Fluctuation theorem revisited

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(Dated: August 30, 2016)

Recently the "fluctuation theorem" has been criticized on the basis of contents incorrectly attributed to it. Here I reestablish, once more, the original so that its substantial difference from other statements that have been given, subsequently, the same name can be better appreciated.

PACS numbers: 47.52, 05.45, 47.70, 05.70.L, 05.20, 03.20

Fluctuations

The *fluctuation theorem* is a property of the phase space contraction of an Anosov map S, called *time evolution*, which is *time reversible*. The connection between the fluctuation theorem and Physics is a different matter that I will not discuss here: there are many places where this is done as full detail as possible with the present state of knowledge, [1], [2], [3].

Denote phase space by Ω (a smooth finite boundaryless manifold), by S the time evolution map and by $\sigma(x)$ the phase space contraction

$$\sigma(x) = -\log|\det\partial_x S(x)|$$

Time reversal is an *isometry* I of phase space such that

$$IS = S^{-1}I, \qquad \sigma(Ix) = -\sigma(x)$$

It has been shown that there exists a unique probability distribution, called the *statistics of the motion* or the *SRB distribution*, μ such that for all points $x \in \Omega$, excepted those in a set of 0 volume, it is

$$\lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=0}^{\tau-1}F(S^t x) \stackrel{def}{=} \langle F\rangle = \int_\Omega F(y)\mu(dy)$$

for all smooth observables F defined on phase space.

It is intuitive that "phase space cannot expand": this is expressed by the following result of Ruelle, [4],

Proposition: If $\sigma_+ \stackrel{def}{=} \langle\sigma\rangle$ *it is* $\sigma_+ \ge 0$

Clearly if S is volume preserving it is $\sigma_+ = 0$. This motivates calling systems for which $\langle\sigma\rangle > 0$ "dissipative" and calling volume preserving systems "conservative". If $\sigma_+ > 0$ the system does not admit any stationary distribution with density with respect to the volume.

For Anosov systems which are "transitive" (*i.e.* with a dense orbit), reversible and dissipative one can define the dimensionless phase space contraction, a quantity related to entropy production rate (see [5, 6]) averaged over a time interval of size τ. This is

$$p = \frac{1}{\sigma_+\tau}\sum_{-\tau/2}^{\tau/2-1}\sigma(S^k x)$$

provided *of course* $\sigma_+ > 0$.

Then for such systems the probability with respect to the stationary state, *i.e.* with respect to the SRB distribution μ, that the variable p takes values in $\Delta = [p, p+\delta p]$ can be written as $\Pi_\tau(\Delta) = e^{\tau\max_{p\in\Delta}\zeta(p)+O(1)}$ where $\zeta(p)$ is a suitable function and $O(1)$ refers to the τ–dependence at fixed p for all intervals Δ of given size δp contained in an open interval (p_1^*, p_2^*) (this is often expressed less rigorously as $\lim_{\tau\to\infty}\frac{1}{\tau}\log\Pi_\tau(p) = \zeta(p)$ for $p_1^* < p < p_2^*$).

The function $\zeta(p)$ is analytic in p in the interval of definition (p_1^*, p_2^*) and *convex*. In fact more is true and one can prove the following *fluctuation theorem*:

Proposition: In transitive time reversible Anosov systems the rate function $\zeta(p)$ for the phase space contraction $\sigma(x)$ is analytic and strictly convex in an interval $(-p^*, p^*)$ with $+\infty > p^* \ge 1$ and $\zeta(p) = -\infty$ for $|p| > p^*$. Furthermore

$$\zeta(-p) = \zeta(p) - p\sigma_+, \qquad \text{for} \qquad |p| < p^*$$

which is called the "fluctuation relation".

Existence of $\zeta(p)$ is a theorem by Sinai who proves analyticity and convexity, [7, 8]. Strict convexity follows from a theorem of Griffiths and Ruelle, [9], which shows that the only way strict convexity could fail is if $\sigma(x) = \varphi(Sx) - \varphi(x) + c$ where $\varphi(x)$ is a smooth function (typically a Lipschitz continuous function) and c is a constant, see Ch. 6 in [8]. The constant vanishes if time reversal holds and $\sigma(x) = \varphi(Sx) - \varphi(x)$ contradicts the assumption that $\sigma_+ > 0$ (because $|\frac{1}{\tau}\sum_{-\tau/2}^{\tau/2-1}\sigma(S^k x)| \le \tau^{-1}\max_y|\varphi(y)|$). The value of p^* must be $p^* \ge 1$ otherwise $p^* < 1$ and the average of p could not be 1 (as it is by its very definition). The proposition is proved in [10**?**], see also [**?**], and called the *fluctuation theorem* a name later used by others attributing it to quite different relations ! [11].

Remark: For finite τ the function $\zeta(p)$ and $\zeta(-p)$ are replaced by $\zeta_\tau(p), \zeta_\tau(-p)$ which differ from their limits as $\tau\to\infty$ by a quantity bounded by a constant uniforlmly in any closed interval of $(-p^*, p^*)$

Rather than the above p one may considers the quantity $a = \tau^{-1}\sum_{j=-\tau/2}^{\tau/2-1}\sigma(S^j x)$, and the result becomes

$$\widetilde{\zeta}(-a) = \widetilde{\zeta}(a) - a, \qquad \text{for } |a| < p^*\sigma_+$$

where $\widetilde{\zeta}(a)$, the large deviations rate function for a, is trivially related to $\zeta(p)$. Note that p^* is certainly $< +\infty$ because the variable $\sigma(x)$ is bounded (being continuous on phase space, *i.e.* on the bounded manifold on which the Anosov map is defined).

The latter form of the fluctuation theorem can be misunderstood to suggest that in the case of systems with $\sigma_+ = 0$ *the distribution of the variable* a *is asymmetric (because the extra condition* $|a| < p^*\sigma_+$ *might be forgotten).* In fact errors appeared in the literature because of this misunderstanding; see [**?**] which is part of several attempts to claim that the above fluctuation theorem is either wrong or a consequence of earlier (and later) statements. The supposedly equivalent statements were given same name by their authors, after the above fluctuation theorem was published, thus creating a remarkable confusion, see [11].

Considering more closely the cases $\sigma_+ = 0$ it follows that $\sigma(x) = \varphi(Sx) - \varphi(x) + c$ again by a result of Griffiths and Ruelle (essentially the same mentioned above) and $c = 0$ by time reversal. Hence the variable

$$a = \frac{1}{\tau}\sum_{j=-\tau/2}^{\tau/2-1} \sigma(S^j x)$$

is *bounded* and tends *uniformly* to 0. One could repeat the theory developed for p when $\sigma_+ > 0$ but one would reach the conclusion that $\widetilde{\zeta}(a) = -\infty$ for $|a| > 0$ and the result would be trivial. In fact in this case it follows that that the system admits an SRB distribution with density on Ω, [1, 4]. The distribution of a is symmetric (trivially by time reversal symmetry) and becomes a delta function around 0 as $t \to \infty$ (*i.e.* "no large deviations" in the usual sense are possible).

Nevertheless the fluctuation relation is *non trivial* in cases in which the map S depends on parameters $\underline{E} = (E_1, \ldots, E_n)$ and becomes volume preserving ("conservative") as $\underline{E} \to 0$: in this case $\sigma_+ \to 0$ as $\underline{E} \to \underline{0}$ and one has to rewrite the fluctuation relation in an appropriate way to take a meaningful limit.

The result is that the limit as $\underline{E} \to 0$ of the fluctuation relation in which both sides are divided by $\underline{E}^2$ makes sense and yields (in transitive Anosov dynamical systems) relations which are non trivial and that can be interpreted as giving Green–Kubo formulae and Onsager reciprocity for transport coefficients associated with the thermodynamic fluxes $\underline{J}$ conjugated with the thermodynamic forces $\underline{E}$, [12, 13].

In fact the very definition of the duality between currents and fluxes so familiar in nonequilibrium thermodynamics since Onsager can be set up in such systems by using σ_+, regarded as a function of $\underline{E}$, as a generating function: $J_i = \partial_{E_i}\sigma_+|_{\underline{E}=\underline{0}}$, [12–14]. Note that the fluxes are usually "currents" divided by temperature: therefore via the above interpretation one can try to define the temperature even in nonequilibrium situations, [5].

Acknowledgements: I am indebted to E.G.D. Cohen, of course, and to F. Bonetto for many enlightening discussions.

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