Elliptic resonances and summation of divergent series

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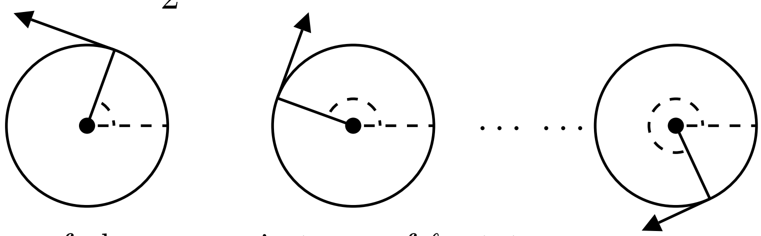
Hamiltonian : $H = \frac{1}{2}\mathbf{I}^2 + \varepsilon f(\boldsymbol{\varphi})$ with $(\mathbf{I}, \boldsymbol{\varphi}) \in \mathbb{R}^\ell \times \mathbb{T}^\ell$



1

Representation of phase space in terms of ℓ rotators.

Resonance : The first r angles $\boldsymbol{\alpha}$ rotate while the remaining s angles $\boldsymbol{\beta}$ do not move ($r + s = \ell$).

Let $\varepsilon = 0$ and $\mathbf{I} = (\mathbf{A}, \mathbf{B})$, $\boldsymbol{\varphi} = (\boldsymbol{\alpha}, \boldsymbol{\beta})$

$$\mathbf{A} = \boldsymbol{\omega}, \quad \mathbf{B} = \mathbf{0}, \quad \boldsymbol{\alpha} = \boldsymbol{\alpha}_0 + \boldsymbol{\omega}\, t, \quad \boldsymbol{\beta} = \boldsymbol{\beta}_0$$

If the $\boldsymbol{\omega}$ are rationally independent, *e.g.* id there are $C_0, \tau > 0$

$$|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| \geq \frac{1}{C_0\, |\boldsymbol{\nu}|^\tau} \qquad \text{for all } \boldsymbol{\nu} \in \mathbb{Z}^r, \boldsymbol{\nu} \neq \mathbf{0}$$

Question: *For $\varepsilon \neq 0$ are there quasi periodic motions which continue the unperturbed motions?*

For instance if $r = 1$ are there periodic orbits? (Yes! in general *but not too many*). The "same" for tori.

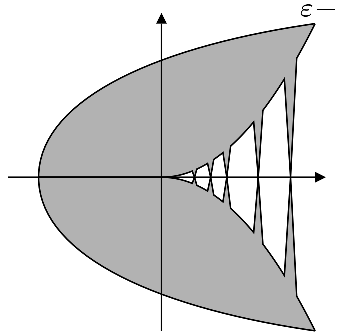
Consider tori with frequencies $(\boldsymbol{\omega},\mathbf{0})$. We prove that in general

Theorem: If $\boldsymbol{\beta}_0\in\mathbb{T}^s$ is a stationarity point for the average (over $\boldsymbol{\alpha}$) $\overline{f}(\boldsymbol{\beta})$ of $f(\boldsymbol{\varphi})\equiv f(\boldsymbol{\alpha},\boldsymbol{\beta})$ and if the $s\times s$ matrix of its derivatives $\partial^2_{\boldsymbol{\beta}\boldsymbol{\beta}}\overline{f}(\boldsymbol{\beta}_0)$ is not degenerate and its eigenvalues are distinct and positive, then there is an invariant torus with parametric equations

$$\boldsymbol{\alpha}=\boldsymbol{\psi}+\mathbf{a}(\boldsymbol{\psi}),\qquad \boldsymbol{\beta}=\boldsymbol{\beta}_0+\mathbf{b}(\boldsymbol{\psi})$$

provided ε is small enough and it is outside a set $\mathcal{E}^c$ which is small near the origin, actually it has zero density there. Motion on such tori is "free": this means

$$\boldsymbol{\psi}\to\boldsymbol{\psi}+\boldsymbol{\omega}t.$$



2

$\mathbf{a},\mathbf{b}$ analytic in $\boldsymbol{\psi}$ and differentiable of arbitrarily high order in ε
analytic also in ε for $\varepsilon<0$ if the matrix $\partial^2_{\boldsymbol{\beta}\boldsymbol{\beta}}\overline{f}(\boldsymbol{\beta}_0)$ is positive

Existence of a formal solution as a power series in ε:
Description of the Lindstedt series.

$$(\mathbf{a}, \mathbf{b}) = \mathbf{h} = \sum_{k=1}^{\infty} \varepsilon^k \mathbf{h}^{(k)}$$

No convergence in general: **however**

Idea: **"There are no divergent series"**

Hence we look for **sum rules**

Split $\mathbf{h}^{(k)}$ as a sum of many terms and recombine them to obtain an absolutely convergent series.

In doing this we shall be forced to sum divergent series by giving their sum by **a prescription**. A typical exemple

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, \qquad z \neq 1$$

even when $|z| > 1$ **!**.

Not "harmless": for instance it means that we **are going to use**:

$$\sum_{k=0}^{\infty} 2^k = 1 + 2 + 4 + 8 + 16 + \ldots \; = \; -1 \qquad !!$$

Classical Lindstedt series for $H = \frac{1}{2}\mathbf{A}^2 + \varepsilon f(\boldsymbol{\alpha})$

$$\mathbf{A} = \mathbf{A}_0, \qquad \boldsymbol{\omega} \stackrel{def}{=} \mathbf{A}_0$$

$$\boldsymbol{\alpha} = \boldsymbol{\psi}, \qquad \boldsymbol{\psi} \in \mathbb{T}^\ell, \qquad \boldsymbol{\psi} \to \boldsymbol{\psi} + \boldsymbol{\omega}\, t$$

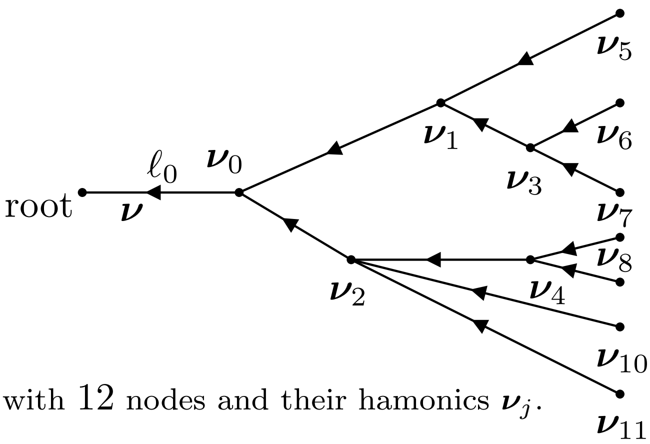
$$\mathbf{A} = \mathbf{A}_0 + \mathbf{k}(\boldsymbol{\psi}), \qquad \boldsymbol{\omega} \stackrel{def}{=} \mathbf{A}_0$$

$$\boldsymbol{\alpha} = \boldsymbol{\psi} + \mathbf{h}(\boldsymbol{\psi}), \qquad \boldsymbol{\psi} \in \mathbb{T}^\ell, \qquad \boldsymbol{\psi} \to \boldsymbol{\psi} + \boldsymbol{\omega}\, t$$

$$\mathbf{h}(\boldsymbol{\psi}) = \varepsilon \mathbf{h}^{(1)}(\boldsymbol{\psi}) + \varepsilon^2 \mathbf{h}^{(2)}(\boldsymbol{\psi}) + \ldots$$

Then $\ddot{\boldsymbol{\alpha}} = -\varepsilon \partial_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha})$ means

$$(\boldsymbol{\omega} \cdot \partial_{\boldsymbol{\psi}})^2 \mathbf{h}(\boldsymbol{\psi}) = -\varepsilon\, \partial_{\boldsymbol{\alpha}} f(\boldsymbol{\psi} + \mathbf{h}(\boldsymbol{\psi}))$$



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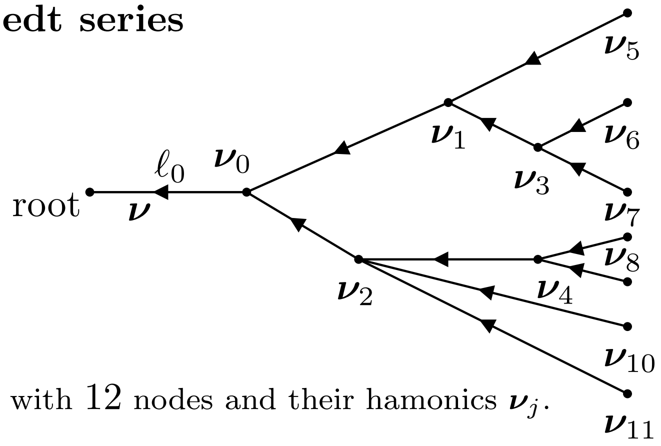
A tree graph θ with 12 nodes and their hamonics $\boldsymbol{\nu}_j$.

The Lindstedt series



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A tree graph θ with 12 nodes and their hamonics $\boldsymbol{\nu}_j$.

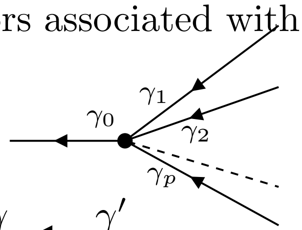
$\boldsymbol{\nu}_v \in \mathbb{Z}^r$ = **node harmonic**. The **current** on line vv' is $\sum_{w\le v} \boldsymbol{\nu}_w$.

Given a graph θ we define its **value**

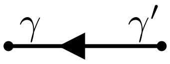
$$\mathrm{Val}(\theta)=\frac{\varepsilon^k}{k!}\Big(\prod_{\mathfrak{v}\in V(\theta)} F_v\Big)\Big(\prod_{\ell\in\Lambda(\theta)} G_\ell\Big),$$

Values are products of factors associated with nodes and with lines.

$$F_v=\partial^{p+1}_{\varphi_{\gamma_0}\ldots\varphi_{\gamma_p}} f_{\boldsymbol{\nu}}(\boldsymbol{\beta}_0) \;=$$

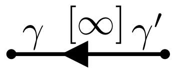


$$G_{\gamma\gamma'}=\delta_{\gamma\gamma'}\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \;=$$



$\boldsymbol{\nu}\neq\mathbf{0}$

$$G_{\gamma\gamma'}=\Big(\frac{1}{\varepsilon\,\partial^2_{\boldsymbol{\beta}\boldsymbol{\beta}} f_{\mathbf{0}}(\boldsymbol{\beta}_0)}\Big)_{\gamma,\gamma'} \;=$$



$\boldsymbol{\nu}=\mathbf{0}$ and $\gamma,\gamma'>r$

$G_{\gamma\gamma'}=0$ if $\boldsymbol{\nu}=\mathbf{0}$ and γ or $\gamma'\le r$

$\partial_{\varphi_{\boldsymbol{\nu}}}\overset{def}{=} i\,\boldsymbol{\nu}_j$ if $j\le r$.

Given θ with k lines (and **without** nodes with $\mathbf{0}$ harmonic and just one entering line carrying a $\mathbf{0}$ current) we define its **value**

$$\mathrm{Val}(\theta)=\frac{\varepsilon^k}{k!}\Big(\prod_{\mathfrak{v}\in V(\theta)}F_v\Big)\Big(\prod_{\ell\in\Lambda(\theta)}G_\ell\Big),$$

with

$$F_v=\prod_j\partial_{\gamma_j}f_{\boldsymbol{\nu}_{\mathfrak{v}}}(\boldsymbol{\beta}_0),$$

$$G_\ell\equiv\delta_{\gamma_\ell,\gamma'_\ell}\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)^2},\qquad\qquad \text{if } \boldsymbol{\nu}_\ell\ne\mathbf{0},$$

$$G_\ell\equiv-\varepsilon^{-1}\,(\partial^2_{\boldsymbol{\beta}}f_{\mathbf{0}}(\boldsymbol{\beta}_0))^{-1}_{\gamma_\ell,\gamma'_\ell},\qquad \text{if } \boldsymbol{\nu}_\ell=\mathbf{0},\ \text{and } \gamma_\ell,\gamma'_\ell>r$$

$$G_\ell\equiv0,\qquad \text{if } \boldsymbol{\nu}_\ell=\mathbf{0},\ \text{and } \gamma_\ell \text{ with } \gamma'_\ell\le r$$

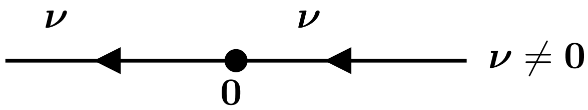
hence division by 0 is forbidden: (*Poincaré*). Value of θ is a monomial of degree q.

If $\Theta^o_{q,\boldsymbol{\nu},\gamma}$ = set of graphs θ with degree q the **Lindstedt series**

$$\varepsilon^q h^{(q)}_{\boldsymbol{\nu},\gamma}=\sum_{\theta\in\Theta^o_{q,\boldsymbol{\nu},\gamma}}\mathrm{Val}(\theta)$$

First resummation

We eliminate the "*trivial nodes*"



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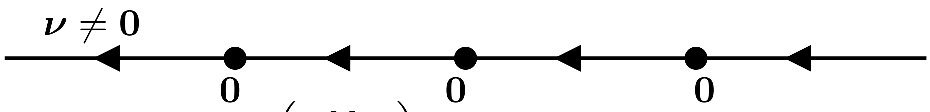
(node factor $M_0 \stackrel{def}{=} \varepsilon \partial^{\,2}_{\boldsymbol{\beta}\boldsymbol{\beta}} f_{\mathbf{0}}(\boldsymbol{\beta}_0)$) bacause chains of k such nodes



4

generate factors $\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\Big(\frac{M_0}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\Big)$ in the tree value

$$\sum_{k=0}^{\infty} \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\Big(\frac{M_0}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\Big)^k \equiv \frac{1}{x^2}\sum_k z^k \stackrel{def}{=} \frac{1}{x^2(1-z)}$$

where $x \stackrel{def}{=} \boldsymbol{\omega}\cdot\boldsymbol{\nu}$, whose sum (provided $\det(x^2 - M_0) \neq 0$), is

$$\overline{g}(x) = \frac{1}{x^2 - M_0}$$

Hence **no trivial nodes** BUT new propagator $(x^2 - M_0)^{-1}$!

Hyperbolic case: $\boldsymbol{\beta}_0$ =is a maximum for $\overline{f}(\boldsymbol{\beta})$ and $x^2 - M_0 \neq 0$ always

Elliptic case: $\boldsymbol{\beta}_0$ = is a minimum or a stationarity point for $\overline{f}(\boldsymbol{\beta})$ and $x^2 - M_0$ *can vanish*

$$\overline{g}(x) \stackrel{def}{=} \frac{1}{x^2 - M_0}$$

Hence **no trivial nodes** BUT new propagator $\overline{g}(x) \stackrel{def}{=} (x^2 - M_0)^{-1}$.

The new series is simpler but now its terms are not even always well defined (in the elliptic or mixed cases at least)

because the divisors can vanish, a case corresponding to $z = 1$ or $\det(x^2 - M_0) = 0$: if $a_1, \ldots a_s$ are eigenvalues of M_0 and $x^2 = \varepsilon a_i$.

One therefore discards the values of ε for which:

$$\min_j \Big| |x| - \sqrt{\underline{\lambda}_j^{[0]}(\varepsilon)} \Big| \ge 2^{-(\overline{n}_0-1)/2} \frac{C_0}{|\boldsymbol{\nu}|^{\tau_1}}$$

here n_0 is a quantity that measures the size of ε: $C_0^2 2^{-n_0-1} < \varepsilon a_s \le C_0^2 2^{-n_0}$. We shall take $\overline{n}_0$ a little larger.

We look at the series as to a sum of *singular functions* in ε: the singularities being given by the zeros of $x^2 - M_0$.

Then we proceed to "slice" the propagators $\overline{g} = (x^2 - M_0)^{-1}$ as sums of quantities which are regular but become larger and larger.

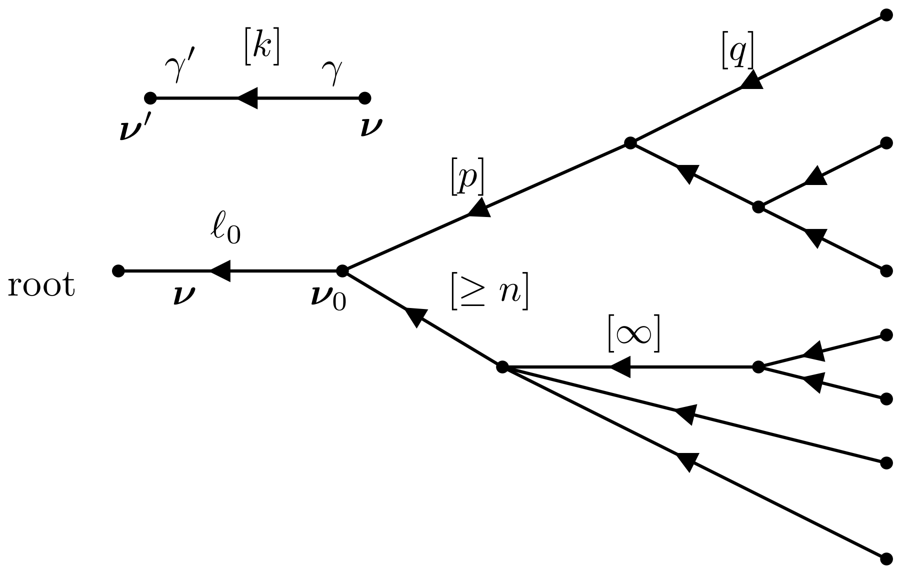
$$\overline{g}(x, \varepsilon) = g^{[0]} + g^{[1]} + \ldots + g^{[n-1]} + g^{[\ge n]}$$

$$\overline{g}(x,\varepsilon) = g^{[0]} + g^{[1]} + \ldots + g^{[n-1]} + g^{[\geq n]}$$

n neing an integer. The splitting is made by defining **essentially**, $g^{[k]}$ as equal to 0 or to $\overline{g}$ **if**

$$2^{-2k} C_0 < x^2 \leq C_0 2^{-2k}$$

The propagators $g^{[k]}$ not only are no longer singular but they have a size which does not vary too much between *max* and *min.*
with the exception of $g^{[\geq n]}$**.**



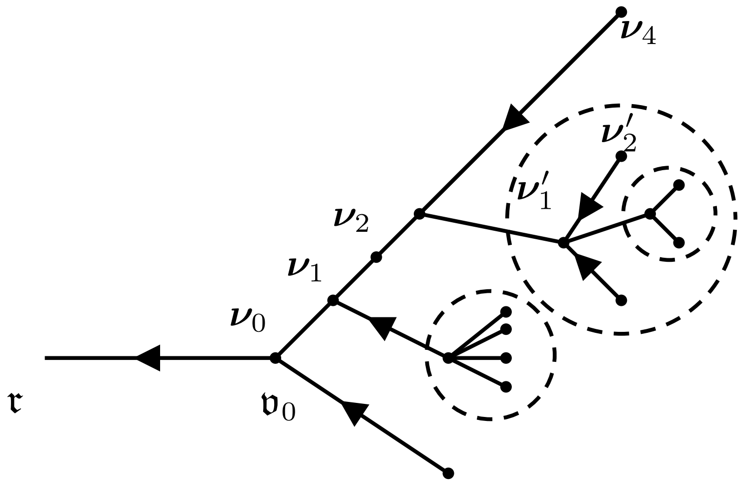
5

A tree graph with some propagators scales marked.

Essentially?: indeed there is an ambiguity in defining $g^{[k]}$.

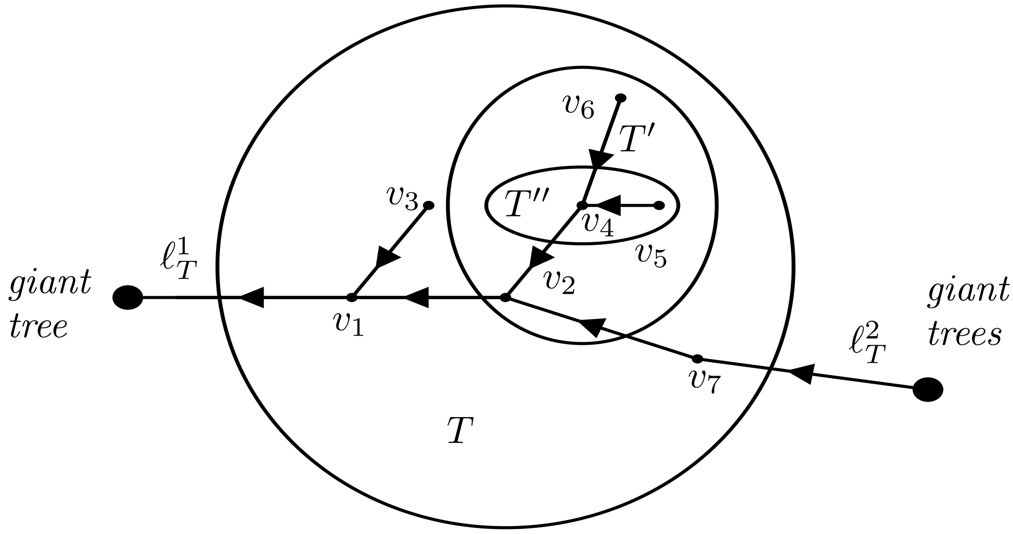
Can define exactly as just said: we would obtain a representation formally giving same result for **h** *provided* the lines carry a "scale index" $[0], [1], \ldots, [n-1], [\geq n]$ and values are evaluated with the corresponding propagators.
The index $[\infty]$ is reserved to lines with $x = 0$ (*i.e.* $\boldsymbol{\nu} = \mathbf{0}$).

This would not be useful. We instead determine $g^{[k]}$ so that graphs become **simpler**: and such that if we only consider graphs in which only scales $[0],\ldots,[n-1]$ ou $[\infty]$ appear *their values sum is convergent.*
Hence we modify ("**a little**") propagators while eliminating graphs containing **self–energy clusters**.



6

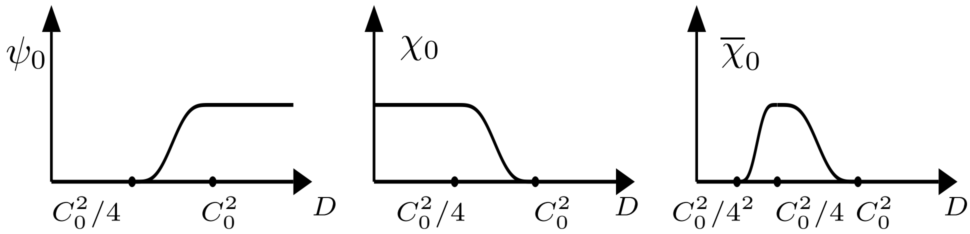
Illustration of the clusters



7

Illustrations of clusters and of self–energy clusters.
"*Responsible for the small divisors problems*"

Multiscale analysis of singularities



8

"cut–offs": $\psi_n(D) = \psi_0(2^{2n}D),\ \chi_n(D) = \chi_0(2^{2n}D)$

$$\chi_n(D) + \psi_n(D) \equiv 1$$

Fixed the size of ε by:

$$\varepsilon \in I \stackrel{def}{=} (2^{-2(n_0+1)}C_0^2, 2^{-2n_0}C_0^2]$$

Define the **distance to a singularity**

$$D(x) = \min_{\varepsilon \in I} \min_j \left| x^2 - \underline{\lambda}_j^{[0]}(\varepsilon) \right|$$

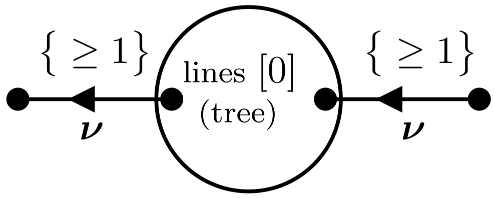
Then we write $\overline{g}(x) \equiv g^{\geq 0](x)}$ and

$$g^{[\geq 0]}(x;\varepsilon) = \psi_0(D(x))\, g^{[\geq 0]}(x;\varepsilon) + \chi_0(D(x))\, g^{[\geq 0]}(x;\varepsilon) =$$

$$\stackrel{def}{=} g^{[0]}(x;\varepsilon) + g^{\left\{\geq 1\right\}}(x;\varepsilon)$$

and we can represent the Lindstedt series simply by adding ascale label [0] or $\{\geq 1\}$ on each line.

Consider a self-energy of scale [0]”

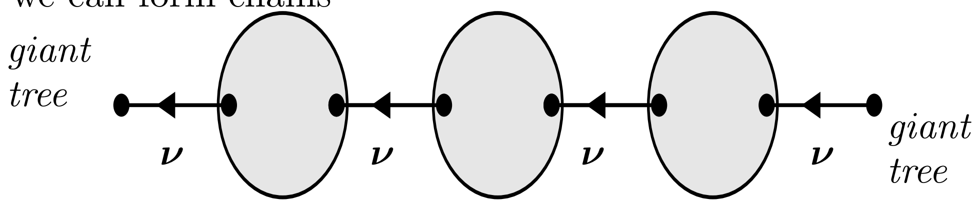


9

$$\mathrm{Val}(\theta) = \text{external factors}\cdot$$

$$\cdot\, g^{\{\geq 1\}} \cdot \Big(\sum \text{possible internal values}\Big)\, g^{\{\geq 1\}}$$

and we can form chains



10

and since $g^{\{\geq 1\}} = \frac{\chi_0(x)}{x^2 - M_0}$, their values can be summed up to give

$$g^{\{\geq 1\}} \cdot \Big((\sum \text{ possible self energies})\, g^{\{\geq 1\}} \Big)^k$$

$$= g^{\{\geq 1\}} \cdot \frac{1}{1 - \sum (s.e.) g^{\{\geq 1\}}}$$

$$= \frac{\chi_0(x)}{x^2 - M_0 - \sum(s.e.)\chi_0} \stackrel{def}{=} g^{[\geq 1]}$$

Self-energies are *eliminated* at the price of trees with scale labels $[\infty], [0]$, and $[\geq 1]$ on the lines.

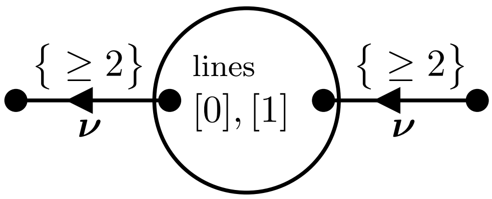
Still singular because the propagator

$$g^{[\geq 1]} \stackrel{def}{=} \frac{\chi_0(x)}{x^2 - \mathcal{M}^{[\leq 1]}}$$

with $\mathcal{M}^{[\leq 1]} \equiv \chi_0\, M_0 + \sum (s.e.)$ can have a denominator close to zero (*and even equal to* 0 *!*).
We now iterate

$$g^{[\geq 1]} \equiv g^{[\geq 1]}\psi_1(D(x)) + g^{[\geq 1]}\chi_1(D(x)) \stackrel{def}{=} g^{[1]} + g^{\{\geq 2\}}$$

Again we generate graphs with lines with scale labels
$[\infty], [0], [1], \{\geq 2\}$. **which can have self energy clusters**



11

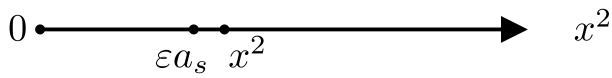
and we proceed as before to eliminate them via resummation.

We thus obtain tree graphs whose lines have scales
$[\infty], [0], [1], \ldots, [n-1], [\geq n]$ **without self energies**

BUT still singular on the lines with scale $[\geq n]$.

Idea: iterate indefinitely: **eliminate** singular lines with label $[\geq n]$.

If $\varepsilon a_s \sim C_0^2 2^{-2n_0}$ (definition of n_0) as soon as we consider propagators $g^{[n]}$



with $n \sim n_0$ we shall no longer be able to bound $x^2 - M_0^{[\leq n]}$ by x^2

Difficulty: once reached scale n_0r divisors are no longer bounded below by *const* $\cdot\, x^2$: because distance to singularities can become much smaller than x^2.

Furthermore singularities move ! *little* but of $O(\varepsilon^2)$: hence risk that cut–off based on distance to initial singularities is no longer good to avoid singularities, even those of the propagators $g^{[n]}$ if $n \gg n_0$, phenomenon of **resonances of the proper frequencies with the eigenvalues of M_0.**

Problem is present only in the elliptic case!

Therefore change strategy: we mesure distance to singularities by

$$\Delta^{[n]}(x;\varepsilon) \stackrel{def}{=} \min \left| x^2 - \underline{\lambda}_j^{[n]}(\varepsilon) \right|,$$

with a reference point which is **adapted** to the scale $[n]$ and follows the varying resonances of the propagator: defined by

$$\underline{\lambda}_j^{[n]}(\varepsilon) \stackrel{def}{=} \lambda_j^{[n]}\left(\sqrt{\underline{\lambda}_j^{[n-1]}(\varepsilon)}, \varepsilon\right), \qquad \underline{\lambda}_j^{[\overline{n}_0-1]}(\varepsilon) \stackrel{def}{=} \lambda_j^{[0]},$$

and we prove that the variations of the matrices $\mathcal{M}^{[\leq n]}$ are **extremely small**; *i.e.* they decrease faster than any power in ε^{-1}.

In this way even at small scales $\Delta^{[n]}(x;\varepsilon)$ is a good estimate of the strength of the singularity.

The sum of the diagrams without self-energies converges: the argument is classic (Eliasson) **provided** one can bound from below the propagators divisors by x^2.

However that is not always true in the elliptic case: as x^2 can be close rather than to 0 to an eigenvalue of $\mathcal{M}^{[\le n]}$.

Nevertheless in the cases the scale is very small because x^2 is close to 0 we still can apply the classical method by Siegel, Bryuno, Pöschel because we can check that, **because of the cancellation whose existence is known already in the case of the combinatorial proof of the KAM theorem** it is $\underline{\lambda}_j(\varepsilon) \equiv 0$ if $j \le r$ and $\lambda_j(x,\varepsilon) = O(\varepsilon\, x^2)$ *instead of the easy but naive* $O(\varepsilon^2)$.

If however $j > r$ *no cancellations* help. **But** when the singularities are due to such resonances the values of the graphs are so small that cancellations are not even necessary, **provided** we discard a setof ε's which do not verify a Diophantine property on $\boldsymbol{\omega}$, *i.e.* do not verify

$$\min\Big\{\Big|x \pm \sqrt{\underline{\lambda}_j^{[m]}(\varepsilon)}\Big|,\ \Big|x \pm \sqrt{\underline{\lambda}_j^{[m]}(\varepsilon)} \pm \sqrt{\underline{\lambda}_i^{[m]}(\varepsilon)}\Big|\Big\} \ge 2^{-\frac12 m}\frac{C_0}{|\boldsymbol{\nu}|^{\tau_1}},$$

$\to$ further restrictions on ε (*infinitely many* on each scale) since $\boldsymbol{\omega}$ is fixed. The importance of such a Diophantine property was elucidated by Melnikov.

Hence the key is that if $x = \boldsymbol{\omega}\cdot\boldsymbol{\nu}$ is large compared to εa_s we do not see the difference between the much easier $\varepsilon < 0$ case and $\varepsilon > 0$ because the divisors are bounded by a constant times x^2, as in the KAM case or as in the hyperbolic case.

For the other (infinitely many) scales we can proceed again as in the KAM case for the terms in which the singularity is due to x^2 being close to 0: otherwise one proves that the contribution to the value of the tree is so small that no cancellations are necessary.

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