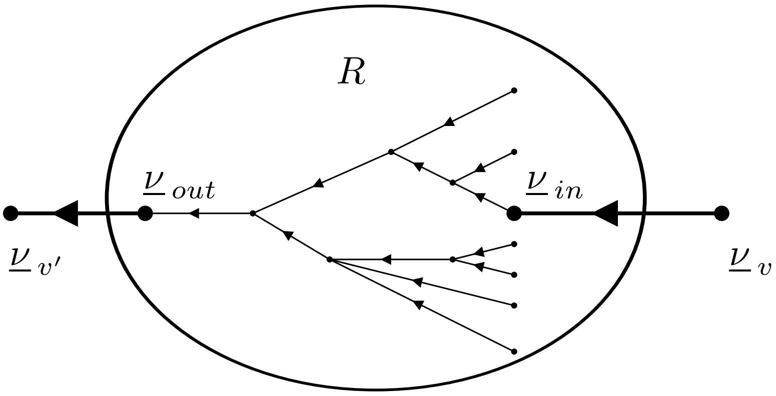
$$\begin{aligned}
&\Big|\frac{\underline{\nu}_{v'} \cdot \underline{\nu}_v}{(\underline{\omega} \cdot \underline{\nu}(v))^2}\Big| \le N^2 2^{-2n} \qquad \text{if} \qquad 2^{n-1} < |\underline{\omega} \cdot \underline{\nu}(v)| \le 2^n \\
&|\mathrm{Val}(\vartheta)| \le \frac{1}{p!} N^{2p} F^p \prod_{n=-\infty}^{0} 2^{-2n\mathcal{N}_n} \\
&\mathcal{N}_n \overset{def}{=} \text{number of lines of scale}\, n
\end{aligned}$$

If $\underline{\nu}(v) \ne \underline{\nu}(w)$ for all $v > w$ then $\mathcal{N}_n$ is “small” $\mathcal{N}_n \le aN2^{n/\tau}p$ for some $a > 0$.
We must use $2^{-n/\tau}N^{-1}$ nodes with momentum $\le N$ to reach a line $v'v$ such that $\underline{\omega} \cdot \underline{\nu}(v) \sim 2^{-n}$, *i.e.* $\underline{\nu}(v) = O(2^{-n/\tau})$. To find another one of the same scale we need as many new ones: hence $\mathcal{N}_n = O(p2^{n/\tau}N)$.

$$\begin{aligned}
\sum_{\vartheta\ con\ p\ nodi} |\mathrm{Val}(\vartheta)| &\le \frac{1}{p!} p! 4^p N^{2p} F^p \Big(\prod_{n=-\infty}^{0} 2^{-2naN2^{n/\tau}p} \Big) = \\
&= \frac{1}{p!} p! 4^p N^{2p} F^p \big(2^{-2aN\sum_{n=-\infty}^{0} n2^{n/\tau}}\big)^p = B^p
\end{aligned}$$

A simple self-energy graph





This is a *self-energy subgraph* if the entering line and the exiting one have the same current $\underline{\nu}$, of scale n, and all the internal lines have scale $m \ge n+3$ and the their number is $< a2^{-n/\tau}$, *i.e.* not too large, **and** $\sum_{w\in R} \underline{\nu}_w = \underline{0}$ and all subgraph lines have different currents (*i.e.* no self-energy sub-subgraph! → "simple").

Resummations of simple self-energy graphs

The contribution to the value of a tree from a self-energy subgraph R inserted on the line $v'v$ is

$$\frac{\underline{\nu}_{v'}\cdot\underline{\nu}_{out}}{(\underline{\omega}\cdot\underline{\nu})^2}\Big(\prod_{\lambda=(w'w)\in R}\frac{\underline{\nu}_{w'}\cdot\underline{\nu}_{w}}{(\underline{\omega}\cdot\underline{\nu}(\lambda))^2}\Big)\frac{\underline{\nu}_{in}\cdot\underline{\nu}_{v}}{(\underline{\omega}\cdot\underline{\nu})^2}\equiv$$

$$\equiv\frac{1}{(\underline{\omega}\cdot\underline{\nu})^2}\underline{\nu}_{v'}\cdot\frac{M_R(\underline{\nu})}{(\underline{\omega}\cdot\underline{\nu})^2}\underline{\nu}_{v}$$

Let $M(\underline{\nu})\overset{def}{=}\sum_R \varepsilon^{|R|}M_R(\underline{\nu})$. We can insert $m=0,1,2,\dots$ self-energy subgraphs on every line of a tree without any such subgraph

$$\sum_{m=0}^{\infty}\frac{1}{(\underline{\omega}\cdot\underline{\nu})^2}\underline{\nu}_{v'}\cdot\Big(\frac{M(\underline{\nu})}{(\underline{\omega}\cdot\underline{\nu})^2}\Big)^m\cdot\underline{\nu}_{v}=$$

$$=\underline{\nu}_{v'}\cdot\frac{1}{(\underline{\omega}\cdot\underline{\nu})^2-M(\underline{\nu})}\cdot\underline{\nu}_{v}$$

that is a *convergent* sum because of the Siegel–Bryuno–Pöschel bound.

Cancellations

This is not enough because $(\underline{\omega}\cdot\underline{\nu})^2 - M(\underline{\nu})$ can vanish!! Nevertheless one shows that

$$M(\underline{\nu}) = (\underline{\omega}\cdot\underline{\nu})^2\, m^1_\varepsilon(\underline{\nu})$$

and the propagator becomes $(\underline{\omega}\cdot\underline{\nu})^{-2}\,(1+m^1_\varepsilon(\underline{\nu}))^{-1} \overset{def}{=} (\underline{\omega}\cdot\underline{\nu})^{-2}\,\underline{\nu}_{v'}G^{(1)}(\underline{\nu})\,\underline{\nu}_{v}$, *i.e.* we have **eliminated** the self-energy subgraphs not containing other self-energy subgraphs.

Elimination of overlapping graphs

Define $m^2_\varepsilon(\underline{\nu})$ in the "same way": considering all trees with simple of self-energy graphs at most and define their value as in the preceding case making use, however, of the new propagators. Then iterate indefinitely: one can check that $G^{(k)}_\varepsilon(\underline{\nu})$ converges to a limit $G^{(\infty)}_\varepsilon(\underline{\nu})$.
The torus invariant equation is therefore obtained by considering all the graphs without self-energies and computing them by means of the new propagator

$$(\underline{\omega}\cdot\underline{\nu})^{-2}\,\underline{\nu}_{v'}\cdot(1+m^\infty_\varepsilon(\underline{\nu}))^{-1}\cdot\underline{\nu}_{v} \overset{def}{=} (\underline{\omega}\cdot\underline{\nu})^{-2}\,\underline{\nu}_{v'}G^{(\infty)}(\underline{\nu})\,\underline{\nu}_{v}$$

which, by the Siegel–Bryuno–Pöschel bound does not present convergence problems and in fact this yields an algorithm to evaluate the sum of the LNP series.

Lower dimensional tori (Resonances)

If $f(\underline{\alpha}\,,\underline{\beta}\,) = \sum_{\underline{\nu}\,,\underline{\mu}} e^{i\underline{\nu}\cdot\underline{\alpha}+i\underline{\mu}\cdot\underline{\beta}} f_{\underline{\nu}\,,\underline{\mu}}$ Feynman's rules undergo some minor changes. After resummation of the self-energy subgraphs (defined in the same way) the propagator is a Hermitian matrix ($n \times n$ as before) which has the form

$$\begin{array}{cc} & \begin{array}{cc} \quad\alpha\qquad\qquad & \qquad\beta \end{array} \\ \begin{array}{c} \alpha \\ \beta \end{array} & \begin{pmatrix} (r\times r) & (r\times(n-r)) \\ (r\times r) & ((n-r)\times(n-r)) \end{pmatrix} = \end{array}$$

$$= \left(\begin{pmatrix} (\underline{\omega}\,\cdot\,\underline{\nu}\,)^2(1+O(\varepsilon^2)) & i(\underline{\omega}\,\cdot\,\underline{\nu}\,)b\varepsilon+O(\varepsilon^2) \\ -i(\underline{\omega}\,\cdot\,\underline{\nu}\,)b\varepsilon+O(\varepsilon^2) & (\underline{\omega}\,\cdot\,\underline{\nu}\,)^2-\varepsilon\,\underline{\partial}_{\,\underline{\beta}}\overline{f}(\underline{\beta}_{\,0})+O(\varepsilon^2) \end{pmatrix} \right)^{-1}$$

where the $\alpha\times\alpha$ elements account for the cancellations discussed in the maximal cases. Also the $\alpha\times\beta$ terms show cancellations (of lower order: 1 instead of 2 when $\underline{\omega}\,\cdot\,\underline{\nu}\to 0$).

Nevertheless the $\beta\times\beta$ elements can vanish on or near the set of infinitely many points ε for which $(\underline{\omega}\,\cdot\,\underline{\nu}\,)^2-\varepsilon\,\underline{\partial}_{\,\underline{\beta}}\overline{f}(\underline{\beta}_{\,0})=0$.

If $\varepsilon>0$ and $\underline{\beta}_{\,0}$ is a *maximum* there is no 0 eigenvalue and the eigenvalues are bounded from below by $(\underline{\omega}\,\cdot\,\underline{\nu}\,)^2$. Hence on falls back in the same situation met in the maximal tori case. Convergence takes place in the domain D_γ ($1>\gamma>0$) of complex ε where $(\underline{\omega}\,\cdot\,\underline{\nu}\,)^2-\varepsilon\,\underline{\partial}_{\,\underline{\beta}}\overline{f}(\underline{\beta}_{\,0})\ge\gamma\,(\underline{\omega}\,\cdot\,\underline{\nu}\,)^2$. The domain has the form

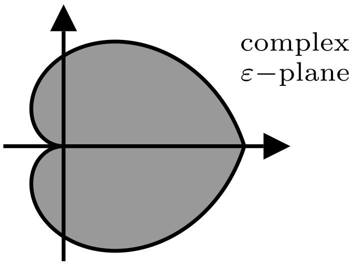


Fig.3: Analyticity domain D_0 for the lower dimensional invariant tori. The cusp at the origin is a second order one. The figure refers to the hyperbolic case.

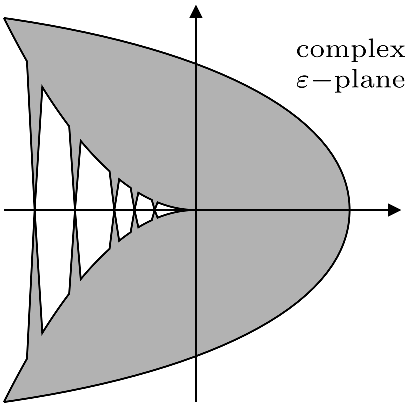


Fig.4: Can the domain D_0 in Fig.3 be extended? the domain might perhaps be (near the origin) as in the picture. It reaches the real axis in cusps with apex at a set I_{ε_0}; in the complex ε–plane they correspond to elliptic tori which would therefore be analytic continuations of the hyperbolic tori. The analytic continuation could be continuous across the real axis on I_{ε_0} and $I_{\varepsilon_0}/\varepsilon_0 \xrightarrow[\varepsilon_0]{} 1$ (*i.e.* I_{ε_0} is very large near 0.

Short Bibliography

Low dimensional tori

S.M. Graff: *On the conservation for hyperbolic invariant tori for Hamiltonian systems*, J. Differential Equations **15** (1974), 1–69.

V.K. Mel'nikov: *On some cases of conservation of conditionally periodic motions under a small change of the Hamiltonian function*, Soviet Math. Dokl. **6**(1965), 1592–1596; *A family of conditionally periodic solutions of a Hamiltonian systems*, Soviet Math. Dokl. **9** (1968), 882–886.

J. Moser: *Convergent series expansions for almost periodic motions*, Math. Ann. **169** (1967), 136–176.

Eliasson, L.H.: *Absolutely convergent series expansions for quasi-periodic motions*, Math. Phys. Electronic J., `<http:// mpej.unige.ch>`, **2** (1996), Paper 4

A. Jorba, R. Llave, M. Zou: *Lindstedt series for lower dimensional tori*, in *Hamiltonian systems with more than two degrees of freedom* (S'Agary, 1995), 151–167, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. Vol. 533, Ed. C. Simy, Kluwer Academic Publishers, 1999.

The bound of Siegel-Bryuno-Pöschel

A.D. Bryuno: *Analytic form of differential equations. The, II. (Russian)*, Trudy Moskov. Mat. Obšč. **25** (1971), 119–262; ibid. **26** (1972), 199–239.

J. Pöschel: Invariant manifolds of complex analytic mappings near fixed points, in *Critical Phenomena, Random Systems, Gauge Theories*, Les Houches, Session XLIII (1984), Vol. II, 949–964, Ed. K. Osterwalder & R. Stora, North Holland, Amsterdam, 1986.

Graph methods

L. Chierchia, C. Falcolini: *Compensations in small divisor problems*, Comm. Math. Phys. **175** (1996), 135–160.

G. Gallavotti: *Twistless KAM tori*, Comm. Math. Phys. **164** nd (1994), no. 1, 145–156. *Twistless KAM tori, quasiflat homoclinic intersections,* *and other cancellations in the perturbation series of certain completely integrable Hamiltonian systems. A review*, Rev. Math. Phys. **6** (1994), no. 3, 343–411.

More recent papers

J. Bricmont, A. Kupiainen, A. Schenkel: *Renormalization group for the Melnikov problem for PDE's*, Comm. Math. Phys. **221** (2001), no. 1, 101–140.

J. Xu, J. You: *Persistence of Lower Dimensional Tori Under the First Melnikov's Non-resonance Condition*, Nanjing University Preprint, 1–21, 2001, J. Math. Pures Appl., in press.

Xiaoping Yuan: *Construction of quasi-periodic breathers via KAM techniques* Comm. Math. Phys., in press.