Divergent series resummations: examples in ODE's & PDE's

Survey of works by
G. Gentile (Roma3),
V. Matropietro (Roma2),
G.G. (Roma1)

http://ipparco.roma1.infn.it

Hamiltonian rotators system

$$\ddot{\underline{\alpha}} = -\varepsilon\partial_{\underline{\alpha}} f(\underline{\alpha}, \underline{\beta}), \quad \ddot{\underline{\beta}} = -\varepsilon\partial_{\underline{\beta}} f(\underline{\alpha}, \underline{\beta})$$

$\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$, $\underline{\beta} = (\beta_1, \ldots, \beta_{n-r})$, f: even trigonometric polynomial of deg. N, $f = \sum_{\underline{\nu}} f_{\underline{\nu}}(\underline{\beta}) e^{i\underline{\nu}\cdot\underline{\alpha}}$, $\underline{\nu} \in \mathcal{Z}^r$. Let

$$f_{\underline{0}}(\underline{\beta}) = \frac{1}{(2\pi)^r} \int f(\underline{\alpha}, \underline{\beta})\, d\underline{\alpha}$$

Special unperturbed motions

$$\underline{\omega} = (\omega_1, \ldots, \omega_r), \ |\underline{\omega} \cdot \underline{\nu}| > \frac{1}{C|\underline{\nu}|^\tau}$$

$$\underline{\alpha} = \underline{\alpha}_0 + \underline{\omega}\, t, \qquad \underline{\beta} = \underline{\beta}_0$$

$r = n \ \longleftrightarrow\ $ *maximal tori* or *KAM tori.*

Are there motions of the "same type" in presence of interaction ?? *i.e.*

Meaning of "same type"

$$\underline{\alpha} = \underline{\psi} + \underline{h}(\underline{\psi}), \qquad \underline{\beta} = \underline{\beta}_0 + \underline{k}(\underline{\psi})$$

$$\underline{\psi} \Rightarrow \underline{\psi} + \underline{\omega}\, t \qquad \underline{\alpha}(t),\ \underline{\beta}(t) \text{ is a solution?}$$

Equations

$$(\underline{\omega}\cdot\underline{\partial}_{\underline{\psi}})^2\,\underline{h}(\underline{\psi}) = -\varepsilon\,\underline{\partial}_{\underline{\alpha}} f(\underline{\psi}+\underline{h}(\underline{\psi}), \underline{\beta}_0 + \underline{k}(\underline{\psi}))$$

$$(\underline{\omega}\cdot\underline{\partial}_{\underline{\psi}})^2\,\underline{k}(\underline{\psi}) = -\varepsilon\,\underline{\partial}_{\underline{\beta}} f(\underline{\psi}+\underline{h}(\underline{\psi}), \underline{\beta}_0 + \underline{k}(\underline{\psi}))$$

$\underline{h}, \underline{k}$ **power series** $\varepsilon \Rightarrow \partial_{\underline{\beta}} f_{\underline{0}}(\underline{\beta}_0) = \underline{0}$

$$(\underline{\omega}\cdot\underline{\partial}_{\underline{\psi}})^2\,\underline{k}^1(\underline{\psi}) = -\underline{\partial}_{\underline{\beta}} f(\underline{\psi}, \underline{\beta}_0) \Rightarrow$$

$$\Rightarrow \qquad \underline{0} = \underline{\partial}_{\underline{\beta}} f_{\underline{0}}(\underline{\beta}_0)$$

Nonlinear wave equ. ("Klein–Gordon")

$$u_{tt} - u_{xx} + \mu u = f(u), \;\; f(u) = u^3$$

$$u(0,t) = u(\pi,t) = 0, \;\; \mu > 0$$

if $\omega_m = \sqrt{\mu + m^2}$, do $f = 0$ periodic solutions like $\varepsilon \cos \omega_1 t \, \sin x$, extend to $f \neq 0$?

$$u(x,t) = \sum_{m,n \in \mathbb{Z}}^{\infty} \hat{u}_{n,m} e^{in\Omega t + imx},$$

with $\hat{u}_{n,m} = -\hat{u}_{n,-m} = \hat{u}_{-n,m}$.

$$\hat{u}_{n,m}\left[-\Omega^2 n^2+\omega_m^2\right]=\hat{f}_{n,m}(u)$$

$$\hat{u}_{n,m}\left[-\tilde{\omega}_1^2 n^2+\tilde{\omega}_m^2\right]=-\nu_m(\varepsilon)\hat{u}_{n,m}+\hat{f}_{n,m}(u),$$

(1 ?) analytic (in ε) exponentially decaying (in m, n) solution *given* $\tilde{\omega}_m$ verifying

$$|\tilde{\omega}_1 n \pm \tilde{\omega}_m| \geq C_0 |n|^{-\tau}, \quad n \neq 0; m, m' \neq 0, \pm 1$$
$$|\tilde{\omega}_1 n \pm (\tilde{\omega}_{m'} \pm \tilde{\omega}_m)| \geq C_0 |n|^{-\tau},$$

(2 ?) for a solution of $\tilde{\omega}_m^2 + \nu_m(\varepsilon) = \omega_m^2$ so that $\Omega^2 = \tilde{\omega}_1^2$.

Problem: construct a series representation of the solutions

Tori: $r \le n$. Lindstedt's series.

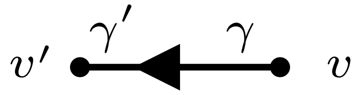
Let ϑ be a tree with

(1) p "**nodes**" $v_0, \ldots, v_{p-1}$,

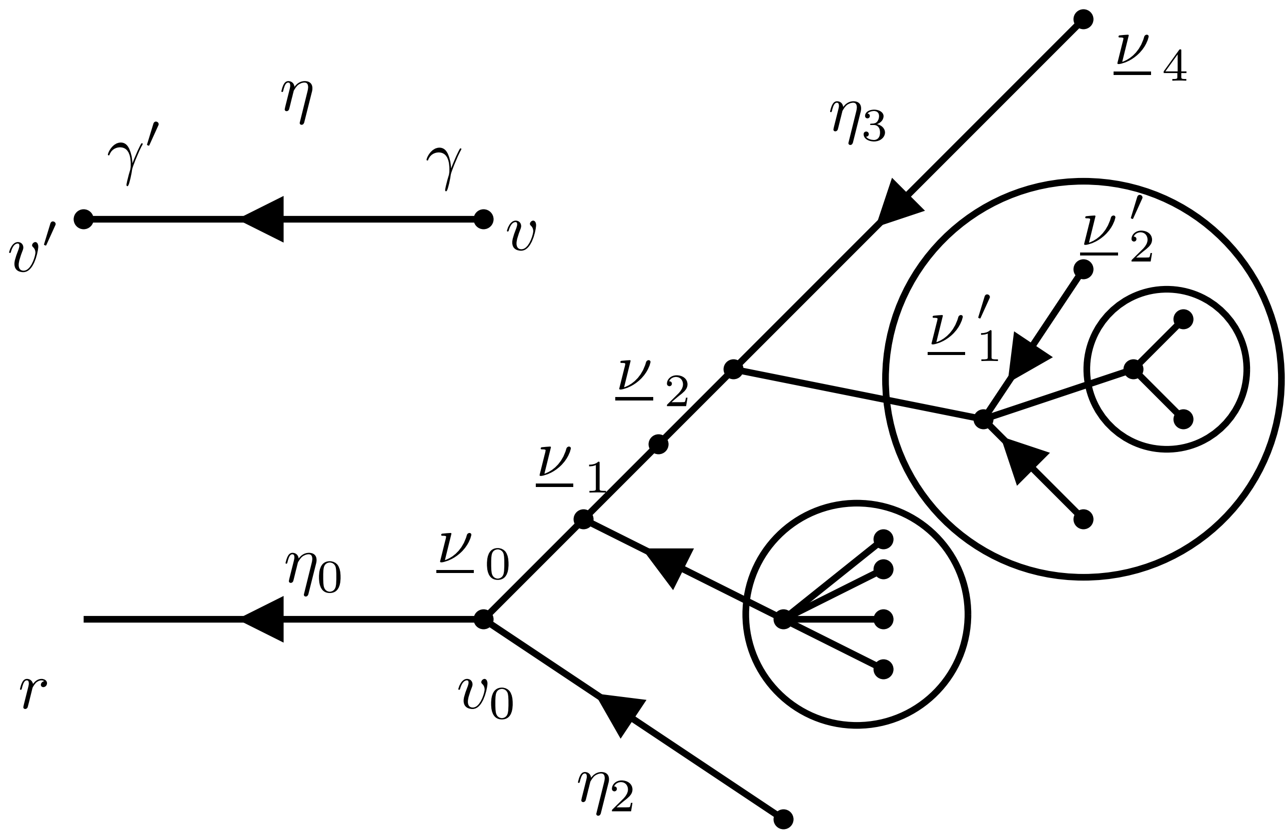
(2) one "**root**" r

(3) possibly "**leaves**" with "**veinage**".

(4)



Tree lines $\ell = \ell_v \equiv (v', v)$ joining nearest nodes (v', v) are oriented towards the root. Total number of nodes $k \ge p$



η
γ'
γ
v'
v
η_3
$\underline{\nu}_4$
$\underline{\nu}'_2$
$\underline{\nu}'_1$
$\underline{\nu}_2$
$\underline{\nu}_1$
$\underline{\nu}_0$
η_0
r
v_0
η_2

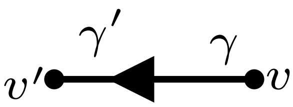
(a) Node v carries "**harmonic**" $\underline{\nu}_v \in \mathcal{Z}^r$.

(b) Line $\ell = (v', v)$ carries **current**

$$\underline{\nu}(\ell) \stackrel{def}{=} \sum_{w \leq v} \underline{\nu}_w \neq \underline{0}$$

(c) Current out of a leaf is $\underline{0}$.

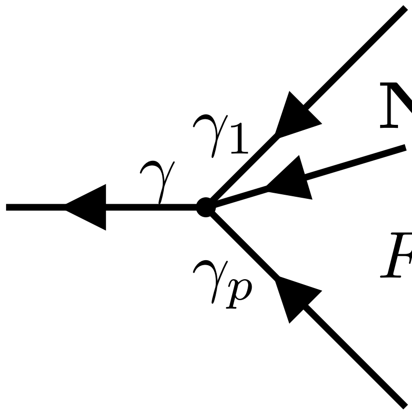
(d) Line $\ell = (v', v)$ carries pair $\eta_\ell = (\gamma_{v'}, \gamma_v)$, $\gamma = 1, \ldots, n$ (components).



Line ℓ has a "**propagator**" matrix

$$G_\ell = \delta_{\gamma,\gamma'}(\underline{\omega} \cdot \underline{\nu}(\ell))^{-2}$$

for leaf lines $G_\ell = \delta_{\gamma,\gamma'}$.



Node v tensor

$$F_v = \frac{1}{p_v!}\partial^{p_v+1}_{\gamma_v,\gamma_{v,1},\ldots,\gamma_{v,p_v}} f_{\underline{\nu}_v}(\underline{\beta}_0)$$

where $\partial_\gamma = i\nu_\gamma$ if $\gamma \le r$ and $\frac{\partial}{\partial \beta_\gamma}$ if $\gamma > r$.

Value, Reduced value, Leaf value

$$\mathrm{Val}(\vartheta)_\gamma = \prod_{v\in nodes(\vartheta)} F_v \prod_{v\in leaves(\vartheta)} L_v \prod_{\ell\in lines(\vartheta)} G_\ell,$$

$$\mathrm{Val}'(\vartheta)_\gamma = \prod_{v\in nodes(\vartheta)} F_v \prod_{v\in leaves(\vartheta)} L_v \prod_{\ell\ne\ell_0} G_\ell$$

$$L_{v,\gamma} = -\partial^2_{\underline{\beta}} f_{\underline{0}}(\underline{\beta}_0)^{-1}\mathrm{Val}'(\vartheta_L),\ \gamma > r$$

Only the root label is not contracted: *i.e.* the value $\mathrm{Val}(\vartheta)$ is a n–vector

Example: no leaves and $r = n$

$$\mathrm{Val}(\vartheta)_\gamma = \prod_{\ell=(v'v)\in\vartheta} \frac{\underline{\nu}_{v'} \cdot \underline{\nu}_v}{(\underline{\omega} \cdot \underline{\nu}(\ell))^2} \prod_{v\in\vartheta} \frac{f_{\underline{\nu}_v}}{p_v!}$$

The contractions of components labels generate the scalar products $\underline{\nu}_{v'} \cdot \underline{\nu}_v$ at each line and $\underline{\nu}_r$ has to be interpreted as the unit vector in the direction γ.

Lindstedt series

The tori parametric equ. $\underline{q}(\underline{\psi})=(\underline{h}(\underline{\psi}),\underline{k}(\underline{\psi}))$, $\underline{\psi}\in\mathcal{T}^r$

$$q^{(k)}_{\underline{\nu},\gamma}=\sum_{\vartheta\in\Theta_{k,\underline{\nu},\gamma}}\varepsilon^k\,\mathrm{Val}(\vartheta)_\gamma$$

$\Theta_{k,\underline{\nu},\gamma}=$

(1) total number of nodes k

(2) with root label γ

(3) root line current $\underline{\nu}$

(4) no line with $\underline{0}$ current (ext. Poincaré's th. on Lindstedt–Newcomb series)

If $r = n$ (maximal tori): no leaves; *by KAM theorems series is convergent* and indeed one can check that k–th order

$$BC^k e^{-\kappa|\underline{\nu}|}$$

Siegel–Bryuno–Pöschel bound

$\ell_v = (v'v)$ has **scale** $n = 0, -1, -2, \ldots$ **if**

$$2^{n-1} < |\underline{\omega} \cdot \underline{\nu}\,(\ell_v)| \le 2^n \Rightarrow$$

$$\Rightarrow |\frac{\underline{\nu}_{v'} \cdot \underline{\nu}_v}{(\underline{\omega} \cdot \underline{\nu}\,(\ell_v))^2}| \le 4N^2 2^{-2n}$$

$$|\mathrm{Val}(\vartheta)| \le \frac{1}{p!}(2N)^{2p} F^p \prod_{n=-\infty}^{0} 2^{-2n\mathcal{N}_n}$$

$$\mathcal{N}_n \stackrel{def}{=} \text{number of lines of scale}\, n$$

Recall

$$\mathrm{Val}(\vartheta)_\gamma = \prod_{\ell_v=(v'v)\in\vartheta} \frac{\underline{\nu}_{v'} \cdot \underline{\nu}_v}{(\underline{\omega} \cdot \underline{\nu}\,(\ell_v))^2} \prod_{v\in\vartheta} \frac{f_{\underline{\nu}_v}}{p_v!}$$

$$|\mathrm{Val}(\vartheta)| \leq \frac{1}{p!}(2N)^{2p}F^{p}\prod_{n=-\infty}^{0}2^{-2n\mathcal{N}_n}$$

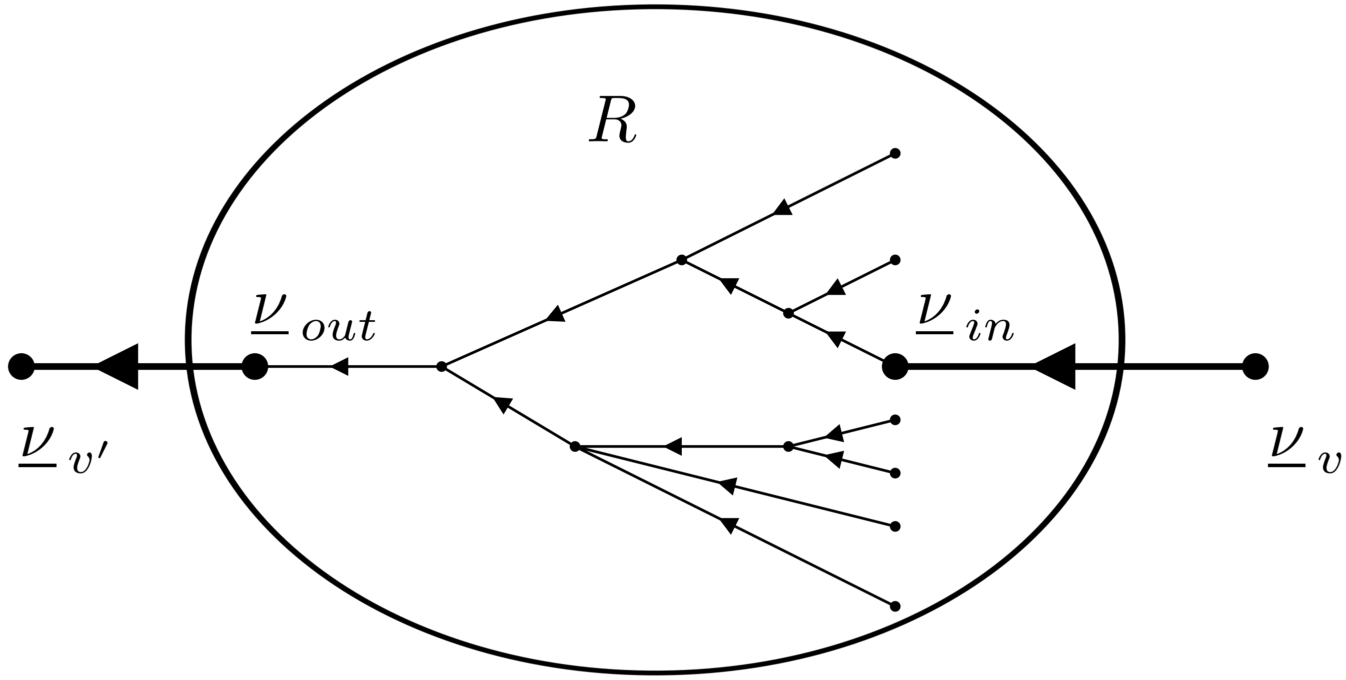
If $\underline{\nu}\,(\ell_v) \neq \underline{\nu}\,(\ell_w)$ for all $v > w \;\Rightarrow$

$$\mathcal{N}_n \leq aN2^{n/\tau}p \Rightarrow$$

$$\Rightarrow \frac{1}{p!}p!4^{p}(2N)^{2p}F^{p}(\prod_{n=-\infty}^{0}2^{-2naN2^{n/\tau}\,p}) =$$

$$= \frac{1}{p!}p!4^{p}(2N)^{2p}F^{p}(2^{-2aN\sum_{-\infty}^{0}n2^{n/\tau}})^{p} = B^{p}$$

Simple self-energy graph (S.E., mass)



$\sum_{w\in R}\underline{\nu}_w=\underline{0}$

This is a **self-energy subgraph** if
(1) the entering line and the exiting one have the same current $\underline{\nu}$, of scale n, and
(2) all internal lines scale $m > n+1$ and their number is $< a2^{-n/\tau}$ **and**
(3) $\sum_{w\in R} \underline{\nu}_w = \underline{0}$

Def: no S.E. sub-subgraph $\Rightarrow$ "simple".
Def: "SE subgr." *synonymous* "mass subgr."

Resummations of simple s.e. graphs

Contribution to a tree value from a mass subgraph R inserted on line $v'v$ is

$$\frac{\underline{\nu}_{v'}\cdot\underline{\nu}_{out}}{(\underline{\omega}\cdot\underline{\nu})^2}\Big(\prod_w \frac{f_{\underline{\nu}_w}}{p_w!}\prod_\ell \frac{\underline{\nu}_{w'}\cdot\underline{\nu}_w}{(\underline{\omega}\cdot\underline{\nu}(\ell))^2}\Big)\frac{\underline{\nu}_{in}\cdot\underline{\nu}_v}{(\underline{\omega}\cdot\underline{\nu})^2}\equiv$$

$$\equiv\frac{1}{(\underline{\omega}\cdot\underline{\nu})^2}\,\underline{\nu}_{v'}\cdot\frac{M_R(\underline{\nu})}{(\underline{\omega}\cdot\underline{\nu})^2}\,\underline{\nu}_v$$

$\prod_w \equiv \prod_{w\in R}$ and $\prod_\ell \equiv \prod_{\ell\in R}$.

Let $M^1(\underline{\nu}) \stackrel{def}{=} \sum_R \varepsilon^{|R|} M_R(\underline{\nu})$,
convergent sum because of the Siegel–Bryuno–Pöschel bound.

Can insert $m = 0, 1, 2, \ldots$ *mass subgr. on every line of a tree without SE*

$$\sum_{m=0}^{\infty} \frac{1}{(\underline{\omega} \cdot \underline{\nu})^2} \underline{\nu}_{v'} \cdot \Big(\frac{M^1(\underline{\nu})}{(\underline{\omega} \cdot \underline{\nu})^2}\Big)^m \cdot \underline{\nu}_v =$$

$$= \underline{\nu}_{v'} \cdot \frac{1}{(\underline{\omega} \cdot \underline{\nu})^2 - M^1(\underline{\nu})} \cdot \underline{\nu}_v$$

Cancellations

BUT $(\underline{\omega}\cdot\underline{\nu})^2 - M^1(\underline{\nu}) = 0$?? need $M^1(\underline{\nu}) = (\underline{\omega}\cdot\underline{\nu})^2\, m^1_\varepsilon(\underline{\nu})$. Up to $(1+O(\varepsilon^2))$

$$\underline{\omega}\cdot\underline{\nu}^{\,2} - M^1 = \begin{pmatrix} (r\times r) & (r\times(n-r)) \\ ((n-r)\times r) & ((n-r)\times(n-r)) \end{pmatrix} =$$

$$= \begin{pmatrix} (\underline{\omega}\cdot\underline{\nu})^2 & i(\underline{\omega}\cdot\underline{\nu})b\varepsilon \\ -i(\underline{\omega}\cdot\underline{\nu})b\varepsilon & (\underline{\omega}\cdot\underline{\nu})^2 - \varepsilon\,\partial^2_{\underline{\beta}} f_{\underline{0}}(\underline{\beta}_0) \end{pmatrix}^{-1}$$

the $r\times r \to$ "KAM cancellations"

BUT $(n-r)\times(n-r)$ elements can vanish on or near the set of infinitely many points ε for which $(\underline{\omega}\cdot\underline{\nu})^2 - \varepsilon\partial^2_{\underline{\beta}} f_{\underline{0}}(\underline{\beta}_0) = 0$.

eliminated simple SE subgraphs **IF**

$$-\varepsilon\partial^2_{\underline{\beta}} f_{\underline{0}}(\underline{\beta}_0) > 0.$$

Elimination of overlapping graphs

Define $M^2(\underline{\nu})$ in the "same way": considering trees with simple SE graphs at most and define value as in the preceding case making use, however, of the new propagators $x^2 - M(\underline{\nu})$. Iterate indefinitely: $G^{(k)}_\varepsilon(\underline{\nu}) \xrightarrow[k\to\infty]{} G^{(\infty)}_\varepsilon(\underline{\nu})$.

Invariant torus equ.: just consider all graphs without SE and new propagator

$$((\underline{\omega}\cdot\underline{\nu})^2 - M^{\infty}(\underline{\nu}))^{-1} \stackrel{def}{=} G^{(\infty)}(\underline{\nu})$$

by Siegel–Bryuno–Pöschel **no converg. prob.**: hence algorithm for LNP series.

If $\varepsilon > 0$ and $\underline{\beta}_0$ is a *maximum* the popagator matrix has no 0 eigenvalue. Hence "no difference from maximal case". Convergence in complex ε domain D where $(\underline{\omega} \cdot \underline{\nu})^2 - \varepsilon\partial^2_{\underline{\beta}} f_{\underline{0}}(\underline{\beta}_0) \geq \gamma(\underline{\omega} \cdot \underline{\nu})^2$.

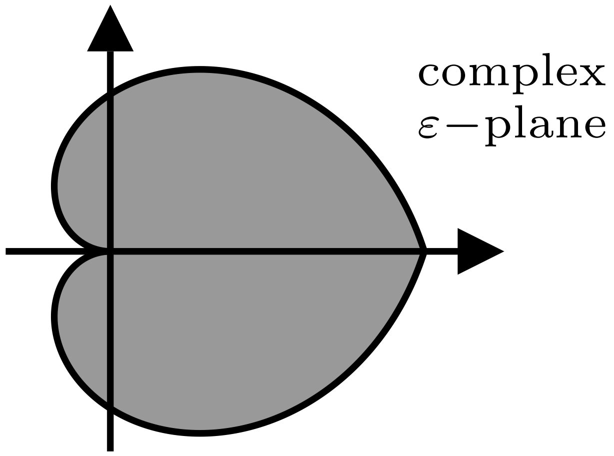


Fig.3: Lower dimensional tori analyt. domain D_0 tori. Cusp at the origin is of ond order .

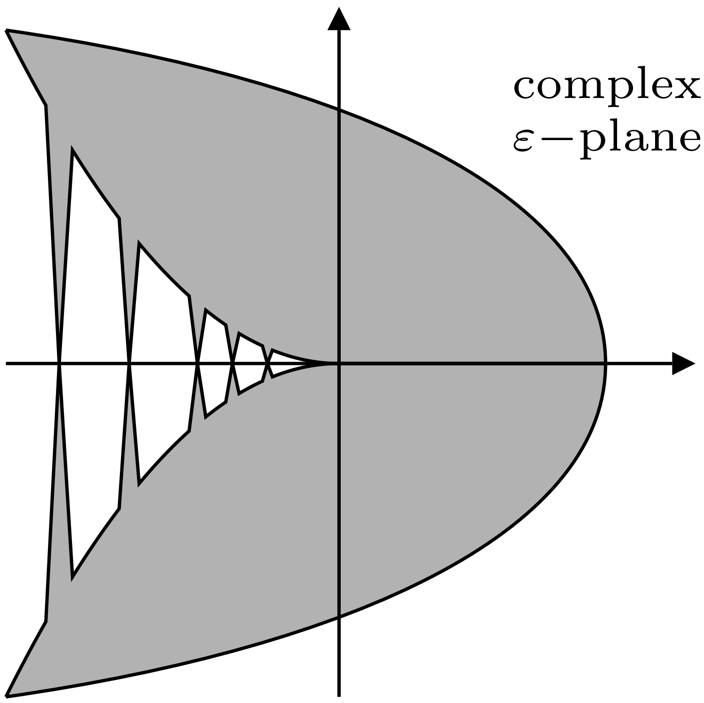


Fig.4: Can D_0 in Fig.3 be extended? Perhaps be (near the origin) as in the picture? Real axis reached in cusps with apex at a set I; for $\varepsilon \in I$ the parametric eq. correspond to elliptic tori which would be analytic continuations of the hyperbolic tori.

Short Bibliography

Low dimensional tori

S.M. Graff: *On the conservation for hyperbolic invariant tori for Hamiltonian systems*, J. Differential Equations **15** (1974), 1–69.

V.K. Mel'nikov: *On some cases of conservation of conditionally periodic motions under a small change of the Hamiltonian function*, Soviet Math. Dokl. **6**(1965), 1592–1596; *A family of conditionally periodic solutions of a Hamiltonian systems*, Soviet Math. Dokl. **9** (1968), 882–886.

J. Moser: *Convergent series expansions for almost periodic motions*, Math. Ann. **169** (1967), 136–176.

Eliasson, L.H.: *Absolutely convergent series expansions for quasi-periodic motions*, Math. Phys. Electronic J., `<http:// mpej.unige.ch>`, **2** (1996), Paper 4

A. Jorba, R. Llave, M. Zou: *Lindstedt series for lower dimensional tori*, in *Hamiltonian systems with more than two degrees of freedom* (S'Agary, 1995), 151–167, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. Vol. 533, Ed. C. Simy, Kluwer Academic Publishers, 1999.

The bound of Siegel-Bryuno-Pöschel

A.D. Bryuno: *Analytic form of differential equa-*

tions. The, II. (Russian), Trudy Moskov. Mat. Obšč. **25** (1971), 119–262; ibid. **26** (1972), 199–239.

J. Pöschel: Invariant manifolds of complex analytic mappings near fixed points, in *Critical Phenomena, Random Systems, Gauge Theories*, Les Houches, Session XLIII (1984), Vol. II, 949–964, Ed. K. Osterwalder & R. Stora, North Holland, Amsterdam, 1986.

Graph methods

L. Chierchia, C. Falcolini: *Compensations in small divisor problems*, Comm. Math. Phys. **175** (1996), 135–160.

G. Gallavotti: *Twistless KAM tori*, Comm. Math. Phys. **164** nd (1994), no. 1, 145–156. *Twistless KAM tori, quasiflat homoclinic intersections, and other cancellations in the perturbation series of certain completely integrable Hamiltonian systems. A review*, Rev. Math. Phys. **6** (1994), no. 3, 343–411.

More recent papers

J. Bricmont, A. Kupiainen, A. Schenkel: *Renormalization group for the Melnikov problem for PDE's*, Comm. Math. Phys. **221** (2001), no. 1, 101–140.

J. Xu, J. You: *Persistence of Lower Dimensional*

Tori Under the First Melnikov's Non-resonance Condition, Nanjing University Preprint, 1–21, 2001, J. Math. Pures Appl., in press.

Xiaoping Yuan: *Construction of quasi-periodic breathers via KAM techniques* Comm. Math. Phys., in press.

G. Gallavotti, G. Gentile, *Hyperbolic low-dimensional tori and summations of divergent series*, Comm. Math. Phys. **227** (2002), no. 3, 421–460.

G. Gentile, *Quasi-periodic solutions for two-level systems*, Comm. Math. Phys. **242** (2003), no. 1–2, 221–250.

G. Gentile, V. Mastropietro, *Construction of periodic solutions of nonlinear wave equations with Dirichlet boundary conditions by the Lindstedt series method*, in press.