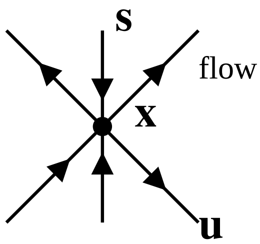
Chaotic motions and developed turbulence

Non equilibrium: *Chaotic hypothesis*

The attractor of a chaotic evolution can be regarded as an Anosov system
$\mathcal{F}$ = phase space



covariant decomposition

$\Rightarrow \exists$ SRB distribution,*i.e.* the "statistics"

$$\frac{1}{T}\int_0^T F(S_t x)dt \xrightarrow[T\to\infty]{} \langle F\rangle \equiv \int \mu(dy)F(y) \qquad a.e.$$

μ is *singular* (in general): $\qquad \dot{x} = f_E(x), \quad x\in\mathcal{F}$

$$\sigma(x) \stackrel{def}{=} -\,\text{divergence} f_E(x), \qquad \sigma_+ \stackrel{def}{=} \langle\sigma\rangle_\mu \,(\geq 0 \text{ Ruelle})$$

Idea: Chaotic hypothesis + symmetries $\rightarrow$ "predictions"

Symmetries: *time reversal*, *symplectic*, ...
Time rev. = isometry, $I^2 = 1$ and *anticommuting*: $\qquad IS_t = S_{-t}I$ or $IS = S^{-1}I$

For *time reversible* Anosov maps or flows:
Fluctuation theorem (Cohen,G, Gentile) if $\sigma_+ > 0$

$$p \stackrel{def}{=} \frac{1}{\tau}\int_0^\tau \frac{\sigma(S_t x)}{\sigma_+}dt, \qquad \text{probab.}(p\in\Delta) = c\, e^{\tau \max_\Delta \zeta(p) + O(1)} \quad \textbf{and}$$

$$\zeta(-p) = \zeta(p) - p\sigma_+, \qquad \text{for all } |p| < p^*, \qquad p^* \geq 1$$

with no free parameters (symmetry relation).
Symplectic symmetry $\Rightarrow$ *pairing rule* (Dettman, Morriss).

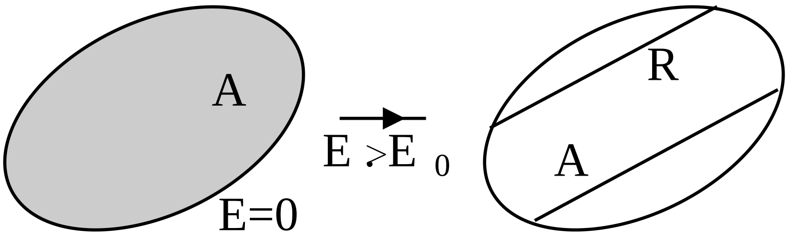
Applications

(1) Not transitive, (*attractor not dense*)

(2) Not reversible, (*viscosity models*)

(3) No pairing, (*lack of symplectic structure*)

Not transitive: Chaotic hypothesis $\to$ attracting set is a manifold.

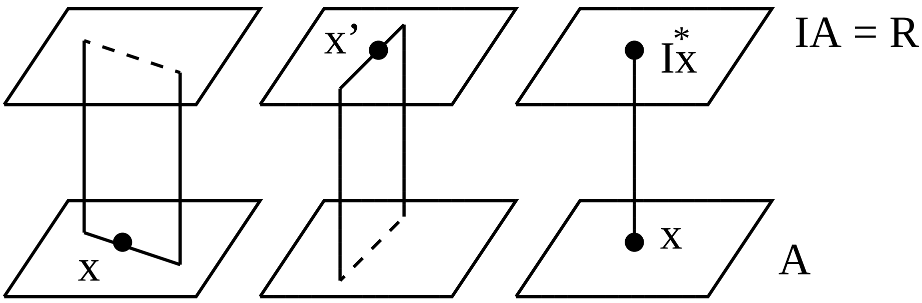


$\overline{R} = I\overline{A} \ne \overline{A}$

Unstable manifold of a point on A is in A

Stable mnf "sticks out" reaching repeller R

If transversal: line from R to A is defines λ defines $ix = x'$ commuting with S and $I^* \stackrel{def}{=} I\,i$ is *local time reversal* on A (Bonetto, G.).



FT holds for the *phase space contraction on the attractor* (quite useless but ...)

Pairing symmetry: (*if holding*) implies

$$\zeta(-p) = \zeta(p) - \sigma_+ \frac{M}{N}$$

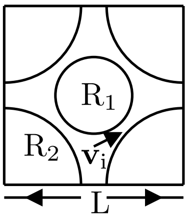
$2M$ = dimension of attractor, $2N$ dimension of phase space.

Lack of reversibility?

Idea: an irreversible dynamics can be equivalent to a reversible one:

same statistics (G.)

Example (Drude's theory of conductivity)



$\ddot{\mathbf{q}}_i = E\mathbf{u} + \textit{collisions}\ - \boldsymbol{\vartheta}_i$: and

speed renormalized to $\sqrt{3k_B\Theta}$

or keep constant speed: $\boldsymbol{\vartheta}_i = \alpha(\dot{\mathbf{q}})\dot{\mathbf{q}}_i,\ \alpha \equiv \frac{\mathbf{E}\cdot\sum\dot{\mathbf{x}}_i}{\sum\dot{\mathbf{x}}_i^2}$

or use viscosity: $\boldsymbol{\vartheta}_i = -\nu\dot{\mathbf{q}}_i$

If $F(\mathbf{p},\mathbf{q}) > 0$ is a local observable: $\frac{\mu_\nu(F)}{\mu_{\mathcal{E}}(F)} \xrightarrow[\substack{L\to\infty,\ N/L=\rho\\ \mathcal{E}=\frac{3}{2}NkT}]{} 1$

provided ν is tuned so that $\langle\mathcal{E}\rangle_{\mu_\nu} = \mathcal{E}_0$.

This is "equivalence of ensembles": analogy

$\nu =$ canonical temperature and $\mathcal{E}$ = microcanonical energy.

Application to **NS** (incompressible $\partial\cdot\mathbf{u} = 0$)

$$\dot{\mathbf{u}} + \underset{\sim}{\mathbf{u}}\cdot\underset{\sim}{\partial}\,\mathbf{u} = \nu\Delta\mathbf{u} - \partial\,p + f\mathbf{g}, \qquad R = \frac{\sqrt{fL}}{\nu}$$

Actually think of : cut off at $|\mathbf{k}| \le K_k = L^{1}R^{\frac{3}{4}}$, $N \simeq R^{\frac{9}{4}}$, *i.e.* OK41 is assumed.

To apply the chaotic hyp. need

(1) *chaos* (yes, if R large).

(2) *reversibility*

(3) *pairing* (because the attractor is very small)

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(1) Equivalence with reversible equations "Gaussian NS eq."

$$\dot{\mathbf{u}} + \underset{\sim}{\mathbf{u}}\cdot\underset{\sim}{\partial}\,\mathbf{u} = \alpha(\mathbf{u})\Delta\mathbf{u} - \partial\,p + f\mathbf{g}, \qquad \alpha = \frac{\int\mathbf{u}\cdot f\,\mathbf{g}}{\int(\partial\,\mathbf{u})^2}\ \Rightarrow\ \int\mathbf{u}^2 = \mathcal{E} = const$$

Same statistics for "local observables": F local $\Rightarrow$ F depends on finitely many Fourier components of $\mathbf{u}$. **Same statistics** as $R\to\infty$ if $\mathcal{E}$ is chosen $= \langle\int\mathbf{u}^2\rangle_{\mu_\nu}$ (equivalence)

Consequence $\langle\alpha\rangle/\nu \to 1$: **only numerical tests in strongly cut off equations and** $d = 2$ (*Rondoni, Segre*).

Earlier *She, Jackson*: large numerical simulations (different reversible equation)

Other tests: are Lyapunov spectra also identical? (Rondoni, Segre, G.). Here are a few graphs in highly truncated equations ($d = 2$)
Also the linear FR relation comes out within the precision: the approximate pairing that can be observed leads to test the slope $(1 - \frac{2M}{2N})\sigma_+$ in the GNS equations: from the theory it is expected a slope $< \sigma_+$ by the ratio of the number of negative pairs to the nuber of total pairs.

Barometric formula:
Consider the equations (incompressible NS and ED)

$$\dot{\mathbf{u}} + \underset{\sim}{\mathbf{u}} \cdot \underset{\sim}{\partial}\mathbf{u} = \nu\Delta\mathbf{u} - \partial\, p + f\mathbf{g}, \qquad \dot{\mathbf{u}} + \underset{\sim}{\mathbf{u}} \cdot \underset{\sim}{\partial}\mathbf{u} = -\chi\mathbf{u} - \partial\, p + f\mathbf{g},$$

here $\mathbf{u} = \sum_{\mathbf{k}} \boldsymbol{\gamma}_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}}$, $\mathcal{E} = L^3 \sum_{\mathbf{k}} |\boldsymbol{\gamma}_{\mathbf{k}}|^2$.
The equivalence idea leads to think that although the statistics of the two equations are certainly different *nevertheless they might coincide on an appropriate scale.* The friction in NS varies with the scale $\mathbf{k}$ and at some scale it might match that of ED.
By OK41 $v_k^3 k = constant = \eta\nu$ in NS: OK41 does not hold for ED: to fix ideas asume that at fixed cut off k_χ there is *equipartition* between the modes. Then $\langle|\boldsymbol{\gamma}_{\mathbf{k}}|^2\rangle \equiv \gamma^2$

$$\begin{aligned}
&\frac{4\pi}{3}\gamma^2\,(k_\chi \frac{L}{2\pi})^3 = \varepsilon, && \textit{energy density at equipartition}\\
&K_E(k) = \frac{3\varepsilon}{4\pi}\frac{k^2}{k_\chi^3}, && \textit{energy density between } k \textit{ and } k+dk\\
&v_k^3\, k \;=\; \big((kL)^3\gamma^2\big)^{\frac32} k = \varepsilon^{\frac32} k_\chi\,\big(\frac{k}{k_\chi}\big)^{\frac{11}{2}}, && \textit{dimensionless dissipation on scale } k
\end{aligned}$$

If $\varepsilon^{\frac32} k_\chi\,\big(\frac{k}{k_\chi}\big)^{\frac{11}{2}} = \eta\nu$ then NS equation and ED equations have the same statistics on scale k: doubling the dissipation in NS the statistics of the two equations agree on scale 1.3 higher.
The k, or better $\log\frac{k}{k_\chi}$, is the analog of the height and the dim.less dissipation is tyhe analogue of the pressure.

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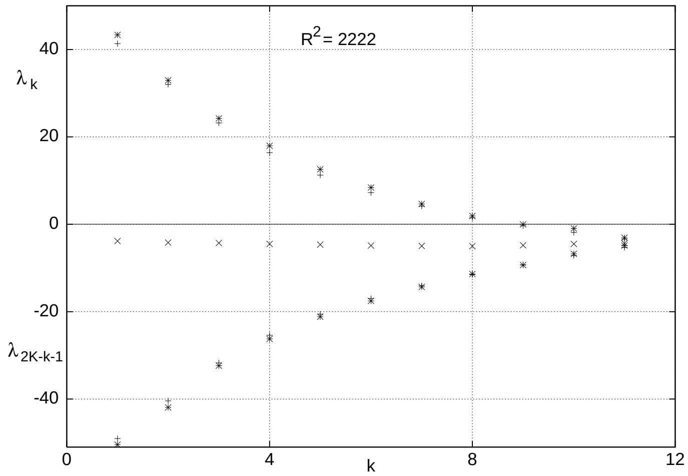

$R^2 = 2222$
λ_k
λ_{2K-k-1}
40
20
0
-20
-40
0
4
8
12
k

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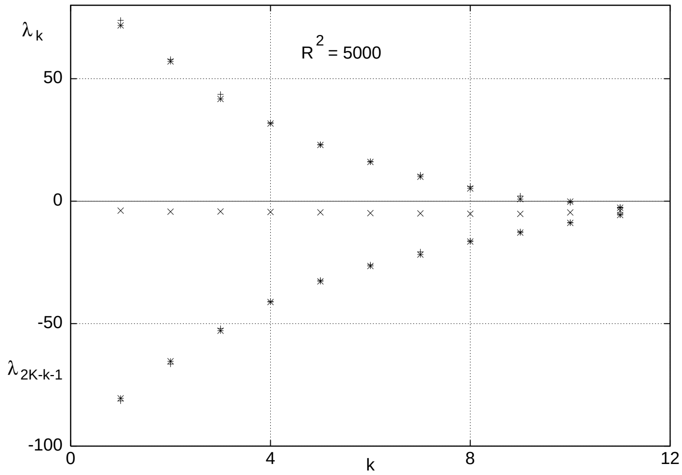

$R^2 = 5000$
λ_k
λ_{2K-k-1}
50
0
-50
-100
0
4
8
12
k

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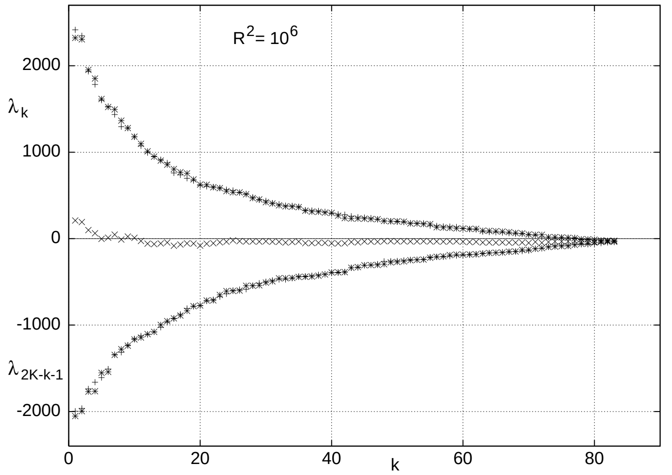
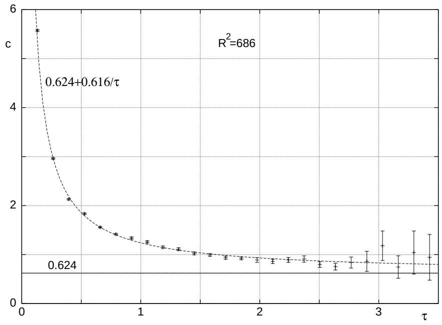


$R^2 = 10^6$
λ_k
λ_{2K-k-1}
2000
1000
0
-1000
-2000
0
20
40
60
80
k

R^2	$\delta Q_0/\langle Q_0\rangle_{NS}$	$\triangle\alpha$	$\triangle Q_1$	$o(M)/M$
800	0.005	0.030	0.053	0.068
1250	0.020	0.018	0.062	0.057
2222	0.002	0.039	0.058	0.077
4444	0.050	0.021	0.093	0.059
5000	0.010	0.008	0.058	0.033

Equivalence NS-GNS dynamics at different Reynolds numbers, column $\Delta\alpha$ to be compared with 1, cfr. [RS99]).



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Evolution towards limiting slope as τ increases