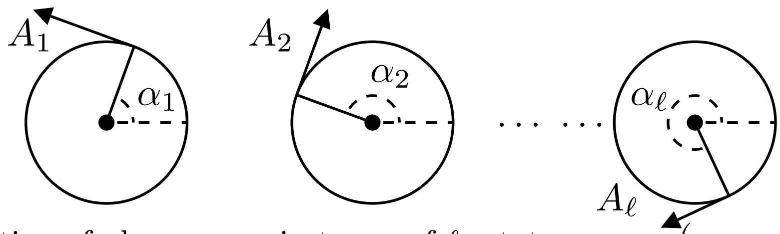
*Resonances, Divergent series and R.G.: (*G. Gentile, G.G.)

Eq. Motion: $\ddot{\boldsymbol{\alpha}} = -\varepsilon \partial_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha})$



1

Representation of phase space in terms of ℓ rotators: $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{T}^\ell$

Potential: $f(\boldsymbol{\alpha}) = \sum_{\boldsymbol{\nu} \in \mathbb{Z}^\ell} f_{\boldsymbol{\nu}}\, e^{i\boldsymbol{\nu}\cdot\boldsymbol{\alpha}}, \quad f_{\boldsymbol{\nu}} \equiv 0$ if $|\boldsymbol{\nu}| > N$

Motions: $\boldsymbol{\alpha} \stackrel{def}{=} (\boldsymbol{\gamma}, \boldsymbol{\beta}) \in \mathbb{T}^r \times \mathbb{T}^s, \quad t \to (\boldsymbol{\gamma} + \boldsymbol{\omega}_0 t, \boldsymbol{\beta})$

$\boldsymbol{\gamma}$= "fast angles", $\boldsymbol{\beta}$="slow angles" $\Rightarrow$ *Resonance* of order s

Independence: $|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}| > \frac{1}{C|\boldsymbol{\nu}|^\tau}, \qquad \forall \quad \mathbf{0} \neq \boldsymbol{\nu} \in \mathbb{Z}^r$

"*Find a resonance copy*", *i.e.* $\mathbf{h} \equiv (\mathbf{g}(\boldsymbol{\psi}), \mathbf{k}(\boldsymbol{\psi})) \in \mathbb{R}^r \times \mathbb{R}^s$ s.t. $\boldsymbol{\alpha} \equiv (\boldsymbol{\gamma}, \boldsymbol{\beta})$, and

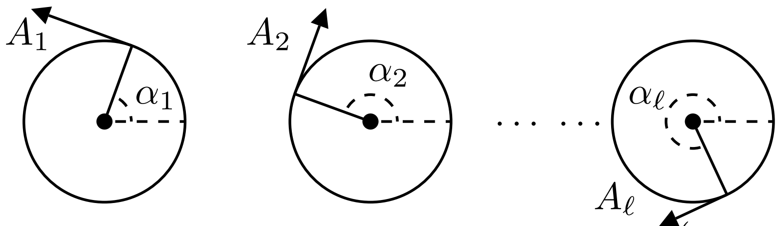
$$\boldsymbol{\gamma} = \boldsymbol{\psi} + \mathbf{g}(\boldsymbol{\psi}), \qquad \boldsymbol{\beta} = \boldsymbol{\beta}_0 + \mathbf{k}(\boldsymbol{\psi}), \qquad \boldsymbol{\psi} \to \boldsymbol{\psi} + \boldsymbol{\omega}_0 t$$

solves $\ddot{\boldsymbol{\alpha}} = -\varepsilon \partial_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha})$, *i.e.*

$$(\boldsymbol{\omega}_0 \cdot \partial_{\boldsymbol{\psi}})^2 \begin{pmatrix} \mathbf{g}(\boldsymbol{\psi}) \\ \mathbf{k}(\boldsymbol{\psi}) \end{pmatrix} = -\varepsilon\, \partial_{\boldsymbol{\alpha}} f\big(\boldsymbol{\psi} + \mathbf{g}(\boldsymbol{\psi}), \boldsymbol{\beta}_0 + \mathbf{k}(\boldsymbol{\psi})\big)$$

If $\overline{f}(\boldsymbol{\beta}) \stackrel{def}{=} \int \frac{d\gamma}{(2\pi)^r} f(\boldsymbol{\gamma}, \boldsymbol{\beta}) \;\Rightarrow\; \partial_{\boldsymbol{\beta}} \overline{f}(\boldsymbol{\beta}_0) = \mathbf{0}$ ($\mathbf{0}$ average force)

There is a power series solution

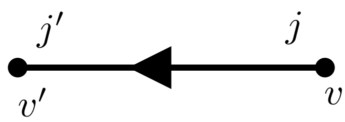


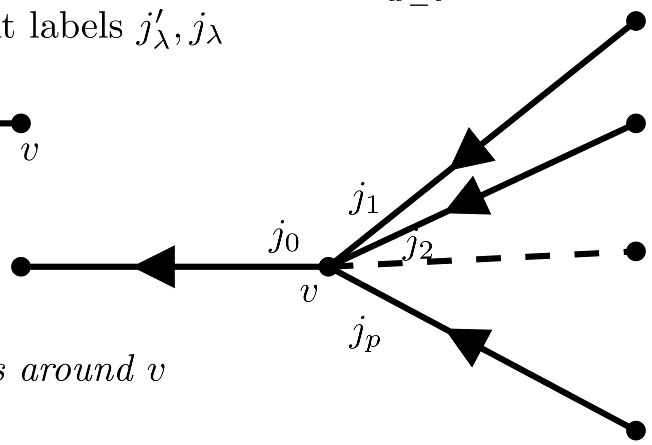
1

Representation of phase space in terms of ℓ rotators: $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{T}^\ell$

Notation: $f(\boldsymbol{\gamma}, \boldsymbol{\beta}) \stackrel{def}{=} \sum_{\boldsymbol{\nu} \in \mathbb{Z}^r} f_{\boldsymbol{\nu}}(\boldsymbol{\beta})\, e^{i\boldsymbol{\gamma}\cdot\boldsymbol{\nu}}$. *Rules*:

(1) To node v attach a *harmonic* $\boldsymbol{\nu}_v \in \mathbb{Z}^r$
(2) To line $\lambda \equiv v'v$ attach *current* $\boldsymbol{\nu}(\lambda) = \sum_{w \le v} \boldsymbol{\nu}_w$
(3) and two component labels j'_λ, j_λ





Component labels around v

$v \to J_v = (j_0, \ldots, j_p)$ and $\partial_{J_v} f_{\boldsymbol{\nu}_v}(\boldsymbol{\beta}_0)$ are defined.

(1) To node v attach a *harmonic* $\boldsymbol{\nu}_v \in \mathbb{Z}^r$
(2) To line $\lambda \equiv v'v$ attach *current* $\boldsymbol{\nu}(\lambda) = \sum_{w \le v} \boldsymbol{\nu}_w$
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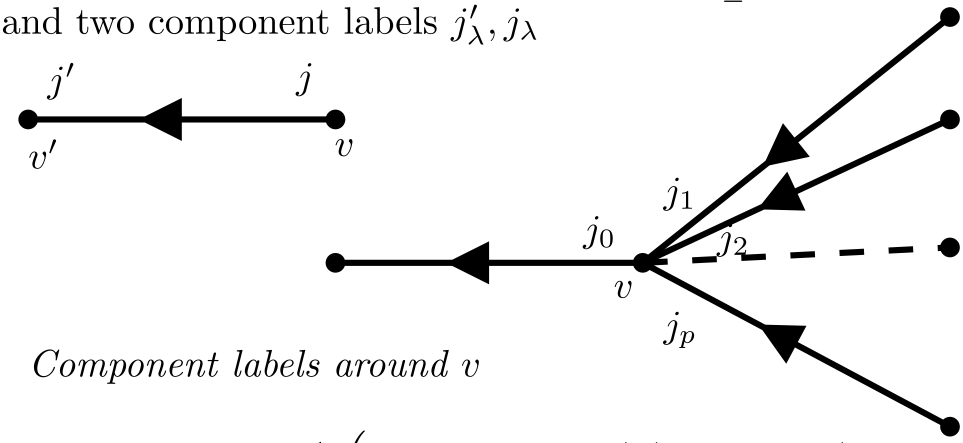


Component labels around v

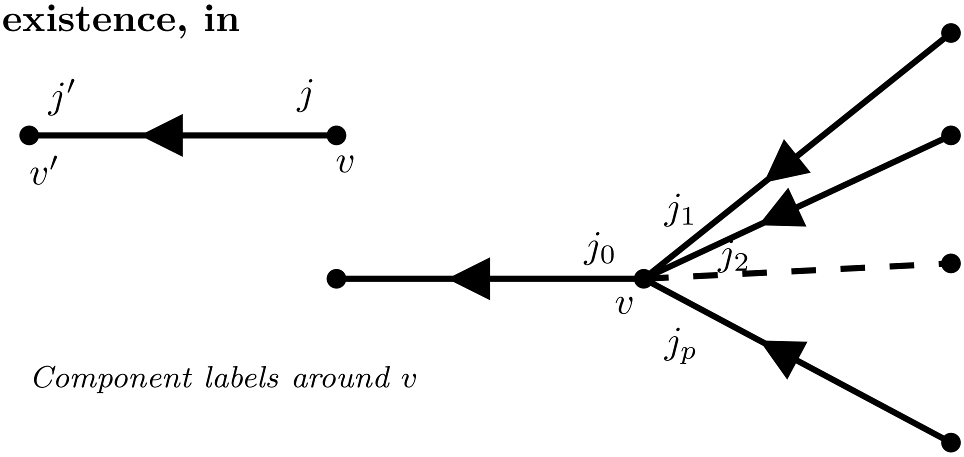
(4) *Value* : $\mathrm{Val}(\theta) = \frac{1}{k!}\Big(\prod_v \varepsilon\partial_{\,J_v} f_{\boldsymbol{\nu}_v}(\boldsymbol{\beta}_0)\Big)\Big(\prod_{lines\lambda} g_{j_\lambda j'_\lambda}\Big)$

$g_{ij} \overset{def}{=} \frac{\delta_{ij}}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu}(\lambda))^2}$, or $g_{ij} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\varepsilon\partial^2_{\boldsymbol{\beta}}\overline{f}(\boldsymbol{\beta}_0))^{-1} \end{pmatrix}$ *if* $\boldsymbol{\nu}(\lambda) = \mathbf{0}$

$\mathbf{h}_{\boldsymbol{\nu}} \equiv \sum^*_\theta \mathrm{Val}(\theta)$: $\ast \longleftrightarrow$ *no trivial node with* $\mathbf{0}$ *incoming current*

Estimate: $|h^{(k)}_{\boldsymbol{\nu}}| \le bB^k\varepsilon^k k!^{2\tau} \to$!! Results:

If $\det \partial^2_{\boldsymbol{\beta}} \overline{f}(\boldsymbol{\beta}_0) \neq 0$ **The tree series can be rearranged to yield a convergent series representation of** $\mathbf{h} = (\mathbf{g}(\boldsymbol{\psi}), \mathbf{k}(\boldsymbol{\psi})$**, hence its existence, in**



Component labels around v

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$\mathcal{E} \subset (-\varepsilon_0, 0]$ **with open dense complement;** 0 **is a** *density point*.

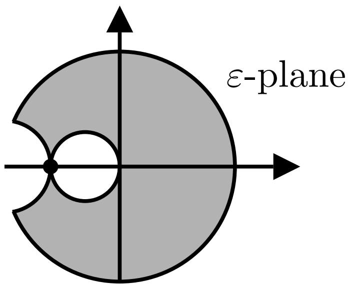

Fig.4: $\varepsilon \in \mathcal{E}$; $\mathcal{E}$ dense at 0. The figure illustrates the analyticity domain $\mathcal{D}(\varepsilon)$ associated with $\varepsilon \in \mathcal{E}$, for $\varepsilon < 0$ (elliptic case), assuming $\partial^2_{\boldsymbol{\beta}\boldsymbol{\beta}} \overline{f}(\boldsymbol{\beta}_0) < 0$

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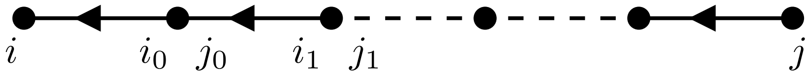
Need $k!^2$ for Borel sum: at 0 derivatives grow as $k!^{2\tau}$.

The $k!$ arise because of “chains in graphs” (self-energy insertions):



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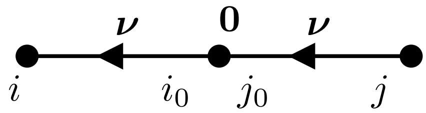
Inserting a single “trivial” node with $\mathbf{0}$ harmonic ($\Rightarrow \boldsymbol{\nu} \neq \mathbf{0}$):



Let $M_{0;i_0j_0} \stackrel{def}{=} \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon\partial^2\overline{f}(\boldsymbol{\beta}_0) \end{pmatrix}$ 5

graph value modified “trivially”: propagator modification $\Rightarrow$

$\frac{\delta_{ij}}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \to \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\big(M_0\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\big)^k$ “Simplify”: NO trivial nodes if

$$\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \Rightarrow \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\sum_{k=0}^{\infty}\big(M_0\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\big)^k \equiv \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2 - M_0}$$

BUT $z = M_0\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} < 1$? NO **!**

using $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$, $z \neq 1$, *e.g.* $\sum_{k=0}^{\infty} 2^k = 1 + 2 + \ldots = -\mathbf{1}$

If accepted $\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \Rightarrow \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \sum_{k=0}^{\infty} \big(M_0 \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\big)^k \equiv \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2 - M_0}$

$\Rightarrow M_0 \geq 0$ ($\boldsymbol{\beta}_0$=maximum) gives $\varepsilon > 0$ "easier" than $\varepsilon < 0$.
For $\varepsilon < 0$ expect to exclude ε s.t $|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| = \pm\sqrt{-\varepsilon\mu_j}$.

PROBLEM: LOTS of other chains ! They cause values $k!^{\eta}$, $\eta > 0$
IDEA: eliminate them all "by resummation" (RG needed).
KEY: Siegel's theorem

Given a tree θ let $\mathcal{N}_n$ be the number of lines of scale n: i.e. s.t.

$$2^{-n} < C|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| \leq 2^{-n+1}$$

$n = 0, 1, \ldots$. *IF no pair lines $\lambda' < \lambda$ with $\boldsymbol{\nu}(\lambda') = \boldsymbol{\nu}(\lambda)$ with only lower scale intermediates THEN*

$$\mathcal{N}_n \leq 4N 2^{-n/\tau}\, k$$

If there were no chains : $\mathcal{N}_n \le 4N2^{-n/\tau}\, k \Rightarrow$

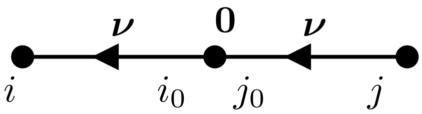
$$\prod_v \frac{1}{(\boldsymbol{\omega}_0\cdot\boldsymbol{\nu})^2} \le C^{2k}\prod_{n=0}^{\infty} 2^{2n\mathcal{N}_n} \le C^{2k}\Big(\prod_{n=0}^{\infty} 2^{2n(4N2^{-n/\tau})}\Big)^k$$

Product of *couplings*: $\prod_v |\partial_{J^v} f_{\boldsymbol{\nu}_v}(\boldsymbol{\beta}_0)| \le \prod_v N^{|J^v|}F^k \le N^{2k}F^k$, *number* of harmonics $\boldsymbol{\nu}$: $\le (2N+1)^k$, n. of *trees* $\le k^{k-1}$

Convergence: $|\varepsilon| < \big(N^2\cdot C^2\cdot (2N+1)^\ell\cdot 3\cdot F\cdot 2^{8N\sum_n n2^{-n/\tau}}\big)^{-1}$

Multiscale analysis (RG): Organize the lines of θ into *clusters*

Definition: A cluster of scale n is a maximal connected set of lines of θ with scales $p \leq n$ and with one line at least of scale n.



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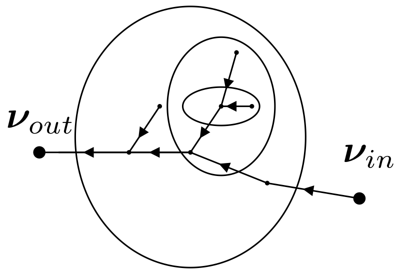
Self energy clusters : 1 line in and 1 line out with $\boldsymbol{\nu}_{in} = \boldsymbol{\nu}_{out}$



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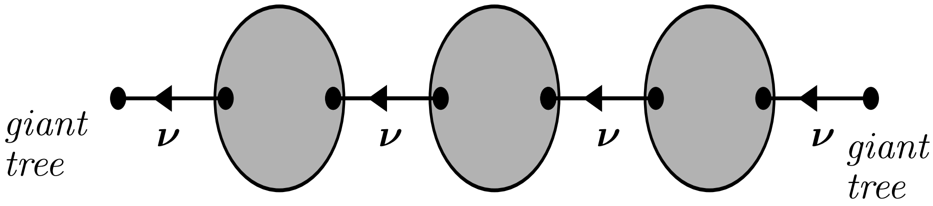
Eliminate self energy clusters by resummations

Necessary multiscale to avoid "overlapping divergences"
First identify the self energy clusters of scale [0]

i.e. with $x \stackrel{def}{=} C\boldsymbol{\omega}\cdot\boldsymbol{\nu}$ with $1 \le x^2$ and resum all chains



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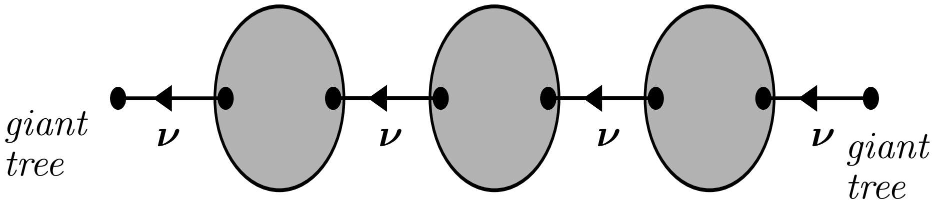
Key remark: *each s.e. cluster does not contain s.e. clusters*

Summing over the contents of each s.e. cluster $\Rightarrow$ convergent sum by Siegel's lemma.

Summing over s.e. clusters of scale [0] leads to
modify propagators of lines on scale [≥ 1]



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$g^{[\geq 1]}(x) = \frac{1}{x^2 - M_0} \;\to\; \frac{1}{x^2-M_0} \sum_{n=0}^{\infty} (M_1 \frac{1}{x^2-M_0})^n \;\to\; \frac{1}{x^2-M_0-M_1}$

$\Rightarrow$ eliminated also s.e. subgraphs of scale [0].

Iterate! at every step only graphs with no s.e. subgraphs have to be considered; $\Rightarrow$ convergent additions made on propagators.
Full resummation of self energy in the hyperbolic case.

In the elliptic the situation is very different.

As the scale decreases the scale $2^{-n} \simeq \varepsilon$ is reached and $x^2 - M_0 - M_1 - \ldots - M_{n-1}$ can vanish $\Rightarrow$

(a) More values of ε excluded

(b)The successive scales must be measured by the size of $x^2 - M_0 - M_1 - \ldots$; analysis becomes delicate:

DIFFICULTY: even in the hyperbolic case it is necessary to check that the renormalized propagators have the same size as the bare ones

Siegel's lemma applies only to graphs in which the propagators size is of the order of $x^2 \equiv (\boldsymbol{\omega} \cdot \boldsymbol{\nu})^{-2}$.

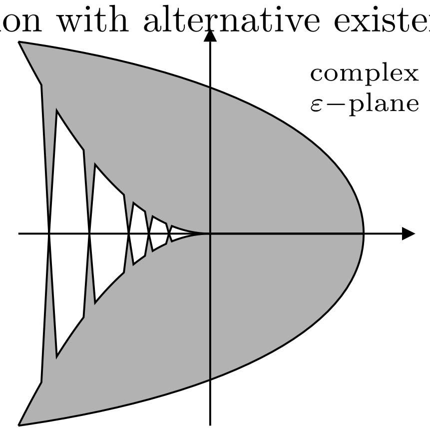
Not automatic but checked via the cancellation mechanism of the KAM theory: this time the cancellations are only partial but still enough.

Case $\det \partial^2_{\boldsymbol{\beta}} \overline{f}(\boldsymbol{\beta}_0) = 0$? If $s = 1$ and $c = \partial^{k_0+1} \overline{f}(\boldsymbol{\beta}_0) \neq 0$

"same but analyticity in $\eta = \varepsilon^{\frac{1}{k_0}}$" (+ A. Giuliani)

Borel Summability? yes if $r = 2$ (perhaps: w. progress +O.Costin,A.Giuliani)

OPEN PROBLEM:
uniqueness (and relation with alternative existence results)



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