Divergent series summation in Hamilton Jacobi equation

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Eq. Motion: $\ddot{\boldsymbol{\alpha}} = -\varepsilon \partial_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha})$

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{T}^\ell$$

$$f(\boldsymbol{\alpha}) \quad \textit{analytic or}$$

$$f(\boldsymbol{\alpha}) = \sum_{\boldsymbol{\nu} \in \mathbb{Z}^\ell} f_{\boldsymbol{\nu}}\, e^{i\boldsymbol{\nu}\cdot\boldsymbol{\alpha}}, \quad f_{\boldsymbol{\nu}} \equiv 0 \text{ if } |\boldsymbol{\nu}| > N$$

$$\textit{Equations of motions} \quad \ddot{\boldsymbol{\alpha}} = -\varepsilon\, \partial_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha})$$

Resonance of order s with frequencies $\boldsymbol{\omega}_0 \in \mathbb{R}^r$ *(unperturbed)*
$\equiv$ *motions with rotation* $\boldsymbol{\omega} = (\boldsymbol{\omega}_0, \mathbf{0}) \in \mathbb{R}^r \times \mathbb{R}^s, \quad \ell = r + s$

$$|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}| > \frac{1}{C|\boldsymbol{\nu}|^\tau}, \qquad \forall \quad \mathbf{0} \neq \boldsymbol{\nu} \in \mathbb{Z}^r$$

$$\boldsymbol{\alpha} \stackrel{def}{=} (\boldsymbol{\gamma}, \boldsymbol{\beta}) \in \mathbb{T}^r \times \mathbb{T}^s, \quad t \to (\boldsymbol{\gamma} + \boldsymbol{\omega}_0 t, \boldsymbol{\beta})$$

$\boldsymbol{\gamma}=$ *"fast angles"*, $\boldsymbol{\beta}=$*"slow angles"*

Hamilton-Jacobi: Find $\mathbf{h}(\boldsymbol{\psi}) \stackrel{def}{=} \binom{\mathbf{g}(\boldsymbol{\psi})}{\mathbf{k}(\boldsymbol{\psi})}$, $\boldsymbol{\psi} \in \mathbb{T}^r$, $\boldsymbol{\beta}_0 \in \mathbb{T}^s$ *with*

$$\mathbf{g}(\boldsymbol{\psi}), \mathbf{k}(\boldsymbol{\psi}) \in \mathbb{R}^r \times \mathbb{R}^s$$

so that $\ddot{\boldsymbol{\alpha}} = -\varepsilon \partial_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha})$, $\boldsymbol{\alpha} \equiv (\boldsymbol{\gamma}, \boldsymbol{\beta})$, *is solved by*

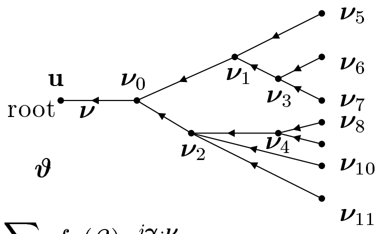
$$\boldsymbol{\gamma} = \boldsymbol{\psi} + \mathbf{g}(\boldsymbol{\psi}), \qquad \boldsymbol{\beta} = \boldsymbol{\beta}_0 + \mathbf{k}(\boldsymbol{\psi}), \qquad \boldsymbol{\psi} \to \boldsymbol{\psi} + \boldsymbol{\omega}_0 t$$

This means

$$(\boldsymbol{\omega}_0 \cdot \partial_{\boldsymbol{\psi}})^2 \begin{pmatrix} \mathbf{g}(\boldsymbol{\psi}) \\ \mathbf{k}(\boldsymbol{\psi}) \end{pmatrix} = -\varepsilon \, \partial_{\boldsymbol{\alpha}} f\big(\boldsymbol{\psi} + \mathbf{g}(\boldsymbol{\psi}), \boldsymbol{\beta}_0 + \mathbf{k}(\boldsymbol{\psi})\big)$$

Resonance $\Rightarrow$ *dimensionality drop from* ℓ *to* r $\Rightarrow$ $\partial_{\boldsymbol{\beta}} \overline{f}(\boldsymbol{\beta}_0) = \mathbf{0}$,
Let $\overline{f}(\boldsymbol{\beta}) \stackrel{def}{=} \int \frac{d\boldsymbol{\gamma}}{(2\pi)^r} f(\boldsymbol{\gamma}, \boldsymbol{\beta})$. **Condition** $\det \partial^2_{\boldsymbol{\beta}\boldsymbol{\beta}} \overline{f}(\boldsymbol{\beta}_0) \neq 0$

Proposition: $\exists$ *power series solution (elementary)*



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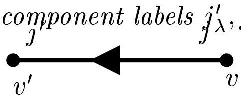
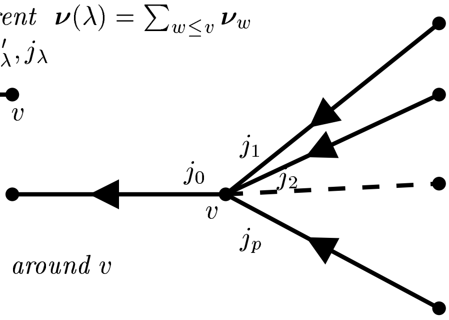
Notation: $f(\boldsymbol{\gamma},\boldsymbol{\beta}) \stackrel{def}{=} \sum_{\boldsymbol{\nu}\in\mathbb{Z}^r} f_{\boldsymbol{\nu}}(\boldsymbol{\beta})\, e^{i\boldsymbol{\gamma}\cdot\boldsymbol{\nu}}$,

$$\partial_j f_{\boldsymbol{\nu}}(\boldsymbol{\beta}_0) \stackrel{def}{=} i\nu_j f_{\boldsymbol{\nu}}(\boldsymbol{\beta}_0), \qquad \partial_j f_{\boldsymbol{\nu}}(\boldsymbol{\beta}_0) \stackrel{def}{=} \partial_{\beta_j} f_{\boldsymbol{\nu}}(\boldsymbol{\beta}_0),$$

$$\partial_{\,J} f_{\boldsymbol{\nu}}(\boldsymbol{\beta}_0) \stackrel{def}{=} \partial_{j_0,\ldots,j_p} f_{\boldsymbol{\nu}}(\boldsymbol{\beta}_0), \quad J = (j_0,\ldots,j_p)$$

(1) To node v *attach a harmonic* $\boldsymbol{\nu}_v \in \mathbb{Z}^r$
(2) To line $\lambda \equiv v'v$ *attach current* $\boldsymbol{\nu}(\lambda) = \sum_{w \le v} \boldsymbol{\nu}_w$
(3) and two component labels j'_λ, j_λ





Component labels around v

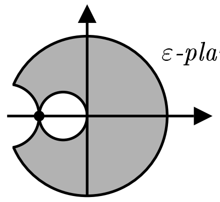
$v \to J_v = (j_0, \ldots, j_p)$ *and* $\partial_{J_v} f_{\boldsymbol{\nu}_v}(\beta_0)$ *are defined.*
(4) Value : $\mathrm{Val}(\theta) = \frac{1}{k!}\Big(\prod_v \varepsilon\partial_{J_v} f_{\boldsymbol{\nu}_v}(\beta_0)\Big)\Big(\prod_{lines\lambda} g_{j_\lambda j'_\lambda}\Big)$

$g_{ij} \stackrel{def}{=} \frac{\delta_{ij}}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu}(\lambda))^2}$, *or* $g_{ij} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\varepsilon\partial^2_{\boldsymbol{\beta}}\overline{f}(\beta_0))^{-1} \end{pmatrix}$ *if* $\boldsymbol{\nu}(\lambda) = \mathbf{0}$

$\mathbf{h}_{\boldsymbol{\nu}} \equiv \sum^*_\theta \mathrm{Val}(\theta)$ *:* $* \longleftrightarrow$ *no trivial node with* $\mathbf{0}$ *incoming current*
Estimate: $|h^{(k)}_{\boldsymbol{\nu}}| \le bB^k\varepsilon^k k!^{2\tau} \;\to$ *!! Results:*

Theorem: **The tree series can be rearranged to yield a convergent series representation of h, hence its existence, in**

$\varepsilon \in \mathcal{E}$; $\mathcal{E}$ dense at 0
elliptic: $\varepsilon < 0$,
$\partial^2_{\beta\beta}\overline{f}(\beta_0) < 0$



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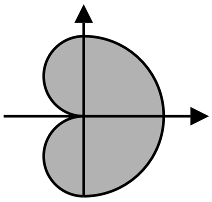
The set $\mathcal{E} \subset (-\varepsilon_0, 0]$ has open dense complement but 0 is a (Lebesgue) density point



$\varepsilon > 0$, hyperbolic case
($\partial^2_{\beta\beta}\overline{f}(\beta_0) < 0$)
analyticity region
common to all $\varepsilon > 0$
BUT estimate $k!^{3\tau}$

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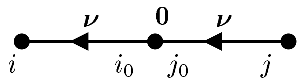
Need $k!^2$ for Borel summability (but $3\tau \geq 3$).

Question: is there uniqueness ? Are others' results the same ? (Delshams, Llave, Zhou $\ell = 3, r = 2$, Treshev $\varepsilon > 0$ only)

Inserting a "trivial" node with $\mathbf{0}$ *harmonic* ($\Rightarrow \boldsymbol{\nu} \neq \mathbf{0}$)

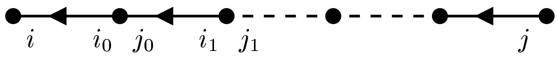


Let $M_{0;i_0j_0} \stackrel{def}{=} \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon\partial^2 \overline{f}(\beta_0) \end{pmatrix}$ 5

get $\frac{\delta_{ij}}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \;\to\; \frac{\delta_{ii_0}}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\big(M_{0;i_0j_0}\frac{\delta_{j_0j}}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\big) \;\Rightarrow$ *propagator modification*

$$M_{0;i_0j_0} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \varepsilon\partial^{\,2}_{i_0j_0} f_{\mathbf{0}}(\beta_0) \end{pmatrix}, \qquad f_{\mathbf{0}}(\beta_0) \equiv \overline{f}(\beta_0)$$

Can form chains of trivial nodes (large values $k!^{2\tau}$)



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$$\frac{\delta_{ij}}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \;\to\; \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\big(M_0\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\big)^k$$

.

"Simplify": NO trivial nodes; *price :*

$$\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \Rightarrow \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\sum_{k=0}^{\infty}\big(M_0\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\big)^k \equiv \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2 - M_0}$$

BUT $z = M_0\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} < 1$ *? NO* **!**

so we are using $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$, $z \neq 1$, *e.g.*

$$\sum_{k=0}^{\infty} 2^k = 1 + 2 + 4 + 8 + 16 + \cdots = \mathbf{-1}$$

If accepted $\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \Rightarrow \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \sum_{k=0}^{\infty} \left(M_0 \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\right)^k \equiv \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2 - M_0}$

$M_0 = \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon \partial_{\boldsymbol{\beta}}^2 \overline{f}(\beta_0) \end{pmatrix}$ *gives* $\varepsilon > 0$ *"easier" than* $\varepsilon < 0$.

For $\varepsilon < 0$ *expect to exclude* ε *s.t* $|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| = \pm\sqrt{-\varepsilon \mu_j}$.

PROBLEM: there are LOTS of other chains ! They cause values $k!^{\eta}$, $\eta > 0$
IDEA: eliminate them "by resummation": left with convergent series

KEY: Siegel's theorem

Given a tree θ *let* $\mathcal{N}_n$ *be the number of lines of scale* n*: i.e. s.t.*

$$2^{-n} < C|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| \le 2^{-n+1}$$

$n = 0, 1, \ldots$. *IF no pair lines* $\lambda' < \lambda$ *with* $\boldsymbol{\nu}(\lambda') = \boldsymbol{\nu}(\lambda)$ *with only lower scale intermediates THEN*

$$\mathcal{N}_n \le 4N 2^{-n/\tau}\, k$$

$$\mathcal{N}_n \le 4N2^{-n/\tau}\, k \Rightarrow$$
$$\Rightarrow \prod_v \frac{1}{(\boldsymbol{\omega}_0\cdot\boldsymbol{\nu})^2} \le C^{2k}\prod_{n=0}^{\infty} 2^{2n\mathcal{N}_n} \le C^{2k}\big(\prod_{n=0}^{\infty} 2^{2n(4N2^{-n/\tau})}\big)^k$$

Trivial bound ($\varepsilon > 0$):

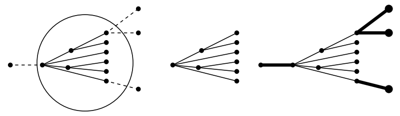
$$\prod_v |\partial_{J^v} f_{\boldsymbol{\nu}_v}(\beta_0)| \le \prod_v N^{|J^v|}F^k \le N^{2k}F^k$$

number of harmonics $\boldsymbol{\nu}$: $\le (2N+1)^k$, number of trees $\le k^{k-1}$
Convergence:

$$|\varepsilon| < \left(N^2\cdot C^2\cdot(2N+1)^\ell\cdot 3\cdot F\cdot 2^{8N\sum_n n2^{-n/\tau}}\right)^{-1}$$

Multiscale analysis: Organize the lines of θ into clusters

Definition: A cluster of scale n is a maximal connected set of lines of θ with scales $p \le n$ and with one line at least of scale n.



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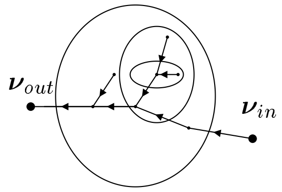
Self energy clusters $\boldsymbol{\nu}_{in} = \boldsymbol{\nu}_{out}$



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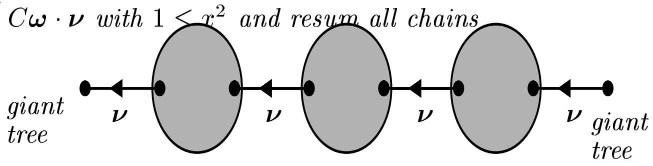
Eliminate self energy clusters by resummations

Necessary multiscale analysis to avoid "overlapping divergences"
First identify the self energy clusters of scale [0]
i.e. with $x \stackrel{def}{=} C\boldsymbol{\omega}\cdot\boldsymbol{\nu}$ *with* $1 < x^2$ *and resum all chains*



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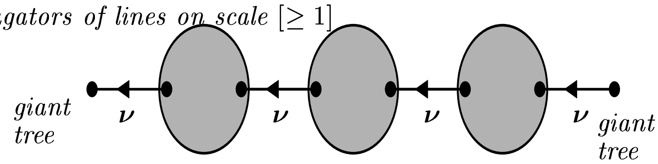
Key remark: each s.e. cluster does not contain s.e. clusters

Summing over the contents of each s.e. cluster $\Rightarrow$ *convergent sum by Siegel's lemma.*

Summing over s.e. clusters of scale [0] *leads to modify propagators of lines on scale* [≥ 1]



$$g^{[\geq 1]}(x) = \frac{1}{x^2 - M_0} \;\to\; \frac{1}{x^2 - M_0} \sum_{n=0}^{\infty} (M_1 \frac{1}{x^2 - M_0})^n \tag{9}$$

which becomes

$$\frac{1}{x^2 - M_0 - M_1}$$

and graphs simplify with no s.e. subgraphs of scale [0].

Iterate! at every step only graphs with no s.e. subgraphs have to be considered; ⇒ *convergent additions made on propagators*

In the hyperbolic case no real new problems arise.

In the elliptic the situation is very different.
As the scale decreases the scale $2^{-n} \simeq \varepsilon$ *is reached and*
$x^2 - M_0 - M_1 - \ldots - M_{n-1}$ *can vanish* $\Rightarrow$

(a) More values of ε *excluded*

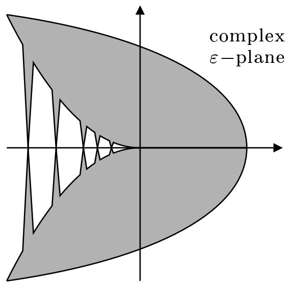
(b) The successive scales must be measured by the size of $x^2 - M_0 - M_1 - \ldots$*; analysis becomes delicate:*

DIFFICULTY: even in the hyperbolic case it is necessary to check that the renormalized propagators have the same size as the bare ones

Siegel's lemma applies only to graphs in which the propagators size is of the order of $x^2 \equiv (\boldsymbol{\omega} \cdot \boldsymbol{\nu})^{-2}$.

Not automatic but checked via the cancellation mechanism of the KAM theory: this time the cancellations are only partial but still enough.

OPEN PROBLEM:
uniqueness (and relation with alternative existence results)



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Precise formulation in HJ (fixed $\boldsymbol{\omega}$ (C,τ)-Diophant.)

$$F(\mathbf{A}',\boldsymbol{\alpha})\stackrel{def}{=}\frac{1}{2}\big(\mathbf{A}'+\partial_{\boldsymbol{\alpha}}\Phi(\mathbf{A}',\boldsymbol{\alpha})\big)^2+\varepsilon f(\boldsymbol{\alpha})$$

$\exists\ \mathbf{A}'_n \xrightarrow[n\to\infty]{} \mathbf{A}'_\infty$ *and* ρ_n,ξ *such that in* $S_{\rho_n}(\mathbf{A}'_n)\times(\mathbb{T}^\ell)_\xi$

$$\Phi_n(\mathbf{A}'_n,\boldsymbol{\alpha})\Rightarrow\begin{cases}\partial_{\boldsymbol{\alpha}}\Phi_n\xrightarrow[n\to\infty]{}\widetilde{H}(\boldsymbol{\alpha}),\ \partial_{\mathbf{A}'}\Phi_n\xrightarrow[n\to\infty]{}\widetilde{\mathbf{h}}(\boldsymbol{\alpha})\\ \partial^2_{\boldsymbol{\alpha}}\Phi_n\xrightarrow[n\to\infty]{}\widetilde{H}'(\boldsymbol{\alpha}),\ \partial^2_{\boldsymbol{\alpha},\mathbf{A}'}\Phi_n\xrightarrow[n\to\infty]{}\widetilde{H}''(\boldsymbol{\alpha})\end{cases}$$

$$\Rightarrow\begin{cases}\frac{1}{2}(\mathbf{A}'_\infty+\widetilde{\mathbf{H}}(\boldsymbol{\alpha}))^2+\varepsilon f(\boldsymbol{\alpha})=E=\boldsymbol{\alpha}-\text{indep.}\\ \partial_{\mathbf{A}'}F(\mathbf{A}'_n,\boldsymbol{\alpha})\xrightarrow[n\to\infty]{}\boldsymbol{\omega}\\ \boldsymbol{\psi}=\boldsymbol{\alpha}+\widetilde{h}(\boldsymbol{\alpha})\longleftrightarrow\boldsymbol{\alpha}=\boldsymbol{\psi}+\mathbf{h}(\boldsymbol{\psi})\\ \boldsymbol{\psi}(t)=\boldsymbol{\psi}+\boldsymbol{\omega}\,t\ \textit{is solution}\end{cases}$$