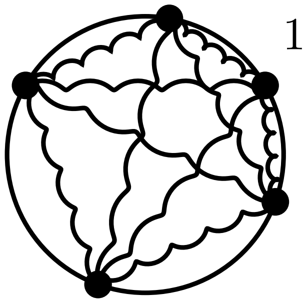
ℓ points on a circle



$1, \ldots, r$ **rotate,** $r+1, \ldots, \ell$ **are fixed**

$$\begin{cases} \dot{\boldsymbol{\alpha}} = \boldsymbol{\omega} \\ \dot{\boldsymbol{\beta}} = \mathbf{0} \end{cases} \Rightarrow \ |\boldsymbol{\omega} \cdot \boldsymbol{\nu}| > \frac{1}{C|\boldsymbol{\nu}|^\tau}, \ \boldsymbol{\nu} \in \mathbb{Z}^r$$

Then $\begin{cases} \boldsymbol{\alpha} = \boldsymbol{\psi} \\ \boldsymbol{\beta} = \boldsymbol{\beta}_0 \end{cases}$ **are data on torus** $\mathbb{T}^\ell$

Motion $\boldsymbol{\psi} \ \to \ \boldsymbol{\psi} + \boldsymbol{\omega} t$**.**

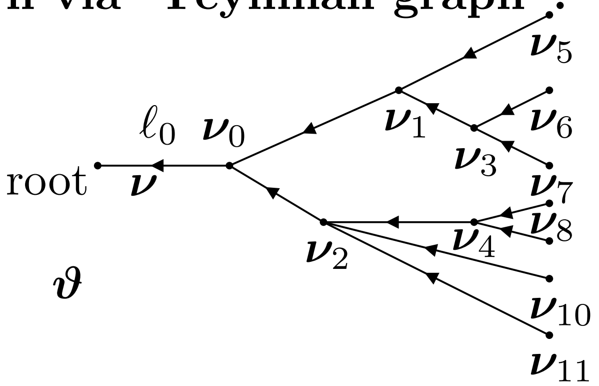
Interaction $\begin{cases} \ddot{\boldsymbol{\alpha}} = -\varepsilon \partial_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha}, \boldsymbol{\beta}) \\ \ddot{\boldsymbol{\beta}} = -\varepsilon \partial_{\boldsymbol{\beta}} f(\boldsymbol{\alpha}, \boldsymbol{\beta}) \end{cases}$ $\overline{f}(\boldsymbol{\beta}) \stackrel{def}{=} \int f(\boldsymbol{\alpha}, \boldsymbol{\beta}) \frac{d\boldsymbol{\alpha}}{2\pi}$

$\partial_{\boldsymbol{\beta}} \overline{f}(\boldsymbol{\beta}_0) = \mathbf{0}$ *expected.* $\varepsilon \neq 0$ **?**

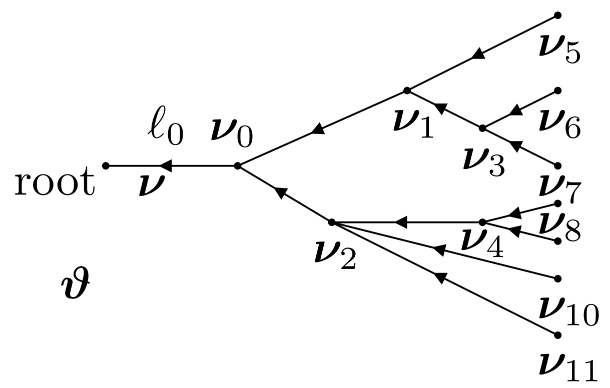
$\begin{cases} \boldsymbol{\alpha} = \boldsymbol{\psi} + \mathbf{h}(\boldsymbol{\psi}) \\ \boldsymbol{\beta} = \boldsymbol{\beta}_0 + \mathbf{k}(\boldsymbol{\psi}) \end{cases}$ **such that** $\boldsymbol{\psi} \to \boldsymbol{\psi} + \boldsymbol{\omega} t$ **??**

Th.: $\mathbf{h}, \mathbf{k}$ admit formal power series: "Lindstedt"

Represent $\mathbf{h}$ via "Feynman graph": $\Rightarrow$ *trees*



A tree graph θ with 12 nodes and their hamonics $\boldsymbol{\nu}_j$.



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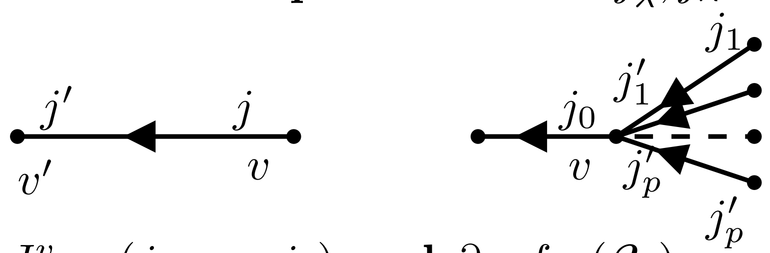
A tree graph θ with 12 nodes and their hamonics $\boldsymbol{\nu}_j$.

(1) To node v **attach a** *harmonic* $\boldsymbol{\nu} \in \mathbb{Z}^\rho$
(2) To line $\lambda \equiv v'v$ **attach** *current* $\boldsymbol{\nu}(\lambda) = \sum_{w \leq (v} \boldsymbol{\nu}_w$
(3) and two component labels j'_λ, j_λ



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$v \to J^v = (j_0, \ldots, j_p)$ **and** $\partial_{J^v} f_{\boldsymbol{\nu}_v}(\boldsymbol{\beta}_0)$ **are defined.**

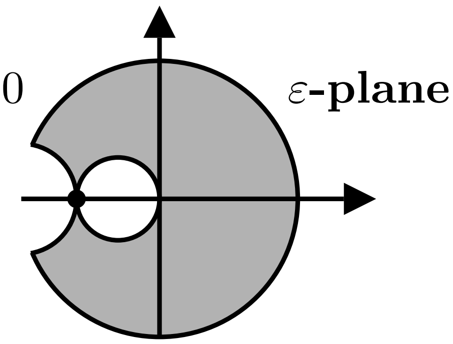
(4) *Value* **:** $\mathrm{Val}(\theta) = \frac{1}{k!} \Big(\prod_v \varepsilon \partial^{\,J_v} f_{\boldsymbol{\nu}_v}(\boldsymbol{\beta}_0) \Big) \Big(\prod_{lines \lambda} g_{i_\lambda j_\lambda} \Big)$

$$g_{ij} \stackrel{def}{=} \frac{\delta_{ij}}{(\boldsymbol{\omega} \cdot \boldsymbol{\nu}(\lambda))^2}, \qquad g_{ij} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \varepsilon \partial^{\,2}_{\boldsymbol{\beta}\boldsymbol{\beta}} f(\boldsymbol{\beta}_0) \end{pmatrix}$$

$\mathbf{h}_{\boldsymbol{\nu}} \equiv \sum^*_\theta \mathrm{Val}(\theta)$ **:**
$*\to$ **no trivial node with 0 harmonic**

Estimate: $|h^{(k)}_{\boldsymbol{\nu}}| \leq bB^k \varepsilon^k k!^{2\tau} \to$ **!! Results:**

$\varepsilon \in \mathcal{E}$**;** $\mathcal{E}$ **dense at** 0
elliptic: $\varepsilon < 0$**,**
$\partial^{\,2}_{\boldsymbol{\beta}\boldsymbol{\beta}} \overline{f}(\boldsymbol{\beta}_0) < 0$



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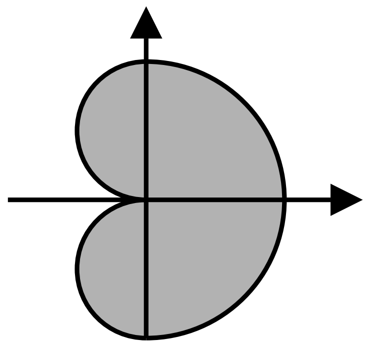
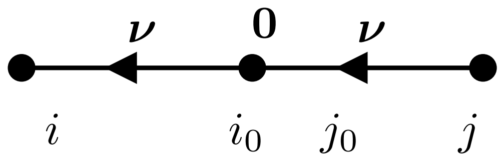


$\varepsilon > 0$, **hyperbolic case**
$(\partial^2_{\boldsymbol{\beta\beta}} \overline{f}(\boldsymbol{\beta}_0) < 0)$
analyticity region
common **to all** $\varepsilon > 0$
BUT **estimate** $k!^{3\tau}$

Need $k!^2$ **for Borel summability (but** $3\tau \geq 3$**).**

Question: *is there uniqueness* **? Are others' results the same ? (Delshams, LLave, Zhou** $\ell = 3, r = 2$**, Treshev** $\varepsilon > 0$ **only)**

Inserting a node with 0 harmonic ($\Rightarrow \boldsymbol{\nu} \neq \mathbf{0}$**) get**

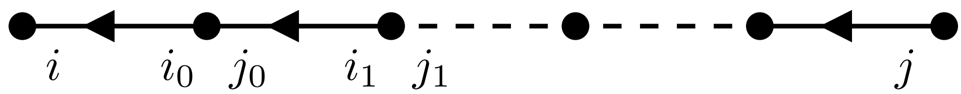


$\Rightarrow$Simple modification of propagator

Let $M_{i_0 j_0} \stackrel{def}{=} \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon \partial^2 \overline{f}(\boldsymbol{\beta}_0) \end{pmatrix}$

$\frac{\delta_{ij}}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \to \frac{\delta_{ij_0}}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \big(M_{i_0 j_0} \frac{\delta_{j_0 j}}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \big)$

Can form *chains*



$\frac{\delta_{ij}}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \to \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \big(M \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \big)^k$ *and* **can sum over** k**.**

"Simplify" trees by *excluding* **trivial nodes: price**

$\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \Rightarrow \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \sum_{k=0}^{\infty} \Big(M \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \Big)^k \equiv \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2 - \varepsilon M}$

BUT $z = M \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} < 1$ **? NO so we are using**

$$\sum 2^k = 1 + 2 + 4 + 8 + 16 + .. = -1$$

If accepted $\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \Rightarrow \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\sum_{k=0}^{\infty}\left(M\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\right)^k \equiv \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2-\varepsilon M}$

gives $\varepsilon > 0$ **"easier" than** $\varepsilon < 0$**.**
For $\varepsilon < 0$ **expect to exclude** ε **s.t** $|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| = \pm\sqrt{-\varepsilon\mu_j}$**.**

Key: Siegel theorem
Given a tree θ *let* $\mathcal{N}_n$ *be the number of lines of scale* n*: i.e.* $2^{-n} < C|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| \le 2^{-n+1}$, $n = 0, 1, \ldots$ *IF no pair lines* $\lambda' < \lambda$ *with* $\boldsymbol{\nu}(\lambda') = \boldsymbol{\nu}(\lambda)$ *with only lower scale intermediates THEN*

$$\mathcal{N}_n \le 4N2^{-n/\tau}\,k$$

Trivial bound ($\varepsilon > 0$**):**

$\prod_v |\partial^{J^v} f_{\boldsymbol{\nu}_v} f(\boldsymbol{\beta}_0)| \le \prod N^{|J^v|}F^k \le N^{2k}$

$\prod_\lambda |\text{propagators}| \le C^{2k}\big(\prod_{n=0}^{\infty} 2^{2nn2^{-n/\tau}}\big)^k$

number of harmonics $\le (2N+1)^k$

number of trees $\le 4^k k!$

<u>**Convergence for** $|\varepsilon| < \big(N^2(2N+1)^\ell F 32^{-8N\sum_n n2^{-n/\tau}}\big)^{-1}$</u>

PROBLEM: there are LOTS of other chains !