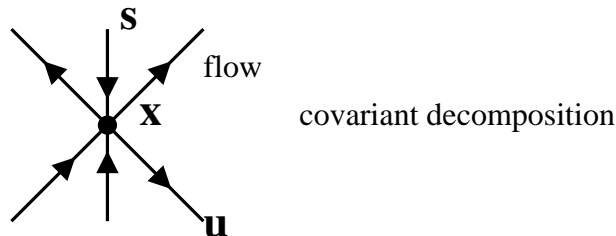


Chaotic motions and developed turbulence

Non equilibrium: *Chaotic hypothesis*

The attractor of a chaotic evolution can be regarded as an Anosov system

\mathcal{F} = phase space



$\Rightarrow \exists$ SRB distribution, *i.e.* the “statistics”

$$\frac{1}{T} \int_0^T F(S_t x) dt \xrightarrow{T \rightarrow \infty} \langle F \rangle \equiv \int \mu(dy) F(y) \quad a.e.$$

μ is *singular* (in general): $\dot{x} = f_E(x), \quad x \in \mathcal{F}$

$$\sigma(x) \stackrel{def}{=} -\text{divergence } f_E(x), \quad \sigma_+ \stackrel{def}{=} \langle \sigma \rangle_\mu (\geq 0 \text{ Ruelle})$$

Idea: Chaotic hypothesis + symmetries \rightarrow “predictions”

Symmetries: *time reversal, symplectic, ...*

Time rev. = isometry, $I^2 = 1$ and *anticommuting*: $IS_t = S_{-t}I$ or $IS = S^{-1}I$

For *time reversible* Anosov maps or flows:

Fluctuation theorem (Cohen,G, Gentile) if $\sigma_+ > 0$

$$p \stackrel{def}{=} \frac{1}{\tau} \int_0^\tau \frac{\sigma(S_t x)}{\sigma_+} dt, \quad \text{probab.}(p \in \Delta) = c e^{\tau \max_\Delta \zeta(p) + O(1)} \quad \mathbf{and}$$

$$\zeta(-p) = \zeta(p) - p\sigma_+, \quad \text{for all } |p| < p^*, \quad p^* \geq 1$$

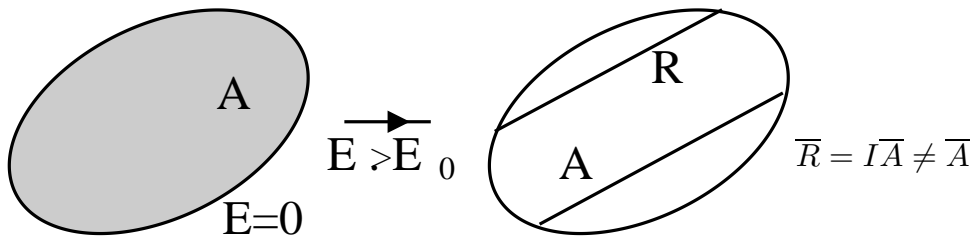
with no free parameters (symmetry relation).

Symplectic symmetry \Rightarrow *pairing rule* (Dettman, Morriss).

Applications

- (1) Not transitive, *(attractor not dense)*
 - (2) Not reversible, *(viscosity models)*
 - (3) No pairing, *(lack of symplectic structure)*
-

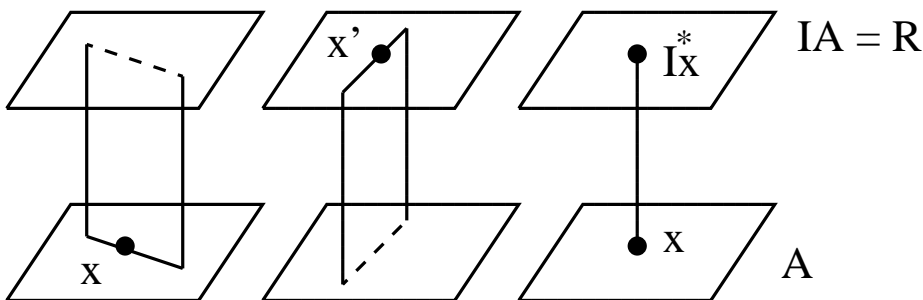
Not transitive: Chaotic hypothesis \rightarrow attracting set is a manifold.



Unstable manifold of a point on A is in A

Stable manifold “sticks out” reaching repeller R

If transversal: line from R to A defines λ defines $ix = x'$ commuting with S and $I^* \stackrel{def}{=} Ii$ is local time reversal on A (Bonetto, G.).



FT holds for the *phase space contraction on the attractor* (quite useless but ...)

Pairing symmetry: (if holding) implies

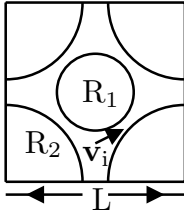
$$\zeta(-p) = \zeta(p) - \sigma_+ \frac{M}{N}$$

$2M =$ dimension of attractor, $2N$ dimension of phase space.

Lack of reversibility?

Idea: an irreversible dynamics can be equivalent to a reversible one:
same statistics (G.)

Example (Drude's theory of conductivity)



$\ddot{\mathbf{q}}_i = E\mathbf{u} + \text{collisions} - \boldsymbol{\vartheta}_i$: and
speed renormalized to $\sqrt{3k_B\Theta}$

or keep constant speed: $\boldsymbol{\vartheta}_i = \alpha(\dot{\mathbf{q}})\dot{\mathbf{q}}_i$, $\alpha \equiv \frac{E \cdot \sum \dot{\mathbf{x}}_i}{\sum \dot{\mathbf{x}}_i^2}$

or use viscosity: $\boldsymbol{\vartheta}_i = -\nu\dot{\mathbf{q}}_i$

If $F(\mathbf{p}, \mathbf{q}) > 0$ is a local observable: $\frac{\mu_\nu(F)}{\mu_\mathcal{E}(F)} \xrightarrow[L \rightarrow \infty, N/L = \rho]{\mathcal{E} = \frac{3}{2}NkT} 1$

provided ν is tuned so that $\langle \mathcal{E} \rangle_{\mu_\nu} = \mathcal{E}_0$.

This is "equivalence of ensembles": analogy

$\nu =$ canonical temperature and $\mathcal{E} =$ microcanonical energy.

Application to NS (incompressible $\partial \cdot \mathbf{u} = 0$)

$$\dot{\mathbf{u}} + \underline{\mathbf{u}} \cdot \underline{\partial} \mathbf{u} = \nu \Delta \mathbf{u} - \partial p + f \mathbf{g}, \quad R = \frac{\sqrt{fL}}{\nu}$$

Actually think of: cut off at $|\mathbf{k}| \leq K_k = L^1 R^{\frac{3}{4}}$, $N \simeq R^{\frac{9}{4}}$, i.e. OK41 is assumed.

To apply the chaotic hyp. need

- (1) chaos (yes, if R large).
 - (2) reversibility
 - (3) pairing (because the attractor is very small)
-

(1) Equivalence with reversible equations "Gaussian NS eq."

$$\dot{\mathbf{u}} + \underline{\mathbf{u}} \cdot \underline{\partial} \mathbf{u} = \alpha(\mathbf{u}) \Delta \mathbf{u} - \partial p + f \mathbf{g}, \quad \alpha = \frac{\int \mathbf{u} \cdot f \mathbf{g}}{\int (\partial \mathbf{u})^2} \Rightarrow \int \mathbf{u}^2 = \mathcal{E} = \text{const}$$

Same statistics for "local observables": F local $\Rightarrow F$ depends on finitely many Fourier components of \mathbf{u} . **Same statistics** as $R \rightarrow \infty$ if \mathcal{E} is chosen = $\langle \int \mathbf{u}^2 \rangle_{\mu_\nu}$ (equivalence)

Consequence $\langle \alpha \rangle / \nu \rightarrow 1$: **only numerical tests in strongly cut off equations and $d = 2$ (Rondoni, Segre).**

Earlier *She, Jackson*: large numerical simulations (different reversible equation)

Other tests: are Lyapunov spectra also identical? (Rondoni, Segre, G.). Here are a few graphs in highly truncated equations ($d = 2$)

Also the linear FR relation comes out within the precision: the approximate pairing that can be observed leads to test the slope $(1 - \frac{2M}{2N})\sigma_+$ in the GNS equations: from the theory it is expected a slope $< \sigma_+$ by the ratio of the number of negative pairs to the number of total pairs.

Barometric formula:

Consider the equations (incompressible NS and ED)

$$\dot{\mathbf{u}} + \underline{\mathbf{u}} \cdot \underline{\partial} \mathbf{u} = \nu \Delta \mathbf{u} - \partial p + f \mathbf{g}, \quad \dot{\mathbf{u}} + \underline{\mathbf{u}} \cdot \underline{\partial} \mathbf{u} = -\chi \mathbf{u} - \partial p + f \mathbf{g},$$

here $\mathbf{u} = \sum_{\mathbf{k}} \gamma_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{x}}$, $\mathcal{E} = L^3 \sum_{\mathbf{k}} |\gamma_{\mathbf{k}}|^2$.

The equivalence idea leads to think that although the statistics of the two equations are certainly different *nevertheless they might coincide on an appropriate scale*. The friction in NS varies with the scale \mathbf{k} and at some scale it might match that of ED.

By OK41 $v_k^3 k = \text{constant} = \eta\nu$ in NS: OK41 does not hold for ED: to fix ideas assume that at fixed cut off k_χ there is *equipartition* between the modes. Then $\langle |\gamma_{\mathbf{k}}|^2 \rangle \equiv \gamma^2$

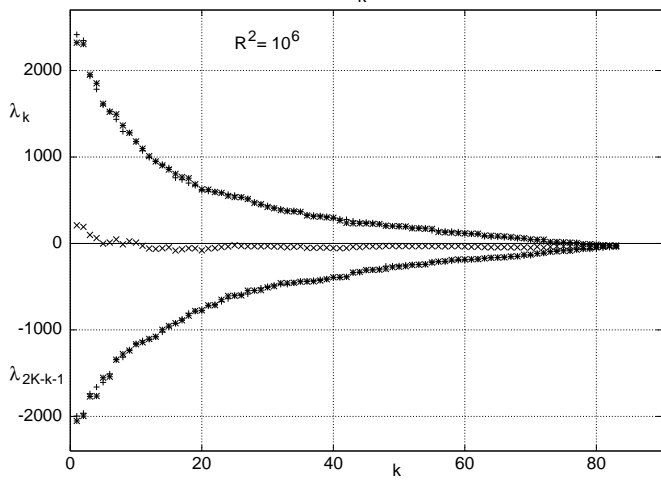
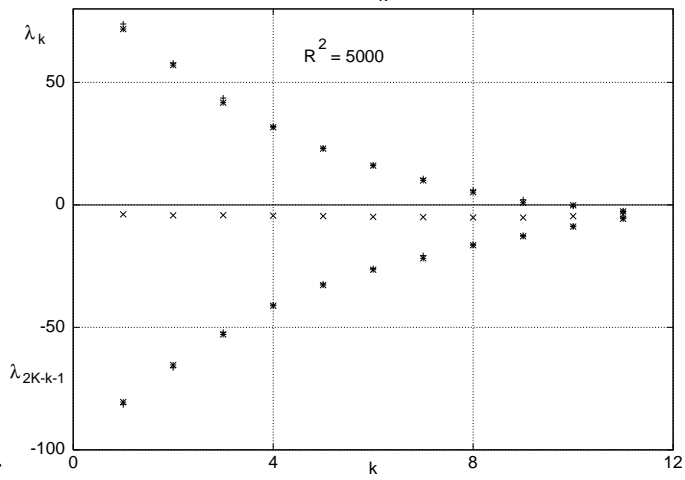
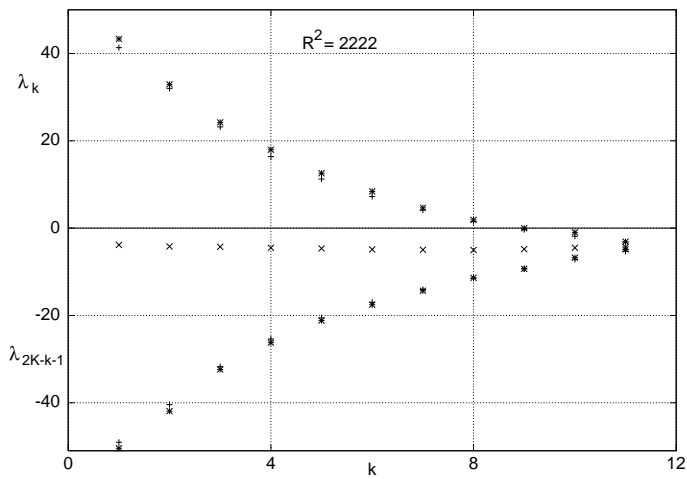
$$\frac{4\pi}{3} \gamma^2 (k_\chi \frac{L}{2\pi})^3 = \varepsilon, \quad \text{energy density at equipartition}$$

$$K_E(k) = \frac{3\varepsilon}{4\pi} \frac{k^2}{k_\chi^3}, \quad \text{energy density between } k \text{ and } k + dk$$

$$v_k^3 k = ((kL)^3 \gamma^2)^{\frac{3}{2}} k = \varepsilon^{\frac{3}{2}} k_\chi \left(\frac{k}{k_\chi}\right)^{\frac{11}{2}}, \quad \text{dimensionless dissipation on scale } k$$

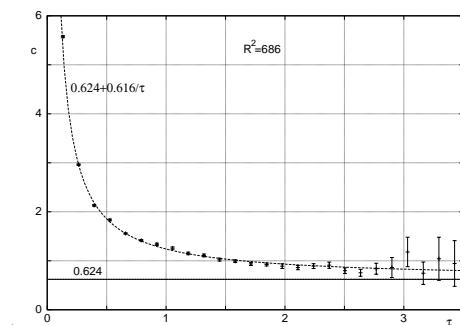
If $\varepsilon^{\frac{3}{2}} k_\chi \left(\frac{k}{k_\chi}\right)^{\frac{11}{2}} = \eta\nu$ then NS equation and ED equations have the same statistics on scale k : doubling the dissipation in NS the statistics of the two equations agree on scale 1.3 higher.

The k , or better $\log \frac{k}{k_\chi}$, is the analog of the height and the dim.less dissipation is the analogue of the pressure.



R^2	$\delta Q_0 / \langle Q_0 \rangle_{NS}$	$\Delta\alpha$	ΔQ_1	$o(M)/M$
800	0.005	0.030	0.053	0.068
1250	0.020	0.018	0.062	0.057
2222	0.002	0.039	0.058	0.077
4444	0.050	0.021	0.093	0.059
5000	0.010	0.008	0.058	0.033

Equivalence NS-GNS dynamics at different Reynolds numbers, column $\Delta\alpha$ to be compared with 1, cfr. [RS99]).



Evolution towards limiting slope as τ increases