Hyperbolic systems and the chaotic hypothesis (Systèmes hyperboliques et hypothèse chaotique)

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Abstract: *Statistical mechanics concerns the probabilistic study of deterministic or random systems with a great number of particles. A deterministic many particles system subject to external non conservative forces is out of equilibrium. However, it may evolve towards a stationary state, when subject also to the action of "thermostatting" forces (forces that keep the total mechanical energy from indefinitely increasing). This stationary state corresponds to a probability distribution and attempts to describeit can be viewed as a nonequilibrium extension of the Boltzmann-Maxwell-Gibbs theory in equilibrium statistical mechanics. Ruelle proposed (1973) a solution to a similar problem in the theory of turbulence. His proposal applies equally well to nonequilibrium statistical mechanics, as suggested in recent papers where it is described under the name of "chaotic hypothesis". It is argued that the latter is a a generalization of the ergodic hypothesis and that it leads to a concrete representation of the stationary distributions. We suggest that the interest of the hypothesis lies in its fundamental nature (i.e. it should hold "essentially without restrictions") and in being, at the same time, compatible with the ergodic hypothesis. We review how in certain special systems it can be used to derive concrete results (the "fluctuation theorem") which have been tested numerically.*

Dynamical Systems
Continuous time (ODE)

$$(M, S_t), \qquad M = \text{phase space manifold}$$
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad S_t = \text{time evolution} \qquad \mathbf{x}(t) \stackrel{def}{=} S_t\mathbf{x}$$

Discrete time (Map)

$$\mathbf{x} \to S\mathbf{x}, \qquad \mathbf{x} \in M, \qquad \mathbf{x}(n) = S^n\mathbf{x}$$

Phase space volume contraction

$$\sigma(\mathbf{x}) = -\mathrm{div}\,\mathbf{f}(\mathbf{x}) = -\sum_i \partial_{x_i} f_i(\mathbf{x}), \quad \text{(cont. time)}$$
$$\sigma(\mathbf{x}) = -\log|\det \partial_{\mathbf{x}} S(\mathbf{x})|, \qquad \text{(discr. time)}$$

Motions can be *regular* or *chaotic*

Regular: $\exists$ coordinates $\mathbf{x} = (\mathbf{A}, \boldsymbol{\alpha}) \in U \times \mathbb{T}^\ell$ s.t.

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \quad \longleftrightarrow \quad \dot{\mathbf{A}} = \mathbf{0},\ \dot{\boldsymbol{\alpha}} = \boldsymbol{\omega}(\mathbf{A})$$
$$S\mathbf{x} \quad \longleftrightarrow \quad \mathbf{A}' \equiv \mathbf{A},\ \boldsymbol{\alpha}' = \boldsymbol{\alpha} + \boldsymbol{\omega}$$

All motions are quasi periodic

Paradigms of regular motions: Harmonic oscillators, Rotators, Rotations

$$\begin{aligned} \dot p_i &= -\omega_i^2 q_i \qquad & \dot A_i &= 0, \qquad & \boldsymbol{\alpha} &\to \boldsymbol{\alpha} + \boldsymbol{\omega}, \\ \dot q_i &= p_i \qquad & \dot\alpha_i &= A_i & & \end{aligned}$$

Chaotic: $\exists$ *natural digital coordinates* $\mathbf{x} \longleftrightarrow \boldsymbol{\varepsilon}(x)$ coding points into sequences $\boldsymbol{\varepsilon} = (\ldots, \varepsilon_{-2}, \varepsilon_{-1}, \varepsilon_0, \varepsilon_1, \ldots)$ of digits $\varepsilon_j = 1, 2, \ldots, k$ so that

(1) $\boldsymbol{\varepsilon}(S\mathbf{x}) =$ transl. of $\boldsymbol{\varepsilon}$ by 1 unit : $\quad \varepsilon_i(S\mathbf{x}) \equiv \varepsilon_{i+1}(\mathbf{x})$

(2) $\exists$ *compatibility matrix* $T_{\varepsilon\varepsilon'} = 0, 1$ s. t.
sequences $\boldsymbol{\varepsilon}$ with $T_{\varepsilon_i \varepsilon_{i+1}} \equiv 1, \forall i$ are in correspondence with points $\mathbf{x}$

(3) map $\boldsymbol{\varepsilon} \to \mathbf{x}$ is continuous
(and $1 - 1.5$ for $\mathbf{x}$ outside a set of zero volume)

Paradigms of Chaotic motions Geodesic flow on constant negative curvature surfaces, Toral maps

$$\boldsymbol{\alpha}' = (\alpha'_1, \alpha'_2) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 + \alpha_2 \\ \alpha_1 \end{pmatrix}$$

In all cases "*time averages exist*" for all smooth observable F

$$\frac{1}{T}\sum_{j=0}^{T-1} F(S^j\mathbf{x})\,dt \xrightarrow[T\to\infty]{} \langle F\rangle(\mathbf{x})$$

Regular $\Rightarrow$ averages exist $\forall\mathbf{x}$ *but* in general *there is memory of the past.* Finite time averages $\to$ limit "slowly" ($O(t^{-1})$ in general).

Chaotic $\Rightarrow$ exist $\forall\mathbf{x}$ but a set N of zero volume. Finite time averages $\to$ limit "quickly" (paradigmatically exponentially); are $\mathbf{x}$=*independent.* Define the *statistics* SRB, μ_{srb}: for $\mathbf{x}$ outside N

$$\lim_{T\to\infty}\frac{1}{T}\sum_{j=0}^{T-1} F(S^j\mathbf{x}) = \int_M F(\mathbf{y})\,\mu_{srb}(d\mathbf{y})$$

Conservative chaotic systems: Hamiltonian flow on surface, their Poincaré sections maps, volume preserving maps of Anosov type.

In such cases there is an *invariant* prob. distr. μ_0 which is abs. cont. w.r.t. volume measure

$$\mu(d\mathbf{p}d\mathbf{q}) = \delta(H(\mathbf{p},\mathbf{q}) - E)\frac{d\mathbf{p}\,d\mathbf{q}}{\text{normal.}}, \quad \text{(Hamilton.)}$$

$$\mu(d\boldsymbol{\alpha}) = \frac{d\boldsymbol{\alpha}}{(2\pi)^\ell}, \qquad \text{(toral maps)}$$

In such cases the *statistics* $\mu_{srb} \equiv \mu_0$.

In equilibrium statistical mechanics the problem is to study the statistics and the *ergodic hypothesis* saying that

"for practical purposes", and in general "physical systems", motions on energy surfaces M_E are so irregular that they can be considered ergodic.

$\Rightarrow$ Boltzmann–Gibbs prescription for statistical mechanics as it prescribes rules to evaluate averages via the (only) abs. cont. inv. distr. μ_0.

Dissipative chaotic systems:

However the SRB distribution μ may not be abs. cont. w.r.t. volume μ_0 (even in the paradigmatic Anosov cases). *Dissipative* if

$$\sigma_+ = \langle\sigma\rangle_{srb} = \lim_{T\to\infty}\frac{1}{T}\sum_{j=0}^{T-1}\sigma(S^j\mathbf{x}) \neq 0\,(\text{hence} > 0\ (\text{R.}))$$

The μ_{srb} **cannot** be abs. cont. w.r.t. volume.

Surprising: a general theory of dissipative motions *does not exist*; at best is just beginning.

Cannot dismiss: many physical problems deal with dissipative systems. *Viscous fluids, systems of particles subject to non conservative forces, glassy materials ...* (all eventually reducible to particles subject to non conservative forces).

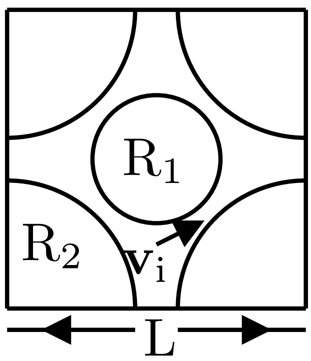
A recent proposal is **Chaotic hypothesis**:

"for practical purposes" in general "physical systems" motions on energy surfaces M_E are so irregular that they can be considered Anosov systems.

This implies that the statistics of motions of all systems exists and it is their SRB distribution.

An example

N particles in a periodic box



$\ddot{\mathbf{q}}_{\rm i} = \mathrm{E}\mathbf{u} - \theta_{\rm i}$ + collision
1) speed renorm to $\sqrt{3k_B\Theta}$

2) or keep constant speed

3) or $\theta_{\rm i} = \alpha(\dot{\mathbf{q}})\dot{\mathbf{q}}_{\rm i}$

Problem: *what to do with the Ch. Hyp.?*.

SRB distributions can be parameterized (particle systems, for instance) V = volume, E = strength of driving forces, thermostat parameter (*e.g.* $\mathcal{E}$ = energy if it fixes energy).

Collection of the SRB distr. $\Rightarrow$ **ensemble** of "*sta-*

tionary states"; averages depend on $V, \mathcal{E}, E$.

There is a possibility of formulating a statistical theory of nonequilinrium stationary states.

Chaotic Hyp. extends the Ergodic Hyp., providing ground for theory of noneq. stationary states.

Ergodic Hyp. does not solve any concrete problem: but gives a frame to formulate problems: it took a very long time to reach the present developments of Statistical Mechanics.

On the basis of the only ergodic hypothesis one can only hope to find general relations between averages of observables as parameters defining their equilibria vary.

However such relations are of great importance in Physics and in the equilibrium cases they are the object of *Thermodynamics*.

First example was the *heat theorem*:

if equilibrium states of N *particles with "arbitrary" interactions are parameterized by* V = *volume,* U = *energy and if* $p(U, V)$ = *(SRB)-average force per unit surface exerted by particles on walls and* $T(U, V)$ = *average kinetic energy and the averages are evaluated by the Boltzmann distr., (Ergodic Hyp.) then changing* U, V *by* du, dV *generates variations such that*

$$\frac{dU + p\,dV}{T} = \text{exact}$$

which is a mechanical version of the second law of thermodynamics.

Likewise from Chaotic Hyp. alone we cannot hope for more than a description of relation between averages of observables as the parameters of the stationary states change. Yet this is an extremely ambitious task: $\rightarrow$ extending Thermodynamics to nonequilibrium stationary states.

Mathematically this means considering families of Anosov systems with elements continuously depending on a few parameters and identify general relations between the average values of *few* observables.

The Fluctuation Theorem

It is the first of a few general results valid for Anosov systems. Combined with the chaotic hypothesis it becomes a "prediction" of the outcome of certain experiments (several of which have been performed).

Let (M,S) be an Anosov map; suppose that it is dissipative ($\sigma_+>0$) and reversible (i.e. there is an isometry of phase space I such that $IS=S^{-1}I$). Let dimensionless average phase space contr. p over τ be

$$p(\mathbf{x},\tau)=\frac{1}{\tau}\sum_{j=0}^{\tau-1}\frac{\sigma(S^j\mathbf{x})}{\sigma_+}$$

This random variable with the distribution inherited from the SRB verifies a large deviations property

$$\text{probab}\big(p(\mathbf{x},\tau)\in\Delta\big)=e^{\tau\max_{p\in\Delta}\zeta(p)+O(1)}$$

where $\zeta(p)$ analytic in $(-p^*,p^*)$ with $p^*\ge1$. Then

$$\zeta(-p)=\zeta(p)-p\,\sigma_+,\qquad |p|<p^* \qquad (FR)$$

Without free parameters and *independent of system considered* (in class of reversible Anosov).

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