Reviews of Modern Physics, **57**, 471–562, 1985

Renormalization theory and ultraviolet stability for scalar fields via renormalization group methods

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A self-contained analysis is given of the simplest quantum fields from the renormalization group point of view: multiscale decomposition, general renormalization theory, resummations of renormalized series via equations of the Callan–Symanzik type, asymptotic freedom, and proof of ultraviolet stability for sin–Gordon fields in two dimensions and for other super–renormalizable scalar fields. Renormalization in four dimensions (Hepp's theorem and De Calan–Rivasseau $n!$ bound) is presented and applications are made to the Coulomb gases in two dimensions and tot he convergence of the planar graph expansion in four dimensional field theories (t'Hooft–Rivasseau theorem).

ΕΤΗΣΙΑΙ ΠΑΥΟΝΤΑΙ[1]

Contents

[1] From the newly discovered Augustus' meridian in Roma.

i. Introduction

The aim of this work is to provide a self–contained introduction to field theory illustrating, at the same time, most of the known properties of the simplest fields.

I shall develop some of the ideas and methods of constructive field theory whenever they exist, providing the construction (nonperturbative) of various field with one of the few methods available (which I consider conceptually the simplest).

While I have no pretension of saying something new, particularly to the theoretical physicists, I hope that this review might be useful, as many mathematical physicists have never worked on field theory and are not familiar with its remarkable problems, and as many physicists never had any wish or need to look at the rigorous version of statements that they consider obvious.

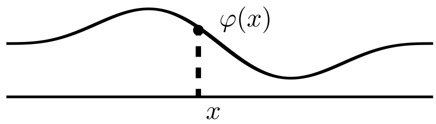
In this section I review some of the philosophy behind the setting of quantum field theory, mostly for completeness and with the hope that this might help some beginners.

The special theory of relativity, in spite of its elegance and simplicity, raises a large number of problems by imposing the rejection of the notion of action at distance to describe interacting mechanical systems.

In fact, the electromagnetic field in the vacuum or the free particles provide simple examples of relativistic systems, but it is difficult to describe relativistically invariant interactions between fields; however, in classical mechanics there is only one field, the electromagnetic field (gravitation is not considered here), and a variety of particles which seem to interact only through it, being charged entities. Their interaction with the electromagnetic field is hard to describe in a fundamental way because of the infinite self energy that it implies.

There has been, and still there is, great hope that the electromagnetic field quantization, or more generally the quantization of systems of fields, would lead to the unification of the field–particle dualism and to the possibility of a description of relativistic quantum interactions between particles and fields. In the remaining part of this section I summarize the heuristic reasoning behind this hope.

Classically a field describes the configurational state of an elastic body. As a primitive example, consider the case of a one–dimensional vibrating string: describe it through the value $\varphi(x)$ of the transversal deformation in the point of abscissa x; see Fig. 1,



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The string parameters will be the density μ, its tension $\mu\, c^2$ (with c the wave propagation speed), and the restoring constant $\mu\,\omega^{2}$': i.e. the string Lagrangian is

$$\mathcal{L} = \frac{\mu}{2}\int_\alpha^\beta \Big(\dot\varphi(x)^2 - c^2\big(\frac{d\varphi}{dx}(x)\big)^2 - \omega^2\varphi(x)^2\Big)\,dx \qquad (1.1)$$

where α, β are the points where the string endpoints are attached ($\alpha = -\infty, \beta = +\infty$) if one wishes relativistic covariance.

The equation of motion is therefore

$$\ddot\varphi(x) - c^2\frac{\partial^2\varphi}{\partial x^2}(x) + \omega^2\varphi(x) = 0 \qquad (1.2)$$

which describes a relativistically invariant field (if $\alpha = -\infty$, $\beta = +\infty$), because if $(x,t) \to \varphi(x,t)$ solves (1.2), so does $(x,t) \to \varphi(R(x,t))$ for any Lorentz transformation R:

$$R = \begin{pmatrix} \cosh y & \sinh y \\ \sinh y & \cosh y \end{pmatrix}, \qquad y \ge 0, c \equiv 1$$

The solutions of (1.2), with $\alpha = -\infty$, $\beta = +\infty$, can be developed in plane waves:

$$e^{i\,(kx - \varepsilon(k)t)}, \qquad k \in R, \qquad (1.3)$$

where

$$\varepsilon(k) = \pm\big(\omega^2 + c^2k^2\big)^{1/2} \qquad (1.4)$$

Via the correspondenceprinciple and Bohr's relations

$$p = \hbar\, k, \quad E = \hbar\varepsilon \qquad (1.5)$$

with $\hbar = \frac{1}{2\pi}\times$ (Planck'sconstant), one sees that the quantized vibrationg string should describe particles for which the relationship between momentum p and velocity v is

$$\begin{aligned} v &= \frac{d\varepsilon}{dk} = \frac{c^2k}{\sqrt{\omega^2 + c^2k^2}} = \frac{c^2}{\hbar}\frac{p}{\sqrt{\omega^2 + (c\,p/\hbar)^2}} \\ p &= v\,\frac{\omega\hbar c^{-2}}{\sqrt{1 - v^2/c^2}} \end{aligned} \qquad (1.6)$$

i.e. the quantized string describes relativistic particles with restmass

$$m_0 = \omega\,\hbar/c^2 \qquad (1.7)$$

It would be easy to convince oneself that such particles do not interact mutually.

However, it is easy to introduce an interaction bewteen them which is relativistically invariant. The simplest way is to modify the classical Lagrangian (1.1) by nonquadratic terms and then quantize it. Consider

$$\begin{aligned} \mathcal{L} =& \frac{\mu}{2}\int_\alpha^\beta \Big(\dot\varphi(x)^2 - c^2\big(\frac{d\varphi}{dx}(x)\big)^2 + \\ & - \big(\frac{m_0c^2}{\hbar}\big)^2\varphi(x)^2 \,-\, I(\varphi(x))\Big)\,dx \end{aligned} \qquad (1.8)$$

where $I(\varphi)$ is some function of φ.

The nonlinearity of the resulting wave equation produces the result that when two or more wave packets collide they emerge out of the collision quite modifies and do not just go through each other as in the case of the linear string, so that their interaction is nontrivial.

It is important to stress one feature of (1.1): in order to describe a particle of mass m_0 it is necessary to consider a string with restoring force constant $\omega^2 = m_0c^2/\hbar$. It is this dependence of ω on $\hbar$ which provides that, in the classical limit $\hbar \to 0$, a particle ith rest mass m_0 is no longer described by a classical solution of the wave equation. The limit $\hbar \to 0$ has to be discussed with more care because of its very singular nature. The actual discussion leads to the natural picture that the classical waves obtaines as limits of quantum states describing a set of freely traveling quantum particles of momenta $p_1, p_@, \ldots$ ("coherent states") are a δ–function wave:

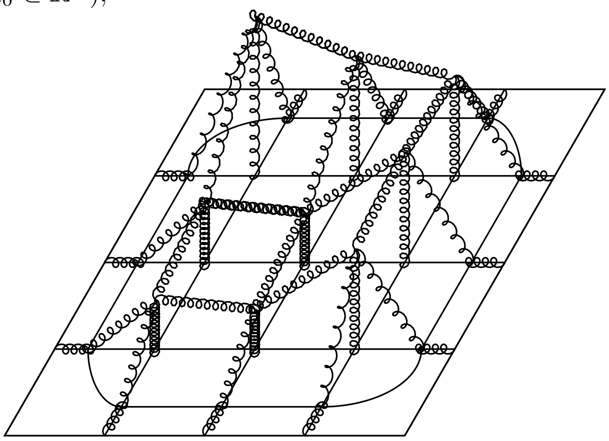
$$\prod_{i=1}^n \delta(x_i - v(p_i)t) \qquad (1.9)$$

with $v(p)$ given by (1.6) ("point particles").

There is, however, an obvious exception: the case $m_0 = 0$. This time the limit as $\hbar \to 0$ does not have the same singular character as before and the classical limits of quantum states are generally correctly described by classical fields verifying the wave equation.

The above discussion, which cannot be developed in more detail here, is the basis for the solution of the "wave–particle dualism" : the calssical waves and particles being in a natural sense the classical limits of quantum fields (respectively massless or massive).

But one should not think that the quantization of the string or of a more general D–dimensional elastic body ($D = 1, 2, 3$; see Fig. 2 for the case $D = 2$, with the body being a discrete set of springs oscillating over the region $\Lambda_0 \subset \mathbb{R}^D$),



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with Hamiltonian

$$\mathcal{H} = \int_{\Lambda_0} \Big(\big(\frac{\pi(\mathbf{x})^2}{2\mu} + \frac{\mu}{2}\Big(c^2(\frac{\partial\varphi}{\partial\mathbf{x}}(\mathbf{x}))^2\Big)^2 + \\ + (\frac{m_0 c^2}{\hbar})^2 \varphi(\mathbf{x})^2 + I(\varphi(\mathbf{x}))\Big)\, d^D\mathbf{x} \tag{1.10}$$

is an easy matter; it is in fact the scope of this paper to review the related problems.

I start with the "naive" quantization: the quantum states will be, by the "nmatural extension of the usual quantization rules", functions of the function φ describing the configurational shape of the elastic deformations; and $\pi(\mathbf{x})$ will have to be thought of as the operator $i\hbar\frac{\delta}{\delta\varphi(\mathbf{x})}$. So the Hamiltonian operator acts on the wave function F as

$$(\mathcal{H}H)(\varphi) = \int_{\Lambda_0} \Big(-\frac{\hbar^2}{2\mu}\frac{\delta^2 F}{\delta\,\varphi(x)^2}(\varphi) + \frac{\mu}{2}\Big(c^2(\frac{\partial\varphi}{\mathbf{x}}(\mathbf{x}))^2 + \\ +(\frac{m_0 c^2}{\hbar})^2\varphi(x)^2 + I(\varphi(\mathbf{x}))\Big)\, F(\varphi)\Big)\; d^D\mathbf{x} \tag{1.11}$$

and it should be defined in the space $L_2(``d\varphi")$, where the scalar product ought to be

$$(F, G) = \int \overline{F(\varphi)}\, G(\varphi)\, "d\varphi" \tag{1.12}$$

and $"d\varphi" = \prod_{\mathbf{x}\in\Lambda_0} d\varphi(\mathbf{x})$.

Even though by now the mathematical meaning that one should try to attach to expressions like (1.11) and (1.12) as "infinite dimensional elliptic operators" and "functional integrals" is quite well understood, particularly when $I \equiv 0$, formulae like the above are still quite shocking for conservative mathematicians, even more so because they turn out to be very useful.

One possible way to give meaning to (1.11) is to go back to first principles and recall the classical interpretation of the vibrating string or of the elastic body as a system of finitely many oscillators, following the brilliant theory of the discretized wave equation and of the related Fourier series due to Lagrange (see for instance (Gallavotti, 1983b), p. 252–283); see Fig. 2.

SUppose that the region is a parallelepiped of side size L and, for the sake of simplicity, with periodic boundary conditions; replace it with a square lattice $\mathbb{Z}_a$ with bonds of size a and such that L/a is an integer . In every point $\mathbf{n}a$ of $\mathbb{Z}_a$ put an oscillator with mass $\mu\, a^D$, described by a coordinate $\varphi_{\mathbf{n}a}$ giving the elongation of the oscillatorover its equilibrium position, and subject to a restoring force with potential energy $\frac{1}{2}\mu a^D\,(\frac{m_0c^2}{\hbar})^2\varphi_{\mathbf{n}a}^2$, to a nonlinear restoring force with potential energy $\frac{1}{2}\mu a^D I(\varphi_{\mathbf{n}a})$. and finally to a linear elastic tension coupling between nearest neighbors $\mathbf{n}a, \mathbf{m}a$ with potential energy $\frac{1}{2}\mu c^2 a^{D-2}(\varphi_{\mathbf{n}a} - \varphi_{\mathbf{m}a})^2$.

Therefore the Lagrangian of the system is

$$\mathcal{L} = \frac{\mu}{2}a^D \sum_{\mathbf{n}a\in\Lambda_0} \Big(\dot\varphi_{\mathbf{n}a}^2 - c^2 \sum_{j=1}^{D} (\varphi_{\mathbf{n}a+\mathbf{e}_j a} - \varphi_{\mathbf{n}a})^2 + \\ - (\frac{m_0c^2}{\hbar})^2 \varphi_{\mathbf{n}a}^2 - I(\varphi_{\mathbf{n}a})\Big) \tag{1.13}$$

where $\mathbf{e}_j$ are D unit vctors oriented as the directions of the lattice.; if $\mathbf{n}a + \mathbf{e}_j a$ is not in Λ_0 but $\mathbf{n}a$ is in Λ_0 then the i-th coordinate, for some i, equals L and $\mathbf{n}a + \mathbf{e}_j a$ has to be interpreted as the point whose i-th coordinate is replaced by i; i.e. (1.13) is interpreted with periodic boundary conditions with coordinates identified modulo L.

Of course there is no conceptual problem in quantizing the system described by (1.13); it is described by the operator on $L_2(\prod_{\mathbf{n}a} d\varphi_{\mathbf{n}a})$:

$$H_{\text{quantum}} = -\frac{\hbar^2}{2\mu a^D}\sum_{\mathbf{n}a\in\Lambda_o}\frac{\partial^2}{\partial\varphi_{\mathbf{n}a}^2} + \frac{\mu a^D}{2}\sum_{\mathbf{n}a\in\Lambda_o} \tag{1.14} \\ \Big(c^2\sum_{j=1}^{D}\frac{(\varphi_{\mathbf{n}a+\mathbf{e}_j a} - \varphi_{\mathbf{n}a})^2}{a^2} + (\frac{m_0c^2}{\hbar})^2\varphi_{\mathbf{n}a}^2 + I(\varphi_{\mathbf{n}a})\Big)$$

with $C_0^\infty(\prod_{\mathbf{n}a\in\Lambda_0}\mathbb{R})$ as domain (of essential self-adjointness) provided $I(\varphi)$ is assumed bounded below, as it should always be.

The properties of the quantum vibrating string, or elastic bosy if D is larger than 1, which will be usually interesting here will be properties of the Hamiltonian (1.14) holding uniformly in the "ultraviolet cut–off a". In fact, in most applications one is actually interested also in properties holding uniformly in the "infrared cut–off L" as well, with L the size of the box Λ_0.

It will become clear that in studying such "ultraviolet stable properties" it will be necessary to put upon the "interaction" $I(\varphi)$ very stringent requirements to avoid that the system becomes trivial in the limit $a \to 0$, the "ultraviolet limit".

Also, last but not least, it should be clear that the objective of field theory is to formulate a relativistically invariant quantum theory of interacting particles, and it might conceivably happen that the above way of trying to give a meaning to (1.11) and (1.12) based on (1.14) may fail: i.e. in the limit $a \to 0$ one is left with a theory describing only free particles. Such a failure, in principle, would not prove the impossibility of giving a non-trivial meaning to (1.11) but only that the way proposed through (1.14) is not appropriate.

In the next section I shall proceed to give a more complete formulation of the ultraviolet problem in connection with (1.14) ("lattice regularization"). Later, in Sec. 3, a different approach naturally emerges: it will be the one which will be really investigated in this work ("Feynman regularization"); in the few cases in which the theory can be really pushed beyond formal perturbation theory the two approaches turn out to be equivalent.

This does not mean that in the case of questions that that are still open the two approaches should be thought as equivalent; however there is no reasonwhy one should be preferred to the other or to any one among many others that one can *a priori* conceive, (Gallavotti and Rivasseau, 1985): therefore, I shall avoid entering nto regularization-dependent questions, and I shall use one well defined rgularization only for definiteness. This will preclude the discussion of some recent deep results based on special regularization assumptions, but the reader is referred to the literature on such questions, (Aizenman, 1982; Frölich, 1982).

ii. Functional integral representation of the Hamiltonian of a quantum field

A very convenient representation of the Hamiltonian and a tool for the analysis of the ultraviolet limit is the functional integral representation (this seems to be a rather old representation; here I follow (Nelson, 1966, 1973c); see also (Glimm and Jaffe, 1981; Guerra *et al.*, 1975; Wilson, 1971, 1972)).

Instead of studying the operator $H_{quantum}$, (1.14), introduce the operator on $L_2(\prod_{\mathbf{n}a} d\varphi_{\mathbf{n}a})$:

$$T_t = e^{-(H_{quantum}-E)t/\hbar}, \qquad t \geq 0 \tag{2.1}$$

where E is the ground–state energy of $H_{quantum}$.

Denoting $\boldsymbol{\varphi} = (\varphi_{\mathbf{n}a})_{\mathbf{n}a\in\Lambda_0}$ and

$$\begin{aligned}
e(\boldsymbol{\varphi}) &= \text{ground state wave function for } H_{quantum},\\
e_0(\boldsymbol{\varphi}) &= \text{ground state wave function for } (H_{quantum})_{I=0}\\
E_0 &= \text{ground state energy for } H_0\\
T_t(\boldsymbol{\varphi},\boldsymbol{\varphi}') &= \text{kernel of } T_t \text{ on } L_2(\prod_{\mathbf{n}a} d\varphi_{\mathbf{n}a})\\
T_t^0(\boldsymbol{\varphi},\boldsymbol{\varphi}') &= \text{kernel of } e^{-t(H_0-E_0)} \text{ on } L_2(\prod_{\mathbf{n}a} d\varphi_{\mathbf{n}a})
\end{aligned} \tag{2.2}$$

it is possible to introduce a probability measure on the space of continuous functions $(t,\mathbf{n}a) \to \varphi_{\mathbf{n}a}(t)$ such that the sets

$$E(A;t_1,\ldots,t_n) = \big\{(\boldsymbol{\varphi}(t_1),\ldots,\boldsymbol{\varphi}(t_n)) \in A\big\}, \tag{2.3}$$

with $A \in (\mathbb{R}^{\Lambda_0})^n$ will have the measure

$$\begin{aligned}
P(E(A;t_1,\ldots,t_n)) = \int_A \prod_{j=1}^n d\boldsymbol{\varphi}(t_j)\cdot e(\boldsymbol{\varphi}(t_1))\cdot\\
\cdot\big(\prod_{j=1}^{n-1} T_{t_{j+1}-t_j}(\boldsymbol{\varphi}(t_j),\boldsymbol{\varphi}(t_{j+1}))\big)\cdot e(\boldsymbol{\varphi}(t_n))
\end{aligned} \tag{2.4}$$

where $d\boldsymbol{\varphi}(t_j) = \prod_{\mathbf{n}a\in\Lambda_0} d\boldsymbol{\varphi}_{\mathbf{n}a}(t_j)$ and $t_1,\ldots,t_n$ play the role of indices.

One readily checks that (2.4) does verify the compatibility conditions necessary to interpret it as a measure on the algebra of sets generated by the sets (2.3) on the space of the continuous functions $t \to \boldsymbol{\varphi}(t) \in \mathbb{R}^{\Lambda_0}$, $\boldsymbol{\varphi}(t) \equiv [\varphi_{\mathbf{n}a}(t)]_{\mathbf{n}a\in\Lambda_0}$, i.e.

(1) $P(E(\cdot)) \geq 0$,
(2) $P(E((\mathbb{R}^{\Lambda_0})^n;t_1,\ldots,t_n)) = 1$,

and if for A, A_0 it is

$$E(A;t_1,\ldots,t_n) = E(A_0;t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_n),$$

in other words if the value of $\boldsymbol{\varphi}(t_j)$ is irrelevant to decide whether $(\boldsymbol{\varphi}(t_1),\ldots,\boldsymbol{\varphi}(t_n))$ is in A, then
(3) $P(E(A;t_1,\ldots,t_n)) = P(E(A_0;t_1,\ldots,t_{j-1},t_{j+1},\ldots,t_n))$.

If $F, G \in L_2(e(\boldsymbol{\varphi})^2 d\boldsymbol{\varphi})$ and if U is the multiplication operator mapping $L_2(e(\boldsymbol{\varphi})^2 d\boldsymbol{\varphi}) \longleftrightarrow L_2(d\boldsymbol{\varphi})$ defined by

$$(UF)(\boldsymbol{\varphi}) = e(\boldsymbol{\varphi})F(\boldsymbol{\varphi}) \tag{2.5}$$

and if $\widetilde{H} = U^{-1}H_{quantum}U$ it is, by (2.4) and (2.5):

$$\begin{aligned}
&(F, e^{-t(\widetilde{H}-E)/\hbar}G)_{L_2(e(\boldsymbol{\varphi})^2 d\boldsymbol{\varphi})} =\\
&= (UF, e^{-tH_{quantum}/\hbar}UG)_{L_2(d\boldsymbol{\varphi})} =\\
&= \int e(\boldsymbol{\varphi})F(\boldsymbol{\varphi})T_t(\boldsymbol{\varphi},\boldsymbol{\varphi}')e(\boldsymbol{\varphi}')G(\boldsymbol{\varphi}')\,d\boldsymbol{\varphi}\,d\boldsymbol{\varphi}' =\\
&= \int F(\boldsymbol{\varphi}(0))\,G(\boldsymbol{\varphi}(t))P(d\boldsymbol{\varphi})
\end{aligned} \tag{2.6}$$

which shows that the measure P contains all the information needed to study the operator $H_{quantum}$ or its equivalent U–image $\widetilde{H}$, (Nelson, 1973a,b,c).

In the above formulae the notation $\boldsymbol{\varphi}$ has been used to denote an element of the space of continuous functions $t \to \boldsymbol{\varphi}(t)$ with values in $\mathbb{R}^{\Lambda_0}$ (while $\boldsymbol{\varphi}$ denotes an element of $\mathbb{R}^{\Lambda_0}$); this notation will be consistently kept.

The object P is of course quite complex and needs, in any event, the theory of $H_{quantum}$ to be really constructed.

It is easy to relate P to the measure P_0, defined as P but with $I(\boldsymbol{\varphi}) = 0$, and to find "explicit" expressions for P_0 itself, citeNe73a,Ne73b,Ne73c.

The measure P_0 is a Gaussian measure because the Green's function $T^0_t(\boldsymbol{\varphi}, \boldsymbol{\varphi}')$, being the heat kernel for the Laplace operator on $L_2(\mathbb{R}^{\Lambda_0})$ plus a quadratic potential, is a Gaussian kernel (in fact the heat kernel for the Laplace operator A is Gaussian, as it is well known, and the addition to A of a quadratic potential B does not change this because of Trotter's formula $e^{A+B} = \lim_{n\to\infty}(e^{A/n}e^{B/n})^n$, and because the composition of several Gaussian kernels is still a Gaussian kernel).

herefore P_0 can be completely described in terms of its "covariance" or "propagator"; if $\xi = (\mathbf{n}a, t) \in \Lambda_0\mathbb{R}$ and $\varphi_\xi \stackrel{def}{=} \varphi_{\mathbf{n}a}(t)$ and $\eta = (\mathbf{m}a, t')$ the covariance is defined as

$$C_{\xi\eta} \stackrel{def}{=} \int_{\mathcal{C}(\Lambda_0\times\mathbb{R})} \varphi_\xi\varphi_\eta\, dP_0, \tag{2.7}$$

where $\mathcal{C}(\Lambda_0 \times \mathbb{R})$ is the set of the continuous functions φ on $\Lambda_0 \times \mathbb{R}$.

A well known elementary calculation allows us to find an explicit formula for C; let $\xi = (\mathbf{x}, t)$ and $\eta = (\mathbf{y}, t')$, then

$$C_{\xi\eta} = \sum_{\mathbf{n}\in\mathbb{Z}^D} \overline{C}_{(\mathbf{x}+\mathbf{n}L),(\mathbf{y},t')}, \qquad \text{with} \tag{2.8}$$

$$\begin{aligned}\overline{C}_{\xi,\eta} =& \frac{\hbar}{(2\pi)^{D+1}\mu}\int_{-\pi/a}^{\pi/a}\int_{-\infty}^{\infty} d^D\mathbf{p}\, dp_0\cdot \\ &\cdot \frac{e^{ip_0^2 0(t-t')}e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{(\frac{m_0c^2}{\hbar})^2 + p_0^2 + 2c^2\sum_{j=1}^D \frac{1-\cos p_j a}{a^2}}\end{aligned} \tag{2.9}$$

see Appendix A1 for a sketchy proof.

Then the measure P is related to P_0 by

$$\begin{aligned}P(d\varphi) =& \lim_{T\to\infty} \frac{e^{-\frac{\mu a^D}{2\hbar}\int_{-t/2}^{t/2} I(\boldsymbol{\varphi}(\tau))\,d\tau}P_0(d\varphi)}{Z(L,T)} \\ Z(L,T) \stackrel{def}{=}& \int e^{-\frac{\mu a^D}{2\hbar}\int_{-t/2}^{t/2} I(\boldsymbol{\varphi}(\tau))}\end{aligned} \tag{2.10}$$

This is the "Feynman–Kac formula ", (Nelson, 1966, 1973c): we recall that here L is the infrared cut–off, i.e. the side size of the cube Λ_0 with opposite sides identified (periodic boundary conditions). The proof of (2.10) is not hard and the rough sketch can be found in Appendix A2.

Rather than using (2.10) to deduce the properties of the measure P when the ultraviolet cut–off a tends to zero it is convenient to study a more explicit representation for P. This representation is a corollary of (2.9) and (2.10) and it is

$$\begin{aligned}P(d\varphi) =& \lim_{T\to\infty}\lim_{b\to 0}\frac{\prod_{\mathbf{n},m} d\boldsymbol{\varphi}_{\mathbf{n}a,mb}}{Z}\cdot\exp\Big[-\frac{\mu\, b\, a^D}{2\hbar}\cdot \\ &\cdot\sum_{\mathbf{n}a\in\Lambda_0}\sum_m \big[\frac{(\varphi_{\mathbf{n}a,mb}-\varphi_{\mathbf{n}a,mb+b})^2}{b^2} + \\ &+ c^2\sum_{j=1}^D \frac{(\varphi_{\mathbf{n}a+\mathbf{e}_ja,mb}-\varphi_{\mathbf{n}a,mb})^2}{a^2} + \\ &+ (\frac{m_0c^2}{\hbar})^2\varphi^2_{\mathbf{n}a,mb}\big] + I(\varphi_{\mathbf{n}a,mb})\Big]\end{aligned} \tag{2.11}$$

where m is an integer varying between $-T/2b$ and $T/2b$ (supposed integer), the points $\pm T/2b$ are identified ("periodic boundary conditions" in the time direction) and Z is a normalization factor depending on L, T, a, b. The p0roof of (2.11) is hinted at the end of Appendix A2.

Call Λ the parallelepiped with sides L, T in $\mathbb{R}^{D+1} \equiv \mathbb{R}^d$ considered with periodic boundary conditions and call $P_{L,T,a,b}$ the measure under the limit sign in (2.11). The "ultraviolet problem" on the lattice is the problem of the theory of the limit:

$$\lim_{a\to 0}\lim_{b\to 0} P_{L,T,a,b} = P_{L,T} \tag{2.12}$$

Here I shall study only questions related to the existence of this limit, which is a problem typical of field theory, while no attention will be devoted to the other fundamental problem of analyzing the limit

$$\lim_{L,T\to\infty} P_{L,T} = P_\infty \tag{2.13}$$

called the "infrared problem". The latter problem can be considered a "thermodynamic limit" problem typical of Statistical Mechanics (which does not mean that it is easy).

The existence of the limit (2.12) will be attacked by trying to establish upper and lower bounds ("ultraviolet stability") uniform in a, b for quantities like

$$\langle e^{\varphi(f)}\rangle \stackrel{def}{=} \int P_{L,T,a,b}(d\varphi)\, e^{a^D b\sum_{\xi\in\Lambda} f(\xi)\varphi_\xi}, \tag{2.14}$$

where f is a C^∞–smooth function with fixed support in the interior of Λ. The expression defined in(2.14) is usually called the "generating function" for the "Schwinger functions" of the measure $P_{L,T,a,b}$.

For simplicity it is convenient to fix the ratio a/b to be equal to c, the speed of sound in our elastic models; also I shall choose $L = cT$ so that the measure $P_{L,L/c,a,a/c}$ can be rewritten

$$P_{L,a}(d\varphi) = \frac{e^{\frac{\mu a^d}{2c\hbar}\sum_{\xi\in\Lambda} I(\varphi_\xi)}\, P^0_{L,a}(d\varphi)}{Z} \tag{2.15}$$

where, if $p\cdot(x-y) \stackrel{def}{=} \mathbf{p}\cdot(\mathbf{x}-\mathbf{y}) + c(t-t')p_d$, $P^0_{L,a}$ is the Gaussian measure on the finite space $\mathbb{R}^\Lambda$ with covariance C

$$\begin{aligned} C_{\xi\eta} &= \sum_{n\in\mathbb{Z}^d} \overline{C}_{\xi+nL,\eta} \\ \overline{C}_{\xi+nL,\eta} &\stackrel{def}{=} \frac{\hbar}{(2\pi)^d \mu c}\int_{-\pi/a}^{\pi/a} \frac{e^{ip(\xi-\eta)}\, d^dp}{m^2 + 2\sum_{j=1}^d \frac{1-\cos p_j a}{a^2}} \end{aligned} \tag{2.16}$$

where $m = m_0 c/\hbar$ (to understand (2.15) note the symmetric role of the d directions in (2.11) once $a = bc, L = cT$), or explicitly

$$\begin{aligned} P_{L,a}(d\varphi) &= \frac{\prod_\xi d\varphi_\xi}{Z}\cdot \\ &\cdot \exp -\frac{\mu a^d}{2\hbar c}\Big[\sum_{\xi\in\Lambda}\big(\sum_{j=1}^d \frac{(\varphi_{\xi+ae_j}-\varphi_\xi)^2}{a^2}\big) + m^2\varphi_\xi^2\Big] \end{aligned} \tag{2.17}$$

The measure (2.17) is called the "lattice free field" and if δ_a^2 denotes the finite difference Laplace operator on the lattice $\mathbb{Z}_a^d$ one sees that $\overline{C}_{\xi\eta}$ in (2.16) is just the kernel of the operator

$$\overline{C} = \frac{1}{\mu c\hbar^{-1}\,(m^2 + \delta_a^2)} \tag{2.18}$$

(by finite difference Laplace operator we mean, here, the nearest neighbor second difference divided by a^2), while if $\delta_{a,L}^2$ denotes the finite difference Laplace operator on the lattice $\mathbb{Z}_a^d$ with periodic boundary conditions on the boundary of the cube Λ, it is

$$C = \frac{1}{\mu c\hbar^{-1}\,(m^2 + \delta_{a,L}^2)} \tag{2.19}$$

i.e. C is the same as $\overline{C}$ apart from the boundary conditions.

The problem of studying the limit as $a \to 0$ of (2.15) is not exactly the same as that of studying the $\lim_{a\to 0}\lim_{b\to 0}$ in (2.12). The really difficult problem being the limit as $a \to 0$, it turns out that setting $b = a/c$ does not make the problem any easier or any harder. All the results that follow could also be obtained if one considered first the limit $b \to 0$ and then the limit $a \to 0$ (or viceversa).

iii. The free field and its multiscale decompositions

It has become clear that the right way to look at the measures (2.17) (free field) is to consider them as stochastic processes indexed by the points of Λ; thus the free field will be thought of as a Gaussian process.

Furthermore, it is convenient to regard the free field as defined everywhere in Λ and not just on the lattice points of $\mathbb{Z}_a^d \cap \Lambda$; this can be done easily by remarking that $C_{\xi\eta}$ makes sense, by (2.16), for all $\xi,\eta \in \mathbb{R}^d$ and therefore we may actually imagine that it describes a family of Gaussian random variables indexed by $\xi \in \Lambda$, whose distribution is still denoted $P^0_{L,a}$.

Since $C_{\xi\eta}$ is infinitely smooth, it follows from the general theory of Gaussian processes that the "sample fields" φ_ξ will be, with probability 1 with respect to $P^0_{L,a}$, C^∞–functions o ξ. However this does not really imply that they are smooth in a physical sense: in fact, the expected values of $\varphi_x^2, (\partial\varphi_\xi), \ldots$ all diverge as $a \to 0$ (if, as will be always supposed, $d \ge 2$).

This means that the fields φ_ξ are indeed smooth but to see that they are such one has to look at them on a scale as small as s. An easy calculation shows that in fact

$$\int (\partial^p\varphi_\xi)^2\, P^0_{L,a}(d\varphi) = \begin{cases} O(a^{-2p-(d-2)}) & d > 2 \\ O(a^{-2p}\,\log a^{-1}) & \text{d=2} \end{cases} \tag{3.1}$$

where ∂^p is any p–th order derivative of φ_ξ.

The relation (3.1) tells us that φ_ξ has to be regarded as a smooth function which can be as large as $a^{(d-2)/2}$, if $d > 2$, or as $(\log a^{-1})^{\frac12}$ if $d = 2$, and which can have k–th order derivative larger by a factor a^{-k}, œthe field looks smooth only on scale a.

In general, in understanding the structure of a stochastic field, two main scales have to be specified: the scale on which the field is smooth and the scale on which the field is without correlations, i.e. the scales on which the field can be regarded as constant and those on which the values that it takes can e regarded as independent random variables.

In our case it would be easy to show that

$$|C_{\xi\eta}| \le M\, e^{-\kappa|\xi-\eta|},\ \forall \xi,\eta \in \mathbb{Z}_a^d,\ \kappa^{-1} \le |\xi-\eta| \le \frac12 L \tag{3.2}$$

where $M, \kappa > 0$ are a–independent; this can be interpreted as saying that the field $|f_\xi$ has an independence scale of $O(\kappa^{-1})$.

If one calls "regular" the random fields with identical smoothness and independence scales it is clear that the fields of interest here (free fields) are *not regular*; and this is the distinctive feature of field theory with respect to statistical mechanics of weakly interacting systems (i.e. away from the critical point). It introduces the new problem of ultraviolet stability, characteristic of field theory.

In fact a regular field is hardly different from a lattice spin system of essentially independent spins located on a lattice with spacing equal to the unique scale of regularity and independence.

Since the techniques for studying lattice spin systems have been well developed in statistical mechanics, at least in some easy cases, the idea arose, (Wilson, 1971, 1972, 1973, 1983) of trying to represent irregular fields as decomposed into sums of regular ones. This leads to the "multiscale decomposition's" of the free field which are discussed below and which are small perturbations of the Gaussian field $P^0_{L,a}$.

It is in fact possible to write the field φ as

$$\varphi_\xi = \sum_{k=0}^\infty \varphi^{(k)}_\xi . \tag{3.3}$$

where $\varphi^{(k)}_\xi$ are independently distributed over the index k and are regular on scale $\gamma^{-k}m^{-1}$. if $\gamma > 1$ is an arbitrarily preassigned number ("scale factor").

The decomposition (3.3) can be done in various ways and with different requirements on $\varphi^{(k)}$.

In general one desires that if $\gamma^{-k}m^{-1} \ge a$, i.e. if the length scale of the field $\varphi^{(k)}$ is larger than the ultraviolet scale a, then the samples of $\varphi^{(k)}$ should be smooth on scale $\gamma^{-k}m^{-1}$, with the h–th derivative being of the order of $\gamma^{k\,h}$ times the size of the field itself, see (3.1), for $h \le p$. Such a decomposition will be called a "class C^p multiscale decomposition" of φ into regular random fields.

There is a simple algorithm to construct multiscale decompositions of the Gaussian random field with covariance operator (2.18). It is based on the following trivial identities:

$$\begin{aligned}
&\frac{1}{m^2+\varepsilon^2} \equiv \sum_{k=0}^\infty \Big(\frac{1}{m^2\gamma^{2k}+\varepsilon^2} - \frac{1}{m^2\gamma^{2(k+1)}+\varepsilon^2}\Big) =\\
&= \sum_{k=0}^\infty \frac{m^2(\gamma^2-1)\gamma^{2k}}{m^4\gamma^{4k+2}+m^2(\gamma^2+1)\gamma^{2k}\varepsilon^2+\varepsilon^4} \equiv \\
&\equiv \sum_{k=0}^\infty\sum_{h=0}^\infty \Big(\frac{m^2(\gamma^2-1)\gamma^{2k}}{m^4\gamma^{4k+2}\gamma^{4h}+m^2(\gamma^2+1)\gamma^{2k}\varepsilon^2+\varepsilon^4} - \\
&- \frac{m^2(\gamma^2-1)\gamma^{2k}}{m^4\gamma^{4k+2}\gamma^{4h+4}+m^2(\gamma^2+1)\gamma^{2k}\varepsilon^2+\varepsilon^4}\Big) =\\
&= \sum_{k=0}^\infty\sum_{h=0}^k \frac{m^2(\gamma^2-1)\gamma^2 m^4(\gamma^4-1)\gamma^{6k}\gamma^{-2h}}{\big[m^4\gamma^2\gamma^{4k}+m^2(\gamma^2+1)\varepsilon^2\gamma^{2k-2h}+\varepsilon^4\big]}\cdot\\
&\cdot \frac{1}{\big[m^4\gamma^6\gamma^{4k}+m^2(\gamma^2+1)\varepsilon^2\gamma^{2k-2h}+\varepsilon^4\big]} \equiv \ldots,
\end{aligned} \tag{3.4}$$

where in the last equality a change of variables is made, changing $k+h$ into k.

The way to read (3.4) is the following: $(m^2+\varepsilon^2)^{-1}$ can be written as a sum of reciprocals of fourth-order polynomials in ε, or as a sum of reciprocals of eight order polynomials, or of sixteenth order, *etc*by the "telescopic" algorithm displayed self–explanatorily in (3.4).

Then to each such decomposition one can associate a decomposition of the random field φ like (3.3). For instance if $\varepsilon_a(p) \stackrel{def}{=} 2\sum_{k=1}^d (1-\cos ap_i)/a^2$ let $\Gamma^k_a(p)$ the expression appearing after the summation symbol in the second line of (3.4) with ε replaced by $\varepsilon_a(p)$, then setting

$$\overline{C}^{(k)}_{\xi\eta} = \frac{\hbar}{(2\pi)^d\,\mu\,c}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} d\,p\,\Gamma^k_a(p)e^{ip(\xi-\eta)} \tag{3.5}$$

and if $C^{(k)}$ is defined as in (2.16), with $\overline{C}$ replaced by $\overline{C}^{(k)}$, one realizes that by the first identity in (3.4) the field φ has the same distribution as the sum of a sequence of fields $\varphi^{(k)}$ with covariances given by $C^{(k)}$.

Similarly, using the last identity in (3.4) and calling $\Gamma^{k,h}_a(p)$ the expression appearing after the summations symbols in (3.4) with $\varepsilon_a(p)$ replacing ε and setting

$$\overline{C}^{(k)}_{\xi\eta} = \frac{\hbar}{(2\pi)^d\,\mu\,c}\sum_{h=0}^k\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} d\,p\,\Gamma^{k,h}_a(p)e^{ip(\xi-\eta)} \tag{3.6}$$

then if $C^{(k)}$ is defined as in (2.16), with $\overline{C}$ replaced by $\overline{C}^{(k)}$ one again finds that the field φ has the same distribution as the sum of a sequence of fields $\varphi^{(k)}$ with covariances given by $C^{(k)}$, *etc*.

The fields $\varphi^{(k)}$ with covariance $C^{(k)}$ related to (3.5) or (3.6) or to the "higher order generalizations" of them, are regular fields for all values of k such that $\gamma^{-k}m^{-1} \ge a$ and, when restricted to the lattice points, are basically independent fields for the larger values of k. Furthermore if $\gamma^{-k}m^{-1} \ge a$ the fields $\varphi^{(k)}$ have essentially the same distribution up to trivial scalings.

To see the above properties of $\varphi^{(k)}$ for $\gamma^{-k}m^{-1} \ge a$ one can heuristically fix k and let $a \to 0$ (so that $\gamma^{-k}m^{-1} \gg a$). Then (3.5) becomes

$$\begin{aligned}
&\overline{C}^{(k)}_{\xi,\eta} \equiv \gamma^{(d-2)k}\overline{C}^{(0)}_{\gamma^k\xi\,\gamma^k\eta},\ C^{(k)}_{\xi\eta} = \sum_{\mathbf{n}\in\mathbb{Z}^d}\overline{C}^{(k)}_{\xi+\mathbf{n}L,\eta} \\
&\overline{C}^{(0)}_{\xi,\eta} = \frac{\hbar}{(2\pi)^d\,\mu\,c}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{m^2(\gamma^2-1)e^{ip(\xi-\eta)}\,d^dp}{m^4\gamma^2+m^2(\gamma^2+1)p^2+p^4},
\end{aligned} \tag{3.7}$$

and it is easy to see that $\overline{C}^{(0)}_{\xi,\eta}$ is well defined and sooth with its derivatives of order $2(1-\varepsilon)$ if $d=2$ and of order $2(\frac12-\varepsilon)$ if $d=3$ and, furthermore, it decays exponentially for $|\xi-\eta|$ large on scale m^{-1} . This means that the field $\varphi^{(k)}$, with covariance $C^{(k)}$ in (3.7) is Hölder continuous on scale $\gamma^{-k}m^{-1}$ with exponent less than 1 if $d=2$ or less than $\frac12$ if $d=3$ (here $\varepsilon>0$ is arbitrary); it is, however, still irregular if $d \ge 4$.

If one uses the second decomposition of φ introduced above, associated with (3.6), one finds

$$\begin{aligned}\overline{C}^{(k)}_{\xi,\eta} &\equiv \gamma^{(d-2)k}\sum_{h=0}^{k}\gamma^{-2h}\,\overline{C}^{(0,h)}_{\gamma^k\xi\,\gamma^k\eta},\\ C^{(k)}_{\xi\eta} &= \sum_{\mathbf{n}\in\mathbb{Z}^d}\overline{C}^{(k)}_{\xi+\mathbf{n}L,\eta}\\ \overline{C}^{(0)}_{\xi,\eta} &= \frac{\hbar}{(2\pi)^d\,\mu\,c}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}}\Gamma^{0,h}(p)e^{ip(\xi-h)},\end{aligned}\tag{3.8}$$

where $\Gamma^{0,h}(p)$ is defined as the similar quantity $\Gamma^{0,h}_a(p)$ appearing in (3.6) with p^2 replacing $\varepsilon_a(p)$; and it is easy to see that $\overline{C}^{(k)}_{\xi\eta}$ has the same qualitative properties of $\gamma^{k(d-2)}C^{(0,0)}_{\gamma^k\xi,\gamma^k\eta}$ and $C^{(0,0)}$ is well defined and smooth together with its derivatives of order $2(2-\varepsilon)$ if $d=4$ (ε being an arbitrary positive number).

This means that the field $\varphi^{(k)}$ has second derivatives which are Hölder continuous with exponent $(1-\varepsilon)$ for $d=2$, of order $\frac{1}{2}-\varepsilon$ for $d=3$, and first derivatives which are Hölder continuous with exponent $1-\varepsilon$ if $d=4$ (for $d=5$ the first derivatives would be Hölder continuous with exponent $\frac{1}{2}-\varepsilon$ while for $d=6,7$ the field itself would be only Hölder continuous with exponent $1-\varepsilon$ or $\frac{1}{2}-\varepsilon$, respectively, and for $d\geq 8$ it would be irregular).

The latter statements can be made more quantitative (see below); their proof is essentially a repetition, adapted tot he circumstances, of the well known proof of Wiener asserting the Hölder continuity of the sample paths of the Brownian motion and it will not be repeated here. One can use the classical method of Wiener as in (Colella and Lanford, 1973); the cases (3.7), (3.8), as well as the others arising from the higher order identities obtained by continuing the decomposition process in (3.4) are specifically treated in (Benfatto *et al.*, 1980b), as a part of a general theory of a class of Markovian Gaussian random fields.

The above discussion on the fields $\varphi^{(k)}$ suggests yet another approach to the ultraviolet stability which will be the one really followed in the upcoming sections.

Namely, choose $\varphi^{(k)}$ to be the random fields with covariance (3.7) (or (3.8) or any other associated with the higher order identities continuing those in (3.4)) and define

$$\varphi^{[\leq N]}_\xi \stackrel{def}{=} \sum_{k=0}^{N}\varphi^{(k)}_\xi \tag{3.9}$$

Then the measure (2.15) can be regarded as obtained by integrating over the $\varphi^{(k)}$'s the measure

$$\begin{aligned}&\frac{1}{Z}e^{-\frac{\mu a^d}{2\,c\,\hbar}\sum_{\xi\in\Lambda}I(\varphi_\xi)}\prod_{k=0}^{\infty}P(d\varphi^{(k)}) =\\ &=\lim_{N\to\infty}\frac{1}{Z_{L,a}}e^{-\frac{\mu a^d}{2\,c\,\hbar}\sum_{\xi\in\Lambda}I(\varphi_\xi)}\prod_{k=0}^{N}P(d\varphi^{(k)})\end{aligned}\tag{3.10}$$

under the condition that $\varphi=\sum_{k=0}^{\infty}\varphi^{(k)}$ is held fixed; the Z'z normalize to 1 the measures in (3.10).

At this point there will be a change in point of view and the fields $\varphi^{(k)}$ will no longer be regarded just as auxiliary fields but as objects interesting in their own right: the stability problem will be extended to the problem of showing that not only φ but also $\varphi^{(k)}$, for each k, have a well defined limit distribution as $a\to 0$ if they are given the distribution (3.10) for $a>0$.

The plan is to attack the ultraviolet stability problem by studying the measure

$$P^{(\leq N)}(d\varphi)=\frac{e^{-\frac{\mu a^d}{2\,c\,\hbar}\sum_\xi I(\varphi^{(\leq N)}_\xi)}}{Z_{N,a}}\prod_{j=0}^{N}P(d\varphi^{(j)}) \tag{3.11}$$

uniformly in a,N allowing $I(\varphi)$ to depend on a,N and on the derivatives of φ if this becomes necessary in order to ensure the existence of an interesting limit as $a\to 0$ after letting $N\to\infty$.

Ultimately "interesting" should mean a field theory susceptible to a physical interpretation as a theory of interacting particles: it should verify various properties among which the possibility of defining an operator formally equal to $\mathcal{H}$ in (1.11). For instance one could require that the field φ verifies the Nelson axioms or the Osterwalder–Schrader axioms or that it should lead in some way or another to the construction of Wightman fields (which undoubtedly is the minimal requirement thought so far) (for a critical discussion and a review of the axioms of various type and their relations see (Simon, 1974), see also (Nelson, 1973a,b,c; Osterwalder and Schrader, 1973a; Wightman, 1956)).

In other words one is free to change the rules of the game provided one eventually succeeds in constructing a Wightman field theory describing nontrivial interactions; see also the comments at the end of Sec. 1 and in Sec. 22.

Of course the more one changes the rules of the game the more one has to work at a later stage. For instance in passing from the lattice regularized–continuous time approach of Sec. 2 to the problem of taking the $N\to\infty$ limit in (3.11), we lose the "unitary character" of the theory because it is no longer clear (and in fact it is not true) that the process $P^{(\leq N)}(d\varphi)$ can be generated by a Hamiltonian, as it was instead the case in Sec. 2, i.e. before starting the chain of "slight" changes leading to (3.11). So once the limit as $N\to\infty$ will have been taken, we shall have to worry whether it has the properties which would allow us to interpret it as generated by a Hamiltonian operator, i.e. whether a formula like (2.6) holds for some operator $H_{quantum}$.

In constructing a field theory it may sometimes be convenient to give up temporarily some of the properties of the final theory; note, on the other hand, that the continuous time lattice regularization although "unitary" is mot translation invariant (a property holding only if $b = 0$ in (2.11)).

At this point in view of the latter remark it is very tempting to simplify the problem by interchanging the limits on a and on N and let $a \to 0$ while keeping N fixed and then let $N \to \infty$. This leads to the measures

$$P^{(\le N)}(d\varphi) = \frac{e^{-\frac{\mu a^d}{2\,c\,\hbar}\int_\Lambda I(\varphi^{(\le N)}_\xi\, d\xi)}}{Z}\prod_{j=0}^{N} P(d\varphi^{(j)}) \qquad (3.12)$$

where now $P(d\varphi^{(j)})$ denotes the distribution of the field $\varphi^{(j)}$ with covariance associated with (3.7) or, alternatively, (3.8) or any higher order regularization of them, and $I(\varphi)$ depends, possibly, on N.

The advantage of studying (3.12) is that it is obviously easier in some respects than (3.11) because the fields $\varphi^{(j)}$ are now related by simple scalings, as the first of (3.7) or (3.8) show (i.e. $\varphi^{(j)}_\xi$ has roughly the same distribution as $\gamma^{j(d-2)}\varphi^{(0)}_{\gamma^j\xi}$; note, however, that even in the case (3.7) there are small corrections because, although in this simple case $\overline{C}^{(j)}$ scales exactly, the covariance $C^{(j)}$ does not so because of the imposition of periodic boundary conditions,

Furthermore, one does not have to distinguish between the cases $\gamma^{-j}m^{-1} \ge a$ and $\gamma^{-j}m^{-1} < a$. However one should note that the fields $\varphi^{(j)}$ with $\gamma^{-j}m^{-1} < a$ are somewhat trivial (i.e. they are approximately independently distributed on the lattice points) and thus one heuristically thinks that the limits of (3.12), as $N \to \infty$, should lead to the same measures as the limit of (3.11) as $N \to \infty$ first and $a \to 0$ second.

This remark could in fact be made more precise to the extent that it can become "all the results discussed in this paper and concerning the existence of formal perturbation theory and concerning the existence of non Gaussian limits of (3.12) as $N \to \infty$, or the existence of formal perturbation expansions of various quantities, could also be obtained considering the limits $\lim_{a\to 0}\lim_{N\to\infty}$ of (3.11)"; this statement should emerge quite clearly from what follows but it will not be explicitly proved (to limit the material presented here).

The theory of the limits as $N \to \infty$ of (3.12) is already complex and interesting enough, and studying (3.11), as far as the problems discussed here are concerned, does not lead to any new ideas but only to rather trivial technical digressions.

Therefore from now on I shall concentrate on the discussion of (3.12) with $\varphi^{(k)}$ being defined by (3.7) or (3.8) or by their higher order analogs depending on the models, the aim being to obtain a limit as $N \to \infty$ in which the distributions of all the variables $\varphi^{(j)}$ are well defined, although they are not Gaussian.

One says that the approach to field theory based on (3.11) is a "nearest neighbor lattice regularization" approach, while the one adopted here, via (3.12), is a "Pauli–Villars regularization" approach of some order; more appropriately it should be called "Feynman regularization", see (Pauli and Villars, 1949). Both approaches are widely used in the literature: see for example (Aizenman, 1982; Boboliubov and Shirkov, 1959; Brydges *et al.*, 1983; Callan, 1976; Frölich, 1982; Park, 1977).

Before starting the analysis of (3.12) it is important to stress once more at the cost of being repetitious and to avoid hiding important issues, that while the theories of (3.11) and (3.12) are equivalent up to technicalities as far as the results presented in this work are concerned, it is by no means clear that they are equally suitable for pursuing the quest of other results that we should like to obtain, first among them that of showing the existence or nonexistence of a non trivial φ^4 field in dimension $d = 4$. Furthermore there are other possible approaches most of which give the same results as the one presented here, if applied to the problems considered here, and which might be better suited for the study of the hard open problems, see comments at the end of Sec. 1 and in Sec. 22 (see also (Gallavotti and Rivasseau, 1985)).

The fields $\varphi^{(k)}$ with covariance (3.7) will e called "first order Pauli–Villars fields" of frequency index k, while those with covariance (3.8) will be "second order Pauli–Villars fields" with frequency index k (shortly "with frequency k"); similarly one can define the n–th order Pauli–Villars fields via the use of higher order identities in (3.4) and with $\varepsilon = p^2$, see below.

Formula (3.12) will define the m–th order regularized interacting measure if $\varphi^{(k)}$ has the meaning of an m–th order Pauli–Villars fie;d (of course we allow only functions $I(\varphi)$ that are such that $I(\varphi^{(\le N)})$ has a meaning, at least with probability 1 with respect to the measure $\prod_{j=0}^{N} P(d\varphi^{(j)})$.

The latter remark is very important: it shows that

$$I(\varphi) \propto \varphi^4 \qquad (3.13)$$

is not admissible for $d \ge 4$ if one uses the Pauli–Villars first order field (because the expected value of $\varphi^{(\le N)}_\xi$ is infinite if $d \ge 4$, by the second of (3.7)). However

$$I(\varphi_\xi) = \lambda\varphi_x^2 + \mu\varphi_\xi^2 + \alpha\,(\partial\,\varphi_\xi)^2 \qquad (3.14)$$

is meaningful if one uses in (3.12) the second order Pauli–Villars regularization even for $d = 4$, because the expected values of $(\varphi^{(\le N)}_\xi)^2$ and $(\partial\,\varphi^{(\le N)}_\xi)^2$ are finite if computed using (3.8) rather than (3.7).

This section will be concluded by listing a more quantitative meaning to be given to the regularity statements about the fields $\varphi^{(k)}$ made after (3.7) and (3.8).

Let $\varphi^{(k)}$ be a sample of a Gaussian random field distributed with covariance $C^{(k)}_{\xi\eta}$ in (3.7). Then if Λ is imagined paved by a lattice Q_k of square boxes Δ with side size $\gamma^{-k}m^{-1}$, one finds that for $d=2,3$ and for all choices of $B_\Delta>0$,

$$|\varphi^{(k)}_\xi|\le B_\Delta\,\gamma^{\frac12 k(d-2)},\qquad \forall\xi\in\Delta\in Q_k \tag{3.15}$$

$$\begin{aligned}|\varphi^{(k)}_\xi-\varphi^{(k)}_\eta|\le& B_\Delta\,\gamma^{\frac12 k(d-2)}(\gamma^k|\xi-\eta||)^{\frac12(4-d)-\varepsilon},\\ &\forall\xi\in\Delta,\ |\xi-\eta|<\gamma^{-k}m^{-1}\end{aligned} \tag{3.16}$$

hold with probability bounded *below* by

$$\prod_{\Delta\in Q_k}(1-\overline{A}\,e^{-\overline{\alpha}B^2_\Delta}) \tag{3.17}$$

if $\overline{A},\overline{\alpha}>0$ are suitable constants depending on the choice of the arbitrary parameter $\varepsilon>0$ but k–independent. Of course one assumes here that the side of Λ is divisible by $\gamma^{-k}m^{-1}$ for all $k\ge0$; this assumption could easily be released for the study of the problems considered in this paper, see however the comments in Sec. 22.

More generally the chain (3.4) can be continued to express $(m^2+p^2)^{-1}$ as a sum of reciprocals of polynomials of degree 2^{n+1} in p^2, $n=0,1,2,\ldots$. In this way one can define a field

$$\varphi^{(\le N)}=\sum_{k=0}^N\varphi^{(k)} \tag{3.18}$$

where $\varphi^{(\le N)}$ is a very smooth Gaussian fields which is decomposed into regular independent fields with covariances $C^{(k)}$ defined by "periodizing" a covariance $\overline{C}^{(k)}$ via the second relation in (3.7) (or (3.8)) and with $\overline{C}^{(k)}$ verifying

$$\begin{aligned}&|\partial^j\overline{C}^{(k)}_{\xi\eta}|\le A_0\gamma^{kj}\gamma^{k(d-2)}e^{-\kappa_0\gamma^k|\xi-\eta|},\quad 0\le j<j_0\\ &|\partial^{j_0-1}\overline{C}^{(k)}_{\xi\eta}-\partial^{j_0-1}\overline{C}^{(k)}_{\xi'\eta}\le\\ &\qquad A_\varepsilon\gamma^{k(j_0-1)}(\gamma^k|\xi-\xi'|)^{1-\varepsilon}e^{-\kappa_0\gamma^k|\xi-\eta|},\end{aligned} \tag{3.19}$$

where $A_0,A_\varepsilon,\kappa_0$ are suitable constants and $\varepsilon>0$ is arbitrary, and where $j_0=2^{n+1}-d$. For instance the case $n=2$ is worked out explicitly in (3.8). The n–th order Pauli–Villars fields defined by $C^{(k)}$ verify, with probability bounded below by (3.17),

$$\begin{aligned}&|\partial^j\varphi^{(k)}_\xi|\le B_\Delta\gamma^{jk}\gamma^{\frac12k(d-2)}\qquad 2j<j_0,\\ &\Big|\varphi^{(k)}_\xi-\sum_{2|\mathbf{a}|<j_0}\frac{1}{\mathbf{a}!}\frac{\partial^{\mathbf{a}}\varphi^{(k)}_\xi}{\partial\xi^{\mathbf{a}}}(\xi-\eta)^{\mathbf{a}}\Big|\le\\ &\le B_\Delta\gamma^{\frac12k(d-2)}\,(\gamma^k|\xi-\eta|)^{\frac12 j_0-\varepsilon}\end{aligned} \tag{3.20}$$

where ∂^j denotes any j–th order derivative and

$$\mathbf{a}!\overset{def}{=}\prod_{i=1}^d a_i!,\qquad |\mathbf{a}|\overset{def}{=}\sum_{i=1}^d a_i,$$

and $\partial^{\mathbf{a}}$ is the derivative of order $a_1+\ldots+a_d$ of order a_i with respect to the first component *etc*, so that the second of (3.2) is an estimate for the remainder of a Taylor series.

For instance if $n=3$, $d=4$ the field $\varphi^{(k)}$ admits five derivatives and the fifth id Hölder continuous with exponent less than $\frac12$ and $C^{(k)}$ admits 11 derivatives.

Since periodic boundary conditions are being used unless explicitly stated otherwise, here as well as in the rest of the paper, $(\xi_2-\xi_1)$ will be a symbolic notation for a periodic function on $\Lambda\times\Lambda$ equal to the vector from ξ_1 to ξ_2 when the distance between ξ_1 and ξ_2 on Λ is small and, for larger distances, equal to $(\xi_2-\_1)\chi(|\xi_2-\xi_1|)$ with $|\xi_2-\xi_1|$ being the distance on the torus and $\chi\in\mathcal{C}^\infty$ is monotonic and $\chi(r)=0$ if $r>1$, $\chi(r)=1$ if $r<\frac12$ (i.e. $(\xi_2-\xi_1)$ is really what the notation suggests if x_1 and x_2 are close enough and (arbitrarily) 0 otherwise, while $|\xi_2-\xi_1|$ is the distance on the torus Λ).

The inequalities (3.19) are elementary consequences of the analysis of the asymptotic behavior of the integrals in (3.7) and (3.8) and of their generalizations to order n. Whereas the inequalities (3.20) and (3.16) follow from the fact that the $\varphi^{(k)}$ form a Markov process and from the regularity properties of $C^{(k)}$ expressed by (3.19) via the classical idea of Wiener for the Hölder continuity of the sample paths of the Brownian motion, see (Benfatto *et al.*, 1980b; Colella and Lanford, 1973).

In the literature other regularizations are also considered which produce infinitely smooth fields $\varphi^{(k)}$ by using "nonpolynomial" decompositions like

$$\frac{1}{1+p^2}=\frac{\chi_0(p)}{1+p^2}+\sum_{k=1}^\infty\frac{\chi_1(\gamma^kp)}{1+p^2} \tag{3.21}$$

where

$$\chi_0(p)+\sum_{k=1}^\infty\chi_1(\gamma^kp)\equiv1 \tag{3.22}$$

and χ_0,χ_1 are $\mathcal{C}^\infty$ nonnegative functions such that χ_1 has support in $1\le p^2\le\gamma^2$. Such decompositions produce $\overline{C}^{(k)}$'s which verify (3.19) with j_0 arbitrarily prefixed but with the modification that the exponential decay factor is replaced by $(1+\gamma^k|\xi-\eta|)^{-w}$ with w arbitrarily prefixed (i.e. the decoupling takes place on the same scale as in (3.19), namely $\gamma^{-k}m^{-1}$, but it is slower thane exponential, although still faster than any prefixed power).

iv. Perturbation theory and ultraviolet stability

I shall try to be very general, not for love of generality, but because perturbation theory is conceptually very simple and if one discusses it in the few examples in which one is really interested one makes it appear more complex, because all the fine details peculiar to each model become most inextricably mixed up with its structure.

The first thing to fix is the interaction $\mathcal{I}(\varphi)$: choose $\mathcal{I}(\varphi)$ to have the form

$$\begin{aligned} &V(\varphi^{(\le N)},\boldsymbol{\lambda},N)=\\ &=\sum_{\alpha=1}^{t}\lambda^{(\alpha)}\int_\Lambda v_N^{(\alpha)}(\varphi_\xi^{(\le N)},\partial\varphi_\xi^{(\le N)})\,d\xi\stackrel{def}{=}I(\varphi)\end{aligned}\tag{4.1}$$

If $\varphi\equiv\varphi^{(\le N)}$ the function $\mathcal{I}(\varphi)$ spans a finite dimensional linear space $\mathcal{I}_N$ as $\boldsymbol{\lambda}=(\lambda^{(\alpha)})_{\alpha=1,\ldots,t}$ spans $\mathbb{R}^t$ or a linear subspace of $\mathbb{R}^t$, fixed *a priori* ; one can regard $\mathcal{I}_N$ as a subspace of $L_2(\prod_{j=0}^N P(d\varphi^{(j)})$.

It is convenient to assume the functions $v^{(\alpha)}(\varphi^{(\le N)})$, $N=0,1,\ldots$, to be so related that for all $N'\le N$ it is

$$\begin{aligned}&\int_\Lambda v^{(\alpha)}(\varphi_\xi^{(\le N')},\partial\varphi_\xi^{(\le N')})\,d\xi=\\ &=\int P(d\varphi^{(N'+1)})\ldots P(d\varphi^{(N)})\times\\ &\times\int_\Lambda v^{(\alpha)}(\varphi_\xi^{(\le N)},\partial\varphi_\xi^{(\le N)})\,d\xi\end{aligned}\tag{4.2}$$

or, in other words, $v_{N'}^{(\alpha)}\in\mathcal{I}_{N'}$ is the projection on $\mathcal{I}_{N'}$ of $v_N^{(\alpha)}\in\mathcal{I}_N$ performed by using as projection operator the integration with respect to the field components of frequency higher than N'. This property, which is verified automatically in all models that are considered here (because every model will be written in "Wick–ordered" form, see below), is very convenient for the exhibition of general structural properties of perturbation theory. In probability the latter property is known as a "martingale" property of the sequence of functions $v_N^{(\alpha)}$.

A sequence $\mathcal{I}=(\mathcal{I}_N)_{N=0,\ldots\infty}$ verifying the martingale relation (4.2) will be called an "interaction".

Of course the choice of the free fields $\varphi^{(j)}$, i.e. of the order of regularization, will always have to be such that the integrals in (4.1) make sense [for instance if $v^{(\alpha)}$ really depends on $\partial\varphi_\xi^{(\le N)}$ we shall use at least a second order regularization for $d\le4$; if $v^{(\alpha)}$ depends only on $\varphi_\xi^{(\le N)}$ then we could also use simply the first order regularization provided $d<4$, see (3.20).

A field theory $\mathcal{I}$ can be defined in two, usually nonequivalent, ways: "nonperturbatively" as a probability measure which is the limit as $N\to\infty$ of measures defined by

$$P_{\mathcal{I},N}(d\varphi^{(\le N)})=\frac{e^{V(\varphi^{(\le N)},\boldsymbol{\lambda}_N,N)}}{Z}\prod_{j=0}^N P(d\varphi^{(j)})\tag{4.3}$$

where $\boldsymbol{\lambda}_N$ is a give sequence of coupling constants called "bare couplings", for which $e^{V(\varphi^{(\le N)},\boldsymbol{\lambda}_N,N)}\in L_1(\prod_{j=0}^N P(d\varphi^{(j)})$, or "perturbatively". The latter sense is based on the following idea, (Dyson, 1949a,b; Feynman, 1948; Schwinger, 1949a,b). Consider the following formal power series in the parameters $\boldsymbol{\lambda}=(\lambda^{(1)},\ldots,\lambda^{(t)}\in\mathbb{R}^t$:

$$\begin{aligned}\boldsymbol{\lambda}_N(\boldsymbol{\lambda})=&\sum_{m_1,\ldots,m_t}\ell_N(\mathbf{m})\lambda^{(1)\,m_1}\ldots\lambda^{(t)\,m_t}\equiv\\ &\equiv\sum_{\mathbf{m}}\ell_N(\mathbf{m})\boldsymbol{\lambda}^{\mathbf{m}},\end{aligned}\tag{4.4}$$

where $\ell(\mathbf{m})\in\mathbb{R}^t$. Then compute

$$\begin{aligned}&\int e^{\varphi^{(\le N)}(f)}P_{\mathcal{I},N}(d\varphi)=\\ &=\frac{\int e^{\varphi^{(\le N)}(f)}e^{V(\varphi^{(\le N)}(f),\boldsymbol{\lambda}_N(\boldsymbol{\lambda}),N)}\prod_{j=0}^N P(d\varphi^{(j)})}{\int e^{V(\varphi^{(\le N)}(f),\boldsymbol{\lambda}_N(\boldsymbol{\lambda}),N)}\prod_{j=0}^N P(d\varphi^{(j)})},\end{aligned}\tag{4.5}$$

formally, by expanding all exponentials in powers and then using (4.4) to express the results as a power series in $\boldsymbol{\lambda}$ by collecting terms with equal powers:

$$\int e^{\varphi^{(\le N)}(f)}P_{\mathcal{I},N}(d\varphi)\equiv\langle e^{\varphi^{(\le N)}(f)}\rangle=\sum_{\mathbf{m}}S(\mathbf{m},N,f)\boldsymbol{\lambda}^{\mathbf{m}}\tag{4.6}$$

Then the *perturbative field theory* with interaction $\mathcal{I}$ and *bare constants* $\boldsymbol{\lambda}_N(\boldsymbol{\lambda})$ given by (4.4) is well defined if the limits

$$S(\mathbf{m},f)=\lim_{N\to\infty}S(\mathbf{m},N,f)\tag{4.7}$$

exist for all smooth test functions f and for all $\mathbf{m}$.

The theory will be called "perturbatively trivial" if the power series

$$\sum_{\mathbf{m}}S(\mathbf{m},f)\boldsymbol{\lambda}^{\mathbf{m}}\tag{4.8}$$

formally converges to the exponential of a quadratic form in f ("Gaussian theory") for $|\boldsymbol{\lambda}|$ small.

Similarly if the limits of (4.3) are Gaussian measures for all possible choices of $\boldsymbol{\lambda}_N$ then one says that the theory is "trivial".

If it is impossible to find a formal power series (4.4) such that the limits (4.7) exist one says that $\mathcal{I}$ is a "non renormalizable" theory.

The power series (4.8) is called the "renormalized series for $\mathcal{I}$", and the parameters $\boldsymbol{\lambda}$ in it are called the "renormalized couplings", while the corresponding formal series (4.4) define the perturbative bare couplings [note that the formal power series (4.4) do not necessarily converge].

It is perhaps worth stressing again that the real objects that one is trying to find are more complex than a probability measure P which is a limit of (4.3) (in a perturbative or nonperturbative sense), so after such limits are constructed one still has to see if they have the right properties to allow their interpretation as relativistically invariant quantum field theories.

However, in the few cases in which the measures P have been constructed as limits for $N \to \infty$ of (4.3) the understanding of the problems remaining before a full interpretation of the results as relativistic quantum fields, has been carried out without excessive difficulties [after the basic techniques to deal with this question were developed in the basic papers, (Glimm, 1968a,b; Glimm and Jaffe, 1968, 1970a,b; Glimm *et al.*, 1973; Guerra, 1972; Nelson, 1966, 1973a,b,c; Osterwalder and Schrader, 1973a,b)], so I shall not develop this question further here, after warning the reader of its paramount importance, on the grounds that it should not be thought of as a part of the main subject of this paper, i.e. of the ultraviolet limit problem.

Perturbation theory plays a major role even in the so called nonperturbative approach (Balaban, 1982a,b, 1983; Benfatto *et al.*, 1978, 1980a,b; Brydges *et al.*, 1983; Federbush and Battle, 1982, 1983; Feldman and Ostervalder, 1976; Gallavotti, 1978, 1979a,b; Gawedski and Kupiainen, 1980, 1983, 1984; Glimm and Jaffe, 1968, 1970a,b; Glimm *et al.*, 1973; Magnen and Seneor, 1976).

Here perturbation theory will be treated from the point of view of the renormalization group, expanding the ideas developed and used in the just quoted (Benfatto *et al.*, 1978, 1980a). I shall follow the theory presented in (Gallavotti and Nicolò, 1985a,b), with some modifications here and there. The first to treat completely, to all orders, perturbation theory by literally applying the renormalization group methods has been (Polchinskii, 1984), who adopts a method slightly different from the one presented here obtaining weaker results —e.g. the $n!$ bounds are not treated in his work, at least not explicitly.

The renormalization group approach to field theory grew out of several earlier works [e.g. (DiCastro and Jona-Lasinio, 1969; Jona-Lasinio, 1975; Kadanoff, 1966; Ma, 1976; Wilson, 1965, 1971, 1972, 1983; Wilson and Kogut, 1973)].

Here "applying the renormalization group method" will mean that one regards the fields $\varphi^{(0)}, \ldots, \varphi^{(k)}, \ldots$ as real entities describing phenomena taking place on their own length scale $\gamma^{-k_m^{-1}}$: and we shall define the *effective interaction on scale* $\gamma^{-k_m^{-1}}$ as

$$e^{V^{(k)}(\varphi^{(k)})} \stackrel{def}{=} \int e^{V(\varphi^{(\leq N)}, \boldsymbol{\lambda}_N(\boldsymbol{\lambda}))} \cdot P(d\varphi^{(N)}) \ldots P(d\varphi^{(k+1)}) \tag{4.9}$$

In perturbation theory one fixes the formal power series $\boldsymbol{\lambda}_N(\boldsymbol{\lambda})$ in such a way that $V^{(k)}$ turns out to be given by a formal power series in $\boldsymbol{\lambda}$ which, order by order, has a limit as $N \to \infty$ if $\varphi^{(0)}, \ldots, \varphi^{(k)}$ verify (3.20) (n being the order of the chosen regularization), and the limit has a short range structure allowing us to interpret $V^{(k)}$ as a statistical mechanics interaction between spins (the $\varphi^{(k)}$'s) located on a lattice of mesh $\gamma^{-k_m^{-1}}$ rather than continuous fields—see Sec. 3).

One might be worried that the fields $\varphi^{(j)}$ do not really have a physical meaning (yet) and that knowing that they are well defined objects even in presence of interaction does not really tell anything about their sum $\varphi^{(\leq N)}$ which is the object with physical meaning (in the limit $N \to \infty$). One could repair this objection by imagining that the last term (say) in (4.1) has the *linear* form

$$\lambda^{(t)} V_N(\varphi^{(\leq N)}, \partial\varphi^{(\leq N)}) \equiv \lambda^{(t)} f(\xi) \varphi^{(\leq N)} \tag{4.10}$$

and show that the effective potentials are well defined with a choice of $\mathbf{l}_N(\mathbf{m})$ leading to $\lambda_N^{(t)} \equiv \lambda^{(t)}$ (i.e. "no renormalization on the linear part of the interaction") and to an expression of the remaining bare couplings $\lambda_N^{(1)}, \ldots, \lambda_N^{(t-1)}$ involving *only* $\lambda^{(1)}, \ldots, \lambda^{(t-1)}$ (*and not* $\lambda^{(t)}$); then the effective potentials and the coefficients $S(\mathbf{m}, N, f)$ would be simply related to the Schwinger functions of the field $\varphi^{(\leq N)}$ and the problem of proving existence and ultraviolet stability of the effective potentials would be in principle harder than that of proving that of the limit (4.7) in absence of a linear term in V_N (although it will be in fact an equivalent problem in the cases studied here).

Alternatively one could decide to worry about this problem after completing the theory of the effective potentials: in fact the formal connection between the effective potentials and the Schwinger functions will be briefly discussed in Sec. 10.

V. Effective potentials: the algorithm for their construction

Given an interaction $\mathcal{I}$ as defined in Sec. 4 (see (4.1) and (4.2)), let

$$V(\varphi^{(\leq N)}) = \sum_{\alpha=1}^{t} \lambda^{(\alpha)} \int_\Lambda v_N^{(\alpha)}(\varphi^{(\leq N)}; \partial\varphi^{(\leq N)})\, d\xi \tag{5.1}$$

The effective interaction on the length scale $\gamma^{-k} m^{-1}$ is defined by

$$e^{V^{(k)}(\varphi^{(\le k)})} = \int e^{V(\varphi^{(\le N)})}\cdot \tag{5.2}$$
$$\cdot P(d\varphi^{(N)})\ldots P(d\varphi^{(k+1)})$$

To be slightly more concrete it is convenient to list the cases which will be treated here or which can be treated easily with the methods reviewed in this paper.

(1) *Polynomial fields in two dimensions*

$$v^{(\alpha)}(\varphi^{(\le N)}) =: (\varphi^{(\le N)})^\alpha : \tag{5.3}$$

where the dots denote the Wick ordering of polynomials and $\alpha > 0$.

In this case as well as in the cases below the property (4.2) is trivially a consequence of the properties of the Wick polynomials. Such properties are remarkable and the reader will be supposed familiar with them. For ease of reference the definitions, their main properties and the ideas from which they are derived are provided in Appendix A3.

The notion of Wick monomial will not be needed in this section nor in the following sections, 6–11, where everything is worked out without referring to Wick ordering or Wick monomials.

(2) *Sine–Gordon field in two dimensions*

$$V(\varphi^{(\le N)}) = \sum_{\sigma=\pm 1} \frac{\lambda}{2}\int_\Lambda : e^{i\sigma\,\alpha\,\varphi_\xi^{(\le N)}} d\xi + \nu\int_\Lambda d\xi =$$
$$= \lambda\int_\Lambda : \cos\varphi_\xi^{(\le N)} : d\xi + \nu\int_\Lambda d\xi, \quad \alpha > 0 \tag{5.4}$$

(3) *Exponential fields, $d \ge 2$*

$$V(\varphi^{(\le N)}) = -\lambda\int_\Lambda : e^{\alpha\,\varphi_\xi^{(\le N)}} d\xi + \nu\int_\Lambda d\xi \tag{5.5}$$

(4) *φ^4 field in three dimensions*

$$V(\varphi^{(\le N)}) = \tag{5.6}$$
$$= -\int_\Lambda \big(\lambda : (\varphi^{(\le N)})^4 : + \mu : (\varphi^{(\le N)})^2 : + \nu\big) d\xi$$

(6) *φ^4 field with wave function renormalization for $d \le 4$*

$$V(\varphi^{(\le N)}) = -\int_\Lambda \big(\lambda : (\varphi^{(\le N)})^4 : + \tag{5.7}$$
$$+ \mu : (\varphi^{(\le N)})^2 : + \alpha : (\partial\varphi^{(\le N)})^2 : + \nu\big) d\xi$$

(7) *φ^6 field with wave function renormalization for $d \le 3$*

$$V(\varphi^{(\le N)}) = -\int_\Lambda \big(\sigma : (\varphi^{(\le N)})^6 : + \lambda : (\varphi^{(\le N)})^4 : +$$
$$+ \mu : (\varphi^{(\le N)})^2 : + \alpha : (\partial\varphi^{(\le N)})^2 : + \nu\big) d\xi \tag{5.8}$$

All the above cases are examples of interactions $\mathcal{I}$ in the sense of (4.1) and (4.2) (see Appendix A3 for the properties of Wick monomials).

In view of the above ambitious models one might thin that it would be very hard to find reasonable expressions for $V^{(k)}$; this is *not* really the case as the algorithm below proves; the reason being that the construction of $V^{(k)}$ can be carried out in general, without using the detailed structures (5.5)–(5.8) or the Wick ordering properties, starting from (5.1),(4.1) and (4.2).

The mathematical basis for the algorithm is a trivial Taylor series. To define it introduce the notations

$$\begin{aligned}\mathcal{E}(\cdot) =& \text{expectation value with respect}\\ &\text{to a probability measure,}\\ \mathcal{E}_k(\cdot) =& \text{expectation value with respect}\\ &\text{to the Gaussian measure } P(d\varphi^{(k)}),\end{aligned} \tag{5.9}$$

and in general, given p random variables $x_1,\ldots,x_p$ and p positive integers $n_1,\ldots,n_p$ one defines the *truncated expectations of* $x_1,\ldots,x_p$ as

$$\mathcal{E}^T(x_1,\ldots,x_p;n_1,\ldots,n_p) = \tag{5.10}$$
$$= \frac{\partial^{n_1+\ldots+n_p}}{\partial\lambda_1^{n_1}\ldots\partial\lambda_p^{n_p}}\log\mathcal{E}(e^{\lambda_1 x_1+\ldots+\lambda_p x_p})\Big|_{\lambda_i=0}$$

The symbol $\mathcal{E}_k^T$ will therefore have a well defined meaning if $x_1,\ldots,x_p$ are p functions depending on $\varphi^{(k)}$. One checks by induction the *Leibnitz rule*: $\forall\omega_1,\ldots,\omega_p\in\mathbb{R}$

$$\mathcal{E}^T(\omega_1 x_1+\ldots+\omega_p x_p;n) = \sum_{\substack{n_1,\ldots,n_p\\ n_1+\ldots+n_p=n}} \frac{n!\,\omega_1^{n_1}\ldots\omega_p^{n_p}}{n_1!\ldots n_p!}\cdot$$
$$\cdot\,\mathcal{E}^T(x_1,\ldots,x_p;n_1,\ldots,n_p) \tag{5.11}$$

and if $n = n_1+\ldots+n_p$

$$\begin{aligned}&\mathcal{E}^T(x;1)\equiv\mathcal{E}(x),\quad \mathcal{E}^T(x;0)\equiv 0\\ &\mathcal{E}^T(x,x,\ldots,x;n_1,n_2,\ldots,n_p)\equiv\mathcal{E}^T(x;n)\end{aligned} \tag{5.12}$$

Therefore the following Taylor expansion, *cumulant expansion*, formally holds

$$\mathcal{E}(e^x) = e^{\sum_{p=1}^\infty \frac{1}{p!}\mathcal{E}^T(x;p)} \tag{5.13}$$

and is convergent for any bounded random variable x. Hence modulo convergence problems

$$\int P(d\varphi^{(N)})e^{V} \equiv e^{\sum_{n=1}^{\infty}\frac{1}{n!}\mathcal{E}_N^T(V;n)} \stackrel{def}{=} e^{V^{(N-1)}} \tag{5.14}$$

and, recursively,

$$\int P(d\varphi^{(N)})P(d\varphi^{(N-1)})e^{V} \equiv e^{\sum_{h=1}^{\infty}\frac{1}{h!}\mathcal{E}_{N-1}^T(V^{(N-1)};h)} =$$

$$= \exp\sum_{h=1}^{\infty}\sum_{n_1+n_2+\ldots=h}\frac{1}{n_1!n_2!\ldots\;1!^{n_1}2!^{n_2}3!^{n_3}\ldots}\cdot \tag{5.15}$$

$$\cdot\,\mathcal{E}_{N-1}^T\big(\mathcal{E}_N^T(V;1),\mathcal{E}_N^T(V;2),\mathcal{E}_N^T(V;3),\ldots;n_1,n_2,n_3,\ldots\big)$$

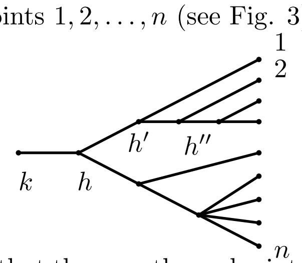
once we have applied the Leibnitz rule (5.11).

It is clear that by combining (5.9) and (5.13) one can find a formal expression for $V^{(k)}$ of the type (5.15): its structure will be elucidated in Sec. 6 by means of a graphical interpretation of the general term arising in the iteration of the above expansion.

vi. A graphical expression for the effective interactions

The structure of $V^{(k)}$, as obtained from V by doing successively integrations over fields of increasing length scale can be described easily i terms of a certain family of planar graphs, actually trees.

Draw n points $1,2,\ldots,n$ (see Fig. 3)



(3)

and imagine that they are the endpoints of a tree γ whose vertices v bear an index h_v, with $k\le h_v\le N$ and $h_v < h_{v'}$ if $v<v'$ in the order of the tree; the lowest vertex r of γ, called the *root* bears the index k, denoted $k(\gamma)$, and out of it emerges one branch only. All the other vertices $v>r$ are branching points with at least two branches. The end points of the tree are not regarded as vertices. The tree can be thought of as a partially ordered set from the root up to the end points.

Two trees will be regarded as identical if they can be superposed, together with the labels appended to their vertices, up to a permutation of the end point labels $(1,2,\ldots,n)$ and up to a change in the lengths of the branches and the location of the vertices which does not alter the topological structure of the tree. In drawing trees we shall agree to think that they are drawn in some standard fashion which always leads to the construction of a given representative in each class.

The number of end points in γ (n in Fig. 3) will be called the *order of* γ. A tree of degree 1 will be called trivial and it ill contain only one line from the root r to the end point 1.

The first vertex after r will be called v_0; it exists if and only if the tree γ is not trival.

Given a nontrivial tree γ, let $\gamma_1,\gamma_2,\ldots,\gamma_s$ be the trees which bifurcate in $|g$ from v_0, i.e. from the first nontrivial vertex (in Fig. 3 it is $s=2$). The s trees can be divided into into q classes of trees whose elements are identical up to the end points labelings and let $\overline{\gamma}_1,\ldots,\overline{\gamma}_q$ be the representatives of each class. Let $p_1,\ldots,p_q$ be the number of elements in each class. Define a *combinatorial factor* $n(\gamma)$ inductively as

$$n(\gamma)=\prod_{i=1}^{q}p_i!\,n(\overline{\gamma}_i)^{p_i} \tag{6.1}$$

setting $n(\gamma)=1$ if γ is the trivial tree.

The index h_v associated with each vertex v wil be called a *frequency index* or the *frequency of* v.

If one stares, for a conveniently long time, at (5.13) and (5.15) it becomes clear that

$$V^{(k)}=\sum_{\gamma,\,k(\gamma)=k}\frac{V(\gamma)}{n(\gamma)} \tag{6.2}$$

where the sum runs over the trees with root at frequency k and with frequency indices $h_v\le N$; $V(\gamma)$ is a function of the field $\varphi^{(\le k)}$ which, although it could be explicitly written, is more conveniently defined by induction. If γ is trivial, let

$$V(\gamma)\stackrel{def}{=}\mathcal{E}_{k+1}\cdots\mathcal{E}_n(V) \tag{6.3}$$

and if γ bifurcates on the first vertex v_0 following its root r into $\gamma_1,\ldots,\gamma_s$ at frequency $h_{v_0}=h$ let

$$\begin{aligned}&V(\gamma)=\\&=\mathcal{E}_{k-1}\ldots\mathcal{E}_{h-1}\mathcal{E}_h^T(V(\gamma_1),\ldots,V(\gamma_s);1,1,\ldots,1)\end{aligned} \tag{6.4}$$

As a result of (6.3),(6.4) one sees that each vertex of γ with index p corresponds to $\mathcal{E}_p^T$, while each line of γ joining two vertices $v<v'$ corresponds to

$$\mathcal{E}_{h_v+1}\ldots\mathcal{E}_{h_{v'}-1}, \tag{6.5}$$

while the lines joining a vertex v to an end poit correspond to

$$\mathcal{E}_{h_v+1}\ldots\mathcal{E}_N, \tag{6.6}$$

and finally each end point corresponds to a function V.

The proof of (6.4) is obtained by combining (6.2) and (6.3) with (5.9)–(5.14): one gets (6.4) immediately by induction on the degree of the tree.

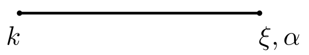
The above algorithm can be modified to obtain more explicit expressions for $V^{(k)}$. Let, see (4.1),(4.2),

$$V = \sum_{\alpha=1}^{t} \lambda^{(\alpha)} \int_{\Lambda} v_N^{(\alpha)}(\varphi_\xi^{(\le N)}, \partial\varphi_\xi^{(\le N)})\, d\xi \tag{6.7}$$

which is the case of interest here and introduce what will be called a *decorated tree* which is a tree whose end points bear labels $\theta(\gamma) = (\xi_1, \alpha_1), \ldots, (\xi_n, \alpha_n)$ instead of $1, \ldots, n$ and $\xi_j \in \mathbb{R}^d$, $\alpha_j \in (1, \ldots, t)$. Then (6.2),(6.4) imply that

$$V^{(k)} = \sum_n \int \sum_{\alpha_1, \ldots, \alpha_n} d\xi_1 \cdots d\xi_n \sum_{\substack{\gamma:\ \text{degree } \gamma = n \\ k(\gamma)=k}} \frac{V(\gamma)}{n(\gamma)} \tag{6.8}$$

where the sum runs over all decorated trees γ with root frequency k and with vertex frequencies h_v with $k < h_v \le N$ for $v > r$, and the value $V(\gamma)$ will have to be computed by using (6.3) and (6.4) except that V has to be replaced in the evaluation of the trivial tree contribution by: $\lambda^{(\alpha)}\, V_N^{(\alpha)}(\varphi_\xi^{(\le N)}, \partial\varphi_\xi^{(\le N)})$ if the trivial tree is



(4)

The third sum in (6.8) is performed by keeping fixed the decoration $\theta(\gamma) = ((\xi_1, \alpha_1), \ldots, (\xi_n, \alpha_n))$. Finally the combinatorial factor of the undecorated tree $\overline{\gamma}$ obtained by stripping γ of its decorations.

In other words one can say that the rule for evaluating a decorated tree is the same as that for evaluating an undecorated tree but with a different interpretation of the end points, which depends on the decorating indices.

For later use it is convenient to define a *tree shape* which is a tree of the above types once stripped of all its indices and decorations, except the index α attached to the end points, which will be called *type indices.*

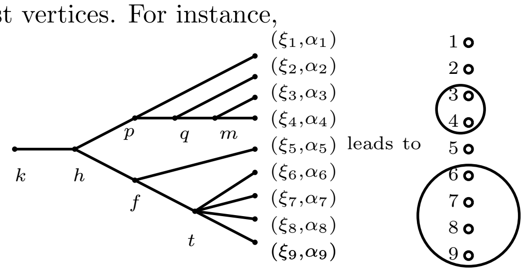
This completes the discussion of the basic graphical algorithm used to build $V^{(k)}$ for $k \ge 0$. It is, however, convenient to define also $V^{(-1)}$. For this purpose one thinks $\varphi^{(\le N)}$ as being given by

$$\varphi^{(\le N)} = \varphi^{(-1)} + \varphi^{(0)} + \ldots + \varphi^{(N)}, \tag{6.9}$$

where the field $\varphi^{(-1)}$ is distributed independently relative to the other $\varphi^{(j)}$, $j \ge 0$, and it has its own covariance $C_{\xi\eta}^{(-1)}$ which need not be specified (because it will eventually be taken to be identically zero whenever it appears in some interesting formulae. The introduction of $V^{(-1)}$ allows us to give a meaning to some expressions that will be met so that the case $k = 0$ can be treated on the same grounds as the cases $k > 0$, and $V^{-1)}$ will be described by trees with root frequency $k = -1$ via (6.8) and (6.9).

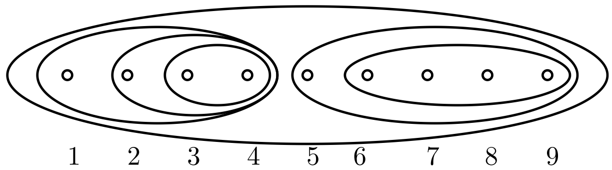
The following interpretation of a decorated tree is interesting and important for later applications. Each vertex v of γ can be interpreted as a cluster of the *end points positions* and the tree provides an organization in a hierarchy of clusters, of the points $\xi_1, \ldots, \xi_n$, which are the position labels of the end points of the tree.

To get a picture of such clusters first draw a box around each point $\xi_1, \ldots, \xi_n$; then consider a vertex v highest on the tree: out of it emerge s lines with labels $(\xi_{j_1}, \alpha_{j_1}), \ldots, (\xi_{j_s}, \alpha_{j_s})$ and do this for all the other highest vertices. For instance,



(5)

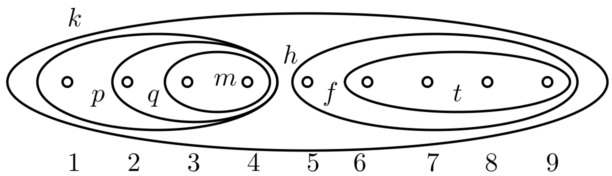
Then consider the next generation vertices and draw boxes around all the end points that can be reached from each of them by climbing the tree, *etc*



(6)

Actually the above cluster representation of γ becomes completely equivalent to the description of γ if inside each box one writes the frequency h_v of the vertex v corresponding to it [(1) attribute, conventionally, index $N+1$ or better no index at all to the innermost boxes enclosing only single points, (2) append to the j-th innermost box the index α_j, and (3) attribute to the outside of the outermost box the index k of the root of γ.]

For instance in the case of Fig. 5 one gets



(7)

where the frequencies $N+1$ have not been marked.

Therefore for each decorated tree one will be able to associate with each vertex a cluster of *points* and to associate with each cluster a frequency index in the above manner; furthermore each point has a position label of γ and a type label attached in the manner described and exemplified in the above pictures.

The *order* of a vertex v will be the number of points in the cluster corresponding to it': it coincides with the number of end points that can be reached from v by climbing the tree. So the degree of the tree coincides with the order of its root vertex as well as with the order of the first nontrivial vertex v_0 (if present).

vii. Renormalizability to second order and enormalization

Consider an interaction $\mathcal{I}$ as defined in Sec. 4, (4.1) and (4.2), and a formal power series like (4.4):

$$\boldsymbol{\lambda}_N(\boldsymbol{\lambda}) = \boldsymbol{\lambda} + \sum_{|\mathbf{m}|\ge 2} \mathbf{l}_N(\mathbf{m})\boldsymbol{\lambda}^{\mathbf{m}} \tag{7.1}$$

and define $V_{1,N} \equiv V_1$, see (5.1), and for $j \ge 2$

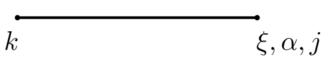
$$\begin{aligned} V_{j,N}^{(\alpha)} &= \big[\sum_{|\mathbf{m}|=j} \mathbf{l}_N(\mathbf{m})\boldsymbol{\lambda}^{\mathbf{m}} \big] \int v^{(\alpha)}(\varphi_\xi^{(\le N)}, \partial\varphi_\xi^{(\le N)})\, d\xi \\ V &= \sum_{j=1}^{\infty}\sum_{\alpha=1}^{t} V_{j,N}^{(\alpha)} \equiv V(\varphi^{(\le N)}; \boldsymbol{\lambda}_N(\boldsymbol{\lambda}), N) \end{aligned} \tag{7.2}$$

From the general theory of the preceding section it is easy to find the rule to compute the effective potential $V^{(k)}$ corresponding to the V in (7.2). The reader who finds the discussion below too abstract for a first reading can compare the abstract steps described here with the concrete corresponding steps done in studying the specific model $]f^4$, as described in Sec. 17 or the sine–Gordon field in Sec. 12.

One only allows trees with end points decorated by

$$(\xi, \alpha, j), \quad \xi \in \mathbb{R}^d, \quad \alpha = 1, \ldots, t, \quad j = 1, 2, 3, \ldots \tag{7.3}$$

Then if the trivial tree



(8)

is interpreted as [see also (4.2)]

$$\begin{aligned} &\mu_j^\alpha(\boldsymbol{\lambda})\mathcal{E}_{k+1}\ldots\mathcal{E}_N\big(v_N^{(\alpha)}(\varphi^{(\le N)}, \partial\varphi^{(\le N)})\big) \equiv \\ &\qquad \equiv \mu_j^\alpha(\boldsymbol{\lambda}) v_k^{(\alpha)}(\varphi^{(\le k)}, \partial\varphi^{(\le k)}) \\ &\mu_j^\alpha(\boldsymbol{\lambda}) \stackrel{def}{=} \sum_{|\mathbf{m}|=j} \mathbf{l}_N^\alpha(\mathbf{m})\, \boldsymbol{\lambda}^{\mathbf{m}} \end{aligned} \tag{7.4}$$

it follows [see Sec. 6] that

$$V^{(k)} = \sum_{n=1}^{\infty} \int d\xi_1 \ldots d\xi_n \sum_{\substack{\alpha_1,\ldots,\alpha_n \\ j_1,\ldots,j_n}} \sum_{\substack{\gamma:\, k(\gamma)=k \\ \text{degree}\, \gamma=n}} \frac{V(\gamma)}{n(\gamma)} \tag{7.5}$$

which expresses $V^{(k)}$ as a power series in $\boldsymbol{\lambda}$: the p-th order term being obtained by selecting in (7.5) the contributions such that $j_1 + \ldots + j_n \stackrel{def}{=} |\mathbf{j}| = p$.

If, given a tree γ with decorations $(\xi_1, \alpha_1, j_1), \ldots, (\xi_n, \alpha_n, j_n)$, one defines the degree $D(\gamma)$ as

$$D(\gamma) = j_1 + \ldots + j_n = |\mathbf{j}|, \tag{7.6}$$

then the contribution to $V^{(k)}$ of order p is obtained by restricting the sum in (7.5) to the trees with $D(\gamma) = p$. If we denote it by $V^{(k),p}$ it is

$$V^{(k),p} = \int d\boldsymbol{\xi} \sum_{\boldsymbol{\alpha}} \sum_{\mathbf{l}} \sum_{\substack{k(\gamma)=k \\ D(\gamma)=p}} \frac{V(\gamma)}{n(\gamma)} \tag{7.7}$$

Define

$$\overline{V}^{(k),p}(\varphi^{(\le k)}) = \lim_{N\to\infty} V^{(k),p}(\varphi^{(\le k)}) \tag{7.8}$$

where $\varphi^{(\le k)} = \sum_{j=0}^k \varphi^{(j)}$ is supposed such that each $\varphi^{(j)}$ verifies the smoothness properties (3.15),(3.16) or (3.20), depending on the regularization used for the free field.

The existence of the limit (7.8) clearly depends upon the choice of the coeficients $\mathbf{l}_N(\mathbf{m})$ in (7.1). According to the discussion of Sec. 4, the theory is renormalizable if there is a choice of the constants $\mathbf{l}_N(\mathbf{m})$ such that the limit (7.8) exists.

It is worth pointing out here that a trivial property of the renormalized series: if $\boldsymbol{\lambda}$ is expressed as a formal power series with N–independent coefficients in terms of new parameters $\boldsymbol{\lambda}'$, then $\boldsymbol{\lambda}_N(\boldsymbol{\lambda})$—see (7.1)—becomes a new formal power series in $\boldsymbol{\lambda}'$ with new coefficients $\mathbf{l}'_N(\mathbf{m})$; it should be clear that if the power series (7.5) in $\boldsymbol{\lambda}$ is renormalized, i.e. if the limts (7.8) exist, then also the power series in $\boldsymbol{\lambda}'$ is renormalized in the same sense (provided the series expressing $\boldsymbol{\lambda}$ in terms of $\boldsymbol{\lambda}'$ has no constant term, of course). This showsthat the coefficients $\mathbf{l}_N(\mathbf{m})$ cannot be uniquely determined by the requirement that the theory is renormalized [i.e. that the limits (7.8) exist].

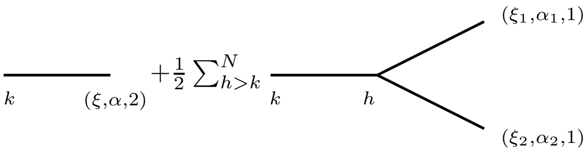
The problem is to decide whether a theory is renormalizable and to estimate in some way the size of $V^{(k),p}$ and, if possible, of $V^{(k)}$ itself.

It is possible to find a general rnormalizability criterion and general renormalization rules [i.e. rules to build the coefficients $\mathbf{l}_N(\mathbf{m})$ in (7.1). The whole theory stems from the simple examples considered below.

Clearly $V^{(k),p}(\varphi^{(\le k)})$ for $p = 1$ will always admit a limit as $N \to \infty$ being N–independent because of the property (4.2) of $\mathcal{I}$.

Therefore the requirement of existence of the limit (7.8) can put nontrivial restrictions only on $V_{2,N}$, $V_{3,N}$, ... (see (7.2)) and one can start by looking at the conditions on $V_{2,N}$ [i.e. on $\mathbf{l}_N(\mathbf{m})$ with $|\mathbf{m}| = 2$] imposed by the requirement of existence of the limit (7.8) for $p = 2$: it should be clear that if the theory is renormalizable it must be possible to fix $V_{2,N}$ so that $\overline{V}^{(k),2}$ (see (7.8)) exists simply because $V_{3,N}, V_{4,N}, \ldots$ do not contribute to $V^{(k),2}$.

Clearly $V^{(k),2}$ is determined by the sum of the contributions of the second order trees, i.e. graphically



(9)

where the summation over the α indices and the integration over the ξ indices is ubderstood. In formulae Fig. 9 becomes

$$\begin{aligned}&\mathcal{E}_{k+1}\ldots\mathcal{E}_N(V_{2,N})+\\&+\frac{1}{2!}\sum_{h>k}^{N}\mathcal{E}_{k+1}\ldots\mathcal{E}_{h-1}\mathcal{E}_h^T(\mathcal{E}_{>h}(V_1),\mathcal{E}_h(V_1))\end{aligned}\tag{7.9}$$

where we have shortened $\mathcal{E}_{>h}$ as $\mathcal{E}_{h+1}\ldots\mathcal{E}_N$ and also $\mathcal{E}^T(x_1,\ldots,x_q)$ as $\mathcal{E}^T(x_1,\ldots,x_q;1,\ldots,1)$.

Two cases can arise (1) the second term in (7.9) converges to a limit as $N\to\infty$ [for $\varphi^{(\le k)}$'s satisfying the smoothness mentioned above, see (3.20)]; or (2) this does not happen.

In the second case one must choose $V_{2,N}$ conveniently, if possible at all, to compensate the divergence present in the second term.

Since $V_{2,N}(\varphi^{(\le N)})$ *will always have to be in the interaction space* $\mathcal{I}_N$ the divergence of the second term in (7.9) can be compensated by a suitavbly chosen $V_{2,N}$ only if such divergence arises because the second term in (7.9) has some very large component on $\mathcal{I}_k$.

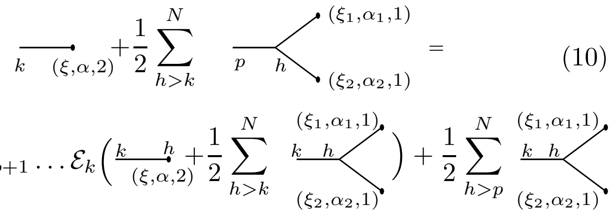
It is however unclear how to define, in an abstract context, the component to be considered: for the time being, and to remain on very general grounds, one can just say that there should be an operation $\mathcal{L}_k$ with range in $\mathcal{I}_k$ such that the two expressions

$$\begin{aligned}&\frac{1}{2!}\sum_{h>k}(1-\mathcal{L}_k)\mathcal{E}_{k+1}\ldots\mathcal{E}_{h-1}\cdot\\&\cdot\mathcal{E}_h^T(\mathcal{E}_{>h}(V_1),\mathcal{E}_{>h}(V_1))\qquad\text{and}\end{aligned}\tag{7.10}$$

$$\begin{aligned}&\mathcal{E}_{k+1}\ldots\mathcal{E}_N(V_{2,N})+\frac{1}{2!}\sum_{h>k}^{N}\mathcal{L}_k\mathcal{E}_{k+1}\ldots\mathcal{E}_{h-1}\cdot\\&\cdot\mathcal{E}_h^T(\mathcal{E}_{>h}(V_1),\mathcal{E}_{>h}(V_1))\end{aligned}\tag{7.11}$$

are convergent as $N\to\infty$ if $\varphi^{(\le k)}$ is smooth in the sense of (3.20).

If such an $\mathcal{L}_k$ exists for each k it is clear that it must depend on k in a special way because we can compute, for $p<k$, the effective potential in two necessarily equivalent ways, as the graphical relation of Fig. 10 explains (in it summation over the indices ξ,α is understood).



(10)

where the righ hand side (*r.h.s.*) is ontained by integrating (to second order in $\boldsymbol{\lambda}$) the exponential of the expression in (7.9), using (5.13).

Since the convergence of (7.10) and (7.11) should imply convergence of both sides of the equation in Fig. 10 for fixed k,p, $k>p$ one finds after a brief calculation that

$$\begin{aligned}&\frac{1}{2!}\sum_{h>k}^{N}\big(\mathcal{L}_p\mathcal{E}_{p+1}\ldots\mathcal{E}_k\,-\,\mathcal{E}_{p+1}\ldots\mathcal{E}_k\mathcal{L}_k\big)\cdot\\&\cdot\mathcal{E}_{k+1}\ldots\mathcal{E}_{h-1}\mathcal{E}_h^T(\mathcal{E}_{>h}(V_1),\mathcal{E}_{>h}(V_1))\end{aligned}\tag{7.12}$$

should admit a limit as $N\to\infty$. A simpl way to enforce such property is, of course, to require that for all $p<k$

$$\mathcal{L}_p\mathcal{E}_{p+1}\ldots\mathcal{E}_k\,\equiv\,\mathcal{E}_{p+1}\ldots\mathcal{E}_k\mathcal{L}_k.\tag{7.13}$$

This leads to the conclusion that one would like $\mathcal{L}_k$ to be definied so that (7.13) holds. Then, proceeding heuristically, note that the limits of (7.11) as $N\to\infty$ exist for all fixed k if they exist for just one k, as the above argument implies. One can thus determine $V_{2,\alpha}$ and $\mathcal{L}_k$ by imposing the existence of the limit as $N\to\infty$ for $k=-1$ and, at the same time, imposing that LL_k make (7.10) convergent as $N\to\infty$.

For instance one can require that

$$\begin{aligned}&(\mathcal{E}_0\ldots\mathcal{E}_NV_{2,N})(\varphi^{(-1)})+\frac{1}{2!}\sum_{h=0}^{N}\cdot\\&\cdot\mathcal{L}_{-1}\mathcal{E}_0\ldots\mathcal{E}_{h-1}\mathcal{E}_h^T\big(\mathcal{E}_{>h}(V_1),\mathcal{E}_{>h}(V_1)\big)(\varphi^{(-1)})=0\end{aligned}\tag{7.14}$$

To continue in reat generality suppose that there isa way of defining the operation $\mathcal{L}_k$ verifying (7.13) and (7.14) and making (7.10) convergent: its existence, or nonexistence, will turn out to be a very easy question in the concrete nodels that we shall examine. Once such a sequence of operations $\mathcal{L}_k$ is found one can produce a new sequence with the same property by setting

$$\begin{aligned}\mathcal{L}'_kF=&\mathcal{L}_kF+\sum_{\alpha=1}^{t}\Big[\sum_{|\mathbf{m}|=2}\widetilde{l}^{(\alpha)}(\mathbf{m})\boldsymbol{\lambda}^{\mathbf{m}}\Big]\cdot\\&\cdot\int v_k^{(\alpha)}(\varphi_\xi^{(\le k)},\partial\varphi_\xi^{(\le k)}),\end{aligned}\tag{7.15}$$

where the coefficients $\widetilde{l}^{(\alpha)}(\mathbf{m})$ are arbitrarily chosen: note that $\mathcal{L}'_k$ verifies (7.13) if $\mathcal{L}_k$ does, because of (4.2).

The remarkable and interesting fact to be pointed out is that if the initial interaction V_1 is changed to $V=V_1+V_{2,N}$ there are very simple graphical rules that allow one to compute the effective interaction generated by $V=V_1+V_{2,N}$ to any order in terms of new types of trees that we call *partially dressed trees*.

The idea of defining such trees comes from computing $V_{2,N}$ defined by equation (7.14) regarded as a (trivial) linear equation for the t coefficients [see (7.4)]

$\mu_2^{(\alpha)} = \sum_{|\mathbf{m}|=2} l_N^{(\alpha)}(\mathbf{m})\boldsymbol{\lambda}^{\mathbf{m}}$ in $V_{2,N}(\varphi^{(-1)})$. Using (4.2) and (7.13) we find that (7.14) becomes

$$\begin{aligned}
0 =& \sum_{\alpha=1}^{t}\Big[\sum_{|\mathbf{m}|=2} l_N^{(\alpha)}(\mathbf{m})\boldsymbol{\lambda}^{\mathbf{m}}\Big]\int v_{-1}^{(\alpha)}(\varphi_\xi^{(-1)},\partial\varphi_\xi^{(-1)})\,d\xi+\\
&+\frac{1}{2!}\sum_{h>-1}^{N}\mathcal{L}_{-1}\mathcal{E}_0\cdots\mathcal{E}_{k-1}\mathcal{E}_h^T(\mathcal{E}_{>h}(V_1),\mathcal{E}_{>h}(V_1)) =\\
=&\sum_{\alpha=1}^{t}\Big[\sum_{|\mathbf{m}|=2} l_N^{(\alpha)}(\mathbf{m})\boldsymbol{\lambda}^{\mathbf{m}}\Big]\int v_{-1}^{(\alpha)}(\varphi_\xi^{(-1)},\partial\varphi_\xi^{(-1)})\,d\xi+\\
&+\frac{1}{2!}\sum_{h>-1}^{k}\mathcal{L}_{-1}\mathcal{E}_0\cdots\mathcal{E}_{k-1}\mathcal{E}_h^T(\mathcal{E}_{>h}(V_1),\mathcal{E}_{>h}(V_1))+\\
&+\frac{1}{2!}\sum_{h>k}^{N}\mathcal{E}_0\cdots\mathcal{E}_k\mathcal{L}_k\mathcal{E}_h^T(\mathcal{E}_{>h}(V_1),\mathcal{E}_{>h}(V_1))
\end{aligned}\tag{7.16}$$

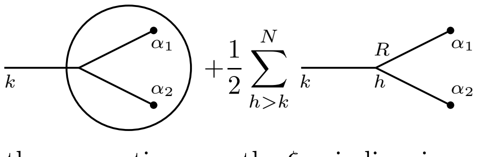
By (7.14) the second term in the *r.h.s.* is $-\mathcal{E}_0\cdots\mathcal{E}_k(V_{2,k})$: this implies, again by (4.2), the following important identity resting on the definitions above

$$\begin{aligned}
&\mathcal{E}_{k+1}\cdots\mathcal{E}_N V_{2,N}\equiv \qquad\qquad (7.17)\\
&=\sum_{\alpha=1}^{t}\Big[\sum_{|\mathbf{m}|=2} l_N^{(\alpha)}(\mathbf{m})\boldsymbol{\lambda}^{\mathbf{m}}\Big]\int v_{-1}^{(\alpha)}(\varphi_\xi^{(<k)},\partial\varphi_\xi^{(<k)})\,d\xi\equiv\\
&\equiv V_{2,k}-\frac{1}{2!}\sum_{h>k}^{N}\mathcal{L}_k\mathcal{E}_{k+1}\cdots\mathcal{E}_{h-1}\mathcal{E}_h^T(\mathcal{E}_{>h}(V_1),\mathcal{E}_{>h}(V_1)).
\end{aligned}$$

The inserting (7.17) in (7.9) one finds that (7.9) equals

$$V_{2,k}+\frac{1}{2!}\sum_{h>k}(1-\mathcal{L}_k)\mathcal{E}_{k+1}\cdots\mathcal{E}_{h-1}\mathcal{E}_h^T(\mathcal{E}_{>h}(V_1),\mathcal{E}_{>h}(V_1))\tag{7.18}$$

The relation (7.18) together with the graphical representation in Fig. 9 suggests representing (7.18) as

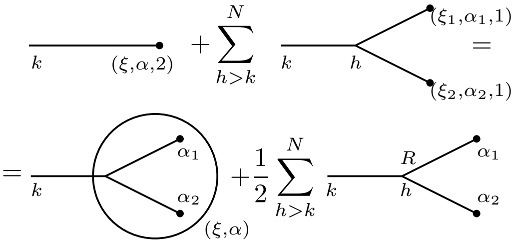


(11)

where the summation over the ξ,α indices is understood; the first graph represents symbolically

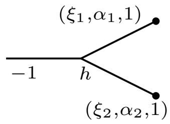
$$\Big[\sum_{|\mathbf{m}|=2} l_k^{(\alpha)}(\mathbf{m})\boldsymbol{\lambda}^{\mathbf{m}}\Big]\int v_{-1}^{(\alpha)}(\varphi_\xi^{(\le)},\partial\varphi_\xi^{(\le)})\tag{7.19}$$

while the second graph represents symbolicaly the second term in (7.18). Hence we have the graphical identity



(12)

where in the *r.h.s.* two new indices α_1,α_2 appear inside a *frame*, reminding us that $V_{2,k}$ is naturally defined as a sum of t^2 terms indxed by α_1,α_2 and the symbols in Fig. 13 allow us to identify them

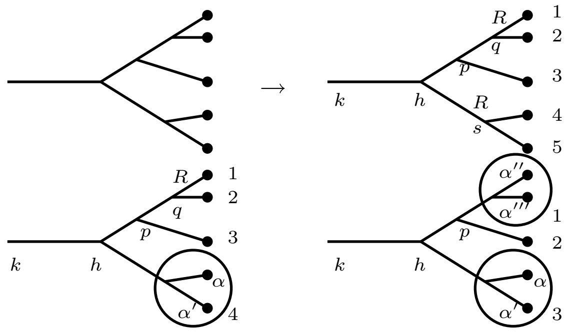
$$\mathcal{E}_0\cdots\mathcal{E}_k(V_{2,k}) = \frac{1}{2}\sum_{\alpha_1,\alpha_2}\sum_{h=0}^{k}\mathcal{L}_{-1}$$ 

(13)

To explain how to compute the higher order effective potentials generated by $V = V_1+V_{2,N}$ the identity in Fig. 12 above suggests introducing the notion of decorated trees *dressed to second order*.

These are objects constructed from an ordinary with no appended labels: first, one considers all the vertices out of which bifurcate exactly two branches ending in end points and one either appends a decorating index R or one encloses the bertex together with the branches emerging from it into a frame; second, to each framed end point one appends an index $\alpha = 1,\ldots,t$; furthermore to each frame and to each unframed endpoint a pair $\xi\in\mathbb{R}^d$ and $\alpha = 1,\ldots,t$ is appended; finally, all the unframed vertices v receive a frequency index h_v.

For instance Fig. 14 illustrates three trees that can arise by dressing to second order an unlabeled tree (out of four possibilities)



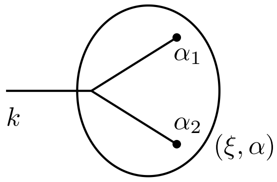
(14)

here the labels $i = 1,2,\ldots$ stand for (ξ_i,α_i).

Given a partially dressed tree γ, dressed to second order, the $V(\gamma)$ will be defined so that

$$V^{(k)} = \int d\boldsymbol{\xi}\sum_{\boldsymbol{\alpha}}\sum_{\gamma:\,k(\gamma)=k}\frac{V(\gamma)}{n(\gamma)}\tag{7.20}$$

where the rule to compute $V(\gamma)$ is simply the same as the one used so far except that the final lines ending in a frame, of the form



(14a)

have to be interpreted as representing the contributions to $V_{2,k}$ descried in connection with Fig. 13.

Also the R over a vertex has to be interpreted as saying that the rule to combine two V_1's in the computation of the truncated expectations $\mathcal{E}^T(\mathcal{E}_{>k}(V_1),\mathcal{E}_{>k}(V_1))$ has to be modified and produces, instead, the term in square brackets of (7.18).

The factor $n(\gamma)$ in (7.20) is now defined as identical to the combinatorial factor $n(\overline{\gamma})$ of the tree $\overline{\gamma}$ obtained from γ by stripping it of all its frames and their contents as well as of all its α decorations.

The above discussion is rather long but conceptually simple; however it has the advantage of suggesting the procedure for the construction of higher order "countert-erms" and for describing the results of their presence in the effective potentials.

viii. Counterterms, effective interaction, and renormalization in a graphical representation (arbitrary order)

The discussion of the preceding section can be naturally extended to provide an algorithm to build $V_{3,N}$, $V_{4,N},\ldots$, i.e. the formal series (7.1).

Again if the reader finds the discussion below too abstract for a first readinbg he can compare the abstract steps described here with the corresposnding ones worked out in Sec. 18 for the φ^4 model.

The basic objects are the *dressed trees* and the trees *dressed to order p*.

A tree dressed to order p will be an object obtained by considering a tree with no labels appended on it and

(1) To each end point append an index $\alpha\in(1,\ldots,t)$.

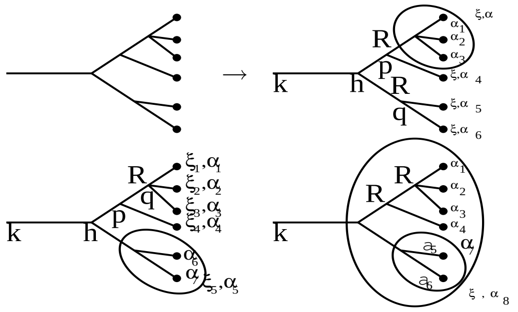
(2) To each vertex different from te root and of order $\le p$ (i.e. followed eventually, though not necessarily immediately, by $\le p$ end points) append an index R or, alternatively enclose it in a "frame" together with the part of the tree following it, excluding the preceding vertices (our convention is that the trees are oriented from the root towards the end points).

(3) Append to each frame an index $\alpha\in(1,\ldots,t)$.

(4) Append to each *outer frame* (note that, in fact, same frames may be inside others) and to each "unframed" end point an index $\xi\in\mathbb{R}^d$.

(5) Append to the unframed vertices a frequency index, increasing along the tree.

Fig. 15 provides a few examples of partial dressing of an unlabeled tree



(15)

where the first is a tree dressed to order 4 (it would be 6 if an extra label R was added on the vertex with frequency h), the second to order 3, the third to order 6. The above notion is a natural extension to $p\ge3$ of the $p=2$ case met in Sec. 7.

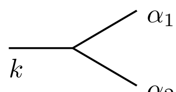
To each partially dressed tree γ one associates a function $V(\gamma)$ so that

$$V^{(k)}=\int\sum_{\boldsymbol{\alpha}}d\boldsymbol{\xi}\sum_{\substack{\gamma:\,k(\gamma)=k\\ \gamma\ dressed\ to\ order\ p}}\frac{V(\gamma)}{n(\gamma)}\tag{8.1}$$

would be the effective potential for $\varphi^{(\le k)}$ obtained starting from $V_1+V_{2,N}+\ldots+V_{p,N}$.

The definition of $V_{3,N},\ldots$ is inductive and so built that the last statement holds. Having already constructed $V_{2,N}$ in Sec. 7, one has to explain how $V_{p+1,N}$ is obtaiend from $V_1,V_{2,N},\ldots,V_{p,N}$ so that (8.1) holds once $V(\gamma)$ is apppropriately defined.

Call γ_0 a shape of a degree two tree



and call $\mathcal{L}_k^{\gamma_0)}$ the sequence of operations introduced in Sec. 7 and called there simply $\mathcal{L}_k$.

In general one looks for a sequence $\mathcal{L}_k^{(\sigma)}$ of operations indexed by the shapes σ of the trees dressed up to order p equal to the degree of σ minus 1 (a *shape* of a tree γ dressed to order p is the tree obtained by stripping γ of all the frequency labels and of all the ξ labels, leaving only the frames, the R labels and the α labels.

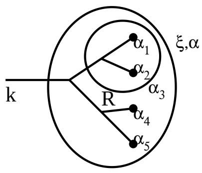
The operation $\mathcal{L}_k^{(\sigma)}$ will be subject to the following requirements (see Sec. 7).

(i) $\mathcal{L}_k^{(\sigma)}$ acts on certain functions of the field $\varphi^{(\le k)}$ and has range in the interaction space $\mathcal{I}_k$. Also if F is in the domain of $\mathcal{L}_k^{(\sigma)}$ then $\mathcal{E}_{q+1}\ldots\mathcal{E}_kF$ is in the domain of $\mathcal{L}_q^{(\sigma)}$ for $q<k$.
(ii) The following extension of (7.13) holds:

$$\mathcal{L}_k^{(\sigma)}\mathcal{E}_k\cdots\mathcal{E}_h\equiv\mathcal{E}_k\cdots\mathcal{E}_h\mathcal{L}_h^{(\sigma)}\tag{8.2}$$

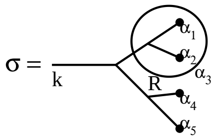
To evaluate the function $V(\gamma)$ associated with a partially dressed tree one will have to interpret a branch of the tree emerging from a vertex with frequency label k and ending in a frame containing a shape σ and carrying frame labels

ξ, α [see conditions (3) and (4) above] as representing a function which,if integrated over ξ, is in $\mathcal{I}_k$. For instance



(16)

encloses a shape σ



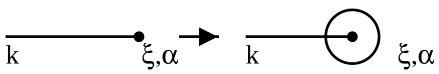
and counts in the evaluation of $V(\gamma)$ as

$$l_{N,\sigma}^{(\alpha)}(k)\, \boldsymbol{\lambda}^\sigma\, v_k^{(\alpha)}(\varphi_\xi^{(\le k)}, \partial\varphi_\xi^{(\le k)}), \tag{8.3}$$

where $l_{N,\sigma}^{(\alpha)}(k)$ are certain coefficients, that we shall call *form factors of the shape* σ, to be defined later and $\boldsymbol{\lambda}^{(\sigma)} \stackrel{def}{=} \prod \lambda^{\overline{\alpha}}$ with the product ranging over the indices appended to the end points of σ (in Fig. 16 they are $\alpha_1, \alpha_2, \alpha_4, \alpha_5$).

In other words once the coefficiants in (8.3) are defined and the meaning of R is specified the meaning of $V(\gamma)$ is essentially the same as would be attributed to a decorated tree (with decorations which are more complicated as they can be framed shapes of trees).

For uniformity of notations it is convenient, in this section, to consider the unframed end points of a partially dressed tree as framed end points containg the "trvial shape", ie

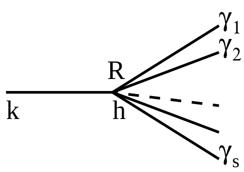


(17)

In this way a partially dressed tree γ can be regarded as always ending in *endframes* containing tree shapes; the name "end point" will be reserved for the end points of the tree obtained from γ by deleting all the frames.

The meaning of the R superscripts, as well as the construction of the coefficients $l_{N,\sigma}^{(\alpha)}(k)$ and of the countert-erms $V_{p,N}$ is described in terms of the operations $\mathcal{L}_k^{(\sigma)}$.

Let γ be a tree dressed to order $p+1$ and with degree $p+1$. Suppose that its first nontrivial vertex v_0 carries a superscript R and is the origin of an s–fold bifurcation into s dressed trees; suppose that the frequency index of v_0 is h; the situation is described in Fig. 18:



(18)

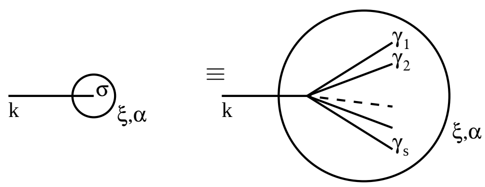
Then if σ is the shape of γ one interprets Fig. 17 as representing

$$V(\gamma) = \mathcal{E}_{k+1} \cdots \mathcal{E}_{h-1} \cdot (1 - \mathcal{L}_{h-1}^{(\sigma)}) \cdot \\ \cdot \mathcal{E}_h^T(V(\gamma_1), \ldots, V(\gamma_s)) \tag{8.4}$$

where $\mathcal{L}_h^{(\sigma)}$ has to be defined so that it verifies the requirements (i) and (ii) above and so that the summation of (8.4) over all trees γ with the same shape σ and root frequency k is ultraviolet finite (i.e. it has a limit whne $N \to \infty$ if $\varphi^{(\le k)}$ is smooth).

In the present general context one cannot discuss the existence or nonexistence of such $\mathcal{L}_h^{(\sigma)}$, although in each model considered in the following sections this will be a very easy problem; here, tocontinue, asume that at least one such $\mathcal{L}_h^{(\sigma)}$ exists. Of course as already remarked the $\mathcal{L}_h^{(\sigma)}$ operation , which basically isolates the "divergent" part of (8.4), will not be uniquely determined, if existing at all.

This completes the definition of the meaning of the R superscripts and the nextstep is to define the coefficients $l_{N,\sigma}(k)$ in (8.3). This will be done via the following prescription. Consider the tree shape σ of degree $p+1$ dressed to order p and enclosed in a frame attached at frequency k:

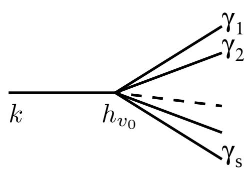


(19)

and assume that σ bifurcates at its first framed vertex v_0 into s completely dressed shapes $\gamma_1, \ldots, \gamma_s$: a *completely dressed* tree is one dressed at least to an order equal to its degree.

As said above, the framed shape in Fig. 18 represents a function of the field $\varphi^{(\le k)}$ of the form (8.3): to define it we follow, in a natural sense, the procedure of the preceding section, as described below.

Delete the outer frame enclosing σ and insert frequency indices at all the unframed vertices of σ as well as pairs ξ, α at all the new external outer frames (formerly internal only to the outer frame and to no others); the root of σ is given the frequency -1 and the indices ξ, α attached to the deleted frame are also deleted (compare Fig. 20 below with Fig. 18):



(20)

(no superscript, R or other,can be above v_0 because σ was supposed dressed only to order p and of degree $p+1$).

Since one is supposing inductively that $V(\gamma_1), \ldots, V(\gamma_s)$ are already defined (being trees of degree $\le p$), one can evaluate the tree in Fig. 20 by giving the usual interpretation of truncated expectation to the vertex v_0

which carries no superscript R. Then one can define the coefficients $l_{N,\sigma}(k)$ in (8.3) in terms of the value of the tree in Fig. 20 (see also Fig. 18) by setting:

$$\begin{aligned}&\sum_{\alpha=1}^{t}\int \boldsymbol{\lambda}^{(\sigma)} l_{N,\sigma}(k) v_{-1}^{(\alpha)}(\varphi_\xi^{(-1)},\partial\varphi_\xi^{(-1)})\,d\xi =\\ &=\int d\boldsymbol{\xi}\sum_{h_{v_0}=0}^{k}\sum\frac{1}{n(\gamma)}\mathcal{L}_{-1}^{(\sigma)}\Big(\mathcal{E}_0\ldots\mathcal{E}_{h_{v_0}-1}\cdot\\ &\cdot\mathcal{E}^T_{h_{v_0}}(V(\gamma_1),\ldots,V(\gamma_k))\Big)\end{aligned}\tag{8.5}$$

where the second sum runs over the frequency assignments to the unframed vertices (if any) of the tree γ.

Finally, again in analogy with the second order case, the *counterterms* $V_{p,N}$ of order p will be sums of contributions $V_{p,N,\sigma}$ coming from tree shapes σ of degree p

$$V_{p,N,\sigma}=\int \boldsymbol{\lambda}^{\sigma}\sum_{\alpha=1}^{t} l_{N,\sigma}^{(\alpha)}(N)\, v_N^{(\alpha)}(\varphi_\xi^{(\le N)},\partial\varphi_\xi^{(\le N)})\,d\xi \tag{8.6}$$

where $\boldsymbol{\lambda}^{\sigma}$ has the meaning described after (8.3).

Proceeding exactly as in Sec. 7, one proves that by using the above rules to interpret Fig. 18 and 19 one determines, via (8.1), the effective potential corresponding to $V_1+V_{2,N}+\cdots+V_{p,N}$ simply by interpreting a partially dressed tree of arbitrary degree as computed by using the above rules starting from the highest vetices and interpreting a lower vertex with no R superscripts as simply representing the truncated expectations of the functionals defined by the s–ple of trees branching out of the vertex.

The proof is, once more, by induction and is left to the reader with the warning that the definitions above have been conceived with the aim of making possible this inductive proof.

It remains to define $\mathcal{L}_k^{(\sigma)}$ in a concrete way in each model (if possible).

As already remarked, the ambiguity in the coefficients of the counterterms [and therefore in the definition of the operations $\mathcal{L}_k^{(\sigma)}$ of identification of the divergent parts] has its deep origin in the trivial fact that if

$$\boldsymbol{\lambda}=\boldsymbol{\lambda}'+\mathbf{L}(\boldsymbol{\lambda}') \tag{8.7}$$

and $\mathbf{L}$ is analytic near the origin with a second order zero, then inserting (8.7) into (7.1) and rearrangin that formal power series in $\boldsymbol{\lambda}$ into a formal power series in $\boldsymbol{\lambda}'$ one necessarily obtains another power series which will enjoy the same proerties as the former one as far as the stability as $N\to\infty$ is concerned.

The situation is very much reminiscent of the stae of perturbation theory in classical mechanics where there are formalpower series, for various objects, which are ambiguously defined for trivial reaons and even "divergent" and which can be "renormalized" by suitable prescriptions, (Gallavotti, 1983a)[2]

An interaction $\mathcal{I}$ for which the operations $\mathcal{L}_k^{(\sigma)}$ can be taken identically zero for trees with, for some $p_0\ge 1$, more than p_0 end points, disregarding the frames, will be called *super-renormalizable* and sometimes we call *bare degree* the number of endpoints in a tree once the frames are deleted.

The basic idea of the above construction of the coutertserms is from (Zimmermann, 1969), where the notion of "forest", here called "tree", is introduced; however here the notion of tree is independent of the notion of Feynman graph, not yet introduced, while in the literature the forests are always asociated with given Feynman graphs. It seems conceptually sinplifying and practically advantageous to be able to introduce the notion of forest without any reference to Feynman graphs.

That perturbation theory can be perhaps done in a neater way by avoiding as much as possible the use of Feynman graphs has been clearly pointed put in (Polchinskii, 1984), who presents a method quite similar to the one introduced here to deal with perturbation theory by using multiscale properties and effective potentials working in momentum space (here configuration space is used instead). The method outlined here has been used in various super-renormalizable cases already in (Benfatto *et al.*, 1978, 1980a,b; Gallavotti, 1978, 1979a,b). In the latter papers, however, the super-renormalizability masks the power of the method [which becomes clearer in (Benfatto *et al.*, 1978, 1980a; Nicolò, 1983), even though the theories treated are still super-renormalizable].

ix. Resummations, form factors and beta function

Before starting the "real work", i.e. the analysis of concrete models, there are still quite a few remarkable abstract considerations that can be made.

If an interaction $\mathcal{I}$ is super-renormalizable, the renormalization leads only to a slightly more complex structure of the trees (which have to be dressed up to a finite order p_0 if the subtraction operations are chosen to be zero when the degree of the trees is larger than the convergence "treshold" p_0—see Sec. 8) and there is little to discuss about them.

But if $\mathcal{I}$ is only renormalizable or even if it is super-renormalizable and yet one chooses to define $\mathcal{L}_k^{(\sigma)}$ to be non zero for σ's of large degree, i.e. if one "oversubtracts", thwe graphical representation of $V^{(k)}$ is enormously more complex and one wishes to simplify it as much as possible by collecting together as many terms a s possible without losing control of what may be going on.

[2] Although the main result of this reference has been previously obtained in (Rüssmann, 1967) the connection with renormalization theory is somewhat new and relevant as a reference here.

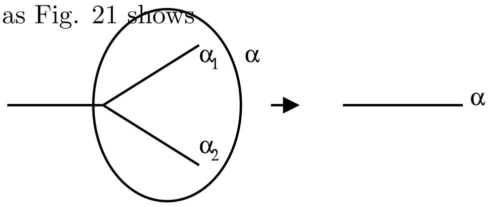
The trouble is that one would naturally like to collect togheter infinitely many trees but, as will become clear, there are no chances that the resulting series will converge in a naive sense. Nevertheless it is possible to devise a simple "summation rule" permitting us to give a meaning to important resummations.

A concrete example on the abstract and general discussion below is in Sec. 20, where the reader who finds too abstract, on first reading, the contents of this section can see the same ideas worked out concretely in the case of φ^4.

The idea leading to such developments can be best illustrated via an example in which it is even mathematically rigorous: the well known *resummation of the leading divergences*—see ('t Hooft, 1974, 1982a,b, 1983, 1984; Landau, 1955; Landau and Pomeranchuck, 1955; Rivasseau, 1985).

One defines a *pruning operation* on the dressed trees, consisting in isolating the final bifurcation of a tree γ which have the form of Fig. 21, called a "most divergent branch" or a "most divergent end–frame" (for reasons that will become clear later)..

The pruning will first delete the "most divergent end-frames" as Fig. 21 shows

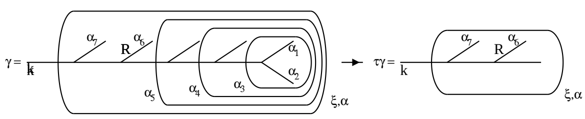


(21)

but this will not be all, because after the deletion of the most divergent end-frames of γ new most divergent frames may appear in what is left of γ: then the pruning will be pursued until no most divergent branches are left. This defines a *pruning mapping* $\tau : \gamma \to \tau\gamma$. The idea is then to define, if γ is a tree with no most divergent branches (i.e. $\tau\gamma \equiv \gamma$),

$$V^R(\gamma) = \sum_{\gamma' : \tau\gamma' = \gamma} V(\gamma'); \tag{9.1}$$

In (9.1) the sum runs over infinitely many trees (*even if the ultraviolet cut–off N is finite*). For instance

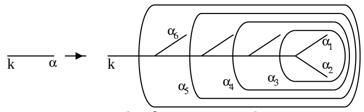


(22)

However the result of the resummation in (9.1), if convergent in any sense, cannot lead to anything other than the conclusion that $V^R(\gamma)$ is evaluated by "slightly modifying" the rules to build $V(\gamma)$: this follows from remarking that the sum (9.1) leads to a change in the meaning of the lines reaching the end points with index ξ, α of a pruned tree γ (i.e. a tree such that $\tau\gamma = \gamma$) and representing, according to the set rules, the function

$$\lambda^{(\alpha)} v^{(\alpha)}(\varphi_\xi^{(\le k)}, \partial\varphi_\xi^{(\le k)}). \tag{9.2}$$

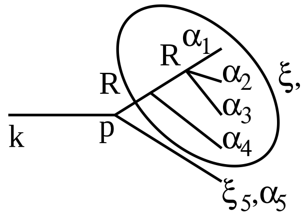
The modification is explained as follows. Consider a tree γ which is pruned: $\tau\gamma = \gamma$. Then all trees γ' with $\tau\gamma' = \gamma$ are obtained from γ simply by considering each end point of γ with index α and growing on it a tree of arbitrary size with simple bifurcations in two branches at each vertex and then drawing a frame around every vertex, as in Fig. 23



(23)

attributing to each frame indices α_j.

An end point of γ can be either "framed", bearing an index α (and no index ξ), or it can be "free", bearing a pair of indices (ξ, α), in Fig. 24 end point of different type are marked on an example of a pruned tree:



(24)

they can be, say, the end points with labels α_1 or ξ_5, α_5.

Consider first the case of an end point which is free and attached to a vertex v of frequency index p (by a branch).

We shall now assume, throughout this section and in the sections following Sec. 16 (where applications of the following considerations are presented), that for a general σ (not necessarily a most divergent one) the operations $\mathcal{L}_k^{(\sigma)}$ depend only on what remains of σ after deleting all the frames that it may contain as well as their contents. This property is very convenient and natural, but it has not been assumed since the beginning in order to develop a flexible enough formalism to permit us to study simoultaneously super-renormalizable as well as just renormalizble cases.

A frame with index α attached to a vertex of frequency p and enclosing a shape σ (whether most divergent, as of interest here, or not) represents, by the general theory of Sec. 8 [see (8.5)], the function

$$\boldsymbol{\lambda}^{(\sigma)}\, l_{N,\sigma}^{(\alpha)} v_p^{(\alpha)}(\varphi_\xi^{(\le p)}, \partial\varphi_\xi^{(\le p)}). \tag{9.3}$$

Hence it appears that by summing over all the γ', with $\tau\gamma' = \gamma$ and obtained by adding to each free vertex of γ any framed most–divegent shape σ just means interpreting the end branches which are like p ——— ξ,α as meaning

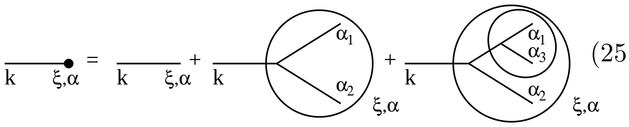
$$\begin{aligned}\Big[\sum_\sigma \boldsymbol{\lambda}^\sigma l_{N,\sigma}^{(\alpha)}(p)\Big]\, v^{(\alpha)}(\varphi_\xi^{(\le p)}, \partial\varphi_\xi^{(\le p)}) \stackrel{def}{=} \\ \stackrel{def}{=} \lambda^{(\alpha)}(p)\, v_p^{(\alpha)}(\varphi_\xi^{(\le p)}, \partial\varphi_\xi^{(\le p)})\end{aligned} \tag{9.4}$$

where the sum runs over all shapes σ that can be attached to the end point and are most divergent.

Similarly consider a framed end point of γ with some index α (like the end point with index α_1 in Fig. 24). In this case the addition of a most divergent tree shape enclosed in a frame and attached to the considered end point just modifies the meaning of the frames of γ as follows. Recall that the form factor $\boldsymbol{\lambda}^{\overline{\sigma}} l_{N,\overline{\sigma}}(p)$ corresponding to a framed shape $\overline{\sigma}$ is evaluated (see Sec. 8, Fig. 19) recursively by, eventually, reducing oneself to the evaluation of the function representing the simple trees q———ξ,α corresponding to the end points of $\overline{\sigma}$ (once, in the evaluation process, they become unframed) and having the meaning of $\lambda^{(\alpha)} v_q^{(\alpha)}$. If to each end point of $\overline{\sigma}$ is added a most divergent framed shape σ and one performs the summation over all possible such σ's it is clear that one simply gets the same result that would be obtained by interpreting q———ξ,α as meaning again (9.4) (with coefficient $\lambda^{(\alpha)}(q)$).

In other words one may consider, in computing the effective potentials, only the trees γ such that $\gamma \equiv \tau\gamma$ provided one interprets the end points of γ attached to a vertex with frequency index p as having the meaning (9.4): the meaning has to be kept, for consistency, even when the endpoints of γ are inside frames (as in Fig. 24). This means that when one computes the form factors for the framed parts of γ and, in doing so, eventually reduces oneself to the case of the tree p———ξ,α one interprets it as meaning (9.4) instead of simply $\lambda^{(\alpha)} v_p^{(\alpha)}(\varphi_\xi^{(\le p)}, \partial\varphi_\xi^{(\le p)})$.

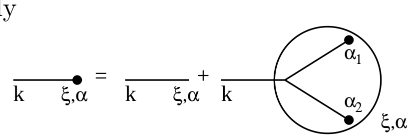
If we give to the end points of a tree $\gamma = \tau\gamma$ the new interpretation and if we represent this graphically by using "heavy dots" at the end points of γ it is clear from the above discussion that the form factors $\lambda^{(\alpha)}(k)$ satisfy the graphical relation of Fig. 25,



(25)

where the left-hand side (*l.h.s.*) can be taken as a symbolic representation of (9.4) and where in the *r.h.s.* , in each term, a summation over the indices α_j is understood.

The equation represented in Fig. 25 can be written, pictorially



(26)

which is actually a simple recursive relation for the form factors $\boldsymbol{\lambda}(k)$:: its iterative solution leads to expressing $\boldsymbol{\lambda}(k)$ as a power series in $\boldsymbol{\lambda}$. The latter power series, once substituted in the $V^R(\gamma)$ defined, as explained above, by interpreting the end points of $|g$ as bearing a heavy dot and with the meaning that in the evaluation of $V^R(\gamma)$ a line k———ξ,α has to be interpreted as in (9.4), yields the representation

$$V^{(k)} = \sum^{*} \frac{V^R(\gamma)}{n(\gamma)} \tag{9.5}$$

where the sum runs over the trees $\gamma = \tau\gamma$ (i.e. over the pruned trees) only.

On the other hand it might happen that the relation in Fig. 26, regarded as an equation for the form factors admits true solutions, not just formal solutions in the form of power series generated by iterating it: then one can use this solution to define the *summation rule* that the sum (9.1) is "by definition" the expression $V^R(\gamma)$ computed with the same rules as $V(\gamma)$ but *restricted to trees* $\gamma = \tau\gamma$ *and interpreting the end points as bearing heavy dots*, which means that they must be interpreted as in the *r.h.s.* of (9.4), with $\boldsymbol{\lambda}(k)$ defined by the given solution of the equation represented by Fig. 26.

In other words the equation in Fig. 26 has two different well defined possible uses. One is to generate by iteration the various terms graphically represented in Fig. 25 [i.e. the formal power series for the form factors $\boldsymbol{\lambda}(k)$ in (9.4)]. The second is to provide a nonperturbative meaning to the sum of the series, in Fig. 25, for the form factors, i.e. a summation rule for the most divergent graphs. The first use is also quite interesting, being equivalent to the direct definition of the various trees in Fig. 25 described in Sec. 8; this is a conceptually simpler way to build the coefficients $\lambda^{(\alpha)}(k)$, although (as a consequence of the principle of conservation of difficulties) this does not really save any work if one wishes to perform a real calculation (the point being, as explicitly illustrated in the models considered later, that Fig. 26 can be converted into an analytic equation only at the price of doing all the calculations necessary to evaluate the trees in Fig. 25, i.e. the formal power series for the form factors).

Following the rules of Secs. 7 and 8 for the evaluation of the coefficients of the element of $\mathcal{I}_k$ associated with a frame (see Fig. 19), one gets for $0 \le k \le N$

$$\lambda^{(\alpha)}(k) = \lambda^{(\alpha)} + \sum_{h=0}^{k} \sum_{\alpha_1,\alpha_2=1}^{t} B^{(\alpha)}_{\alpha_1\alpha_2}(h)\lambda^{(\alpha_1)}(h)\lambda^{(\alpha_2)}(h) \tag{9.6}$$

where $B^{(\alpha)}_{\alpha_1\alpha_2}(h)\lambda^{(\alpha_1)}(h)\lambda^{(\alpha_2)}(h)$ is the coefficient of the integral $\int_\Lambda v_{-1}^{(\alpha)}(\varphi_\xi^{(-1)}, \partial\varphi_\xi^{(-1)})\, d\xi$ in

$$\begin{aligned} &-\frac{1}{2}\int \mathcal{L}_{-1}^{(\sigma_0)} d\xi\, d\eta\, \mathcal{E}_0 \cdots \mathcal{E}_{h-1} \cdot \\ &\cdot \mathcal{E}_h^T\big(v_h^{(\alpha_1)}(\varphi_\xi^{(\le h)}, \partial\varphi_\xi^{(\le h)}), v_h^{(\alpha_2)}(\varphi_\xi^{(\le h)}, \partial\varphi_\xi^{(\le h)})\big) \end{aligned} \tag{9.7}$$

The factor $\frac{1}{2}$ comes from the combinatorial factor associated with the tree shape $\sigma_0 =$ ———<. The coefficients B are manifestly N independent.

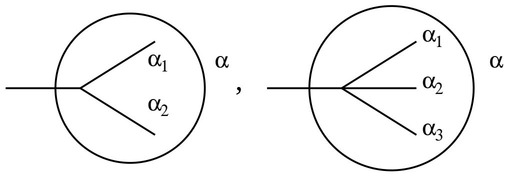
To proceed any further one needs explicit expressions for the B's: the ideal situation arises when the B's have a structure allowing one to conclude that, possibly adjusting the initial values $\lambda^{(\alpha)}(N)$ there is a solution to (9.6) such that

$$\gamma^{-\nu(\alpha)k}\lambda^{(\alpha)}(k)\xrightarrow[k\to\infty]{}0 \tag{9.8}$$

where $\nu(\alpha)$ is some *dimension* (in the concrete models treated later $\nu(\alpha)$ is naturally suggested as one shall expect that the form factors have a k dependence which goes exponentially at a rate characteristic of each form factor and, usually,the order-by-order behavior of the perturbative coefficients of the form factors is estimated to be much worse than the *a priori* guessed exponential).

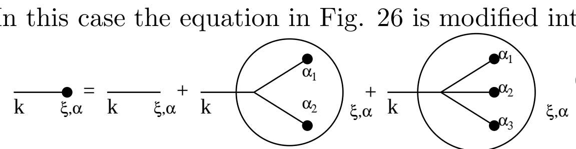
When this is the case, and this depends upon the interaction $\mathcal{I}$, the above simple resummation can produce a great gain in the expressions of $V(\gamma)$ and in their estimates because it may introduce a damping in the contributions from trees having in them bifurcations at too high frequencies. Furthermore such damping results as a consequence of summing, by a well defined summation rule, a series which might be divergent (as in fact it happens in the simplest application that we shall describe here).

The above method to build resummation rules can be extended to cover more complicated sets of trees by modifying the pruning operation. The latter can be extended by prescribing the pruning of a given set of shapes; for instance one can prune all the framed end branches like



(27)

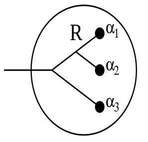
In this case the equation in Fig. 26 is modified into



(28)

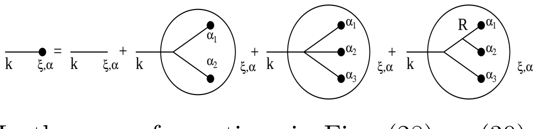
where a sum over the α_j's is understood in each term.

Or if one prunes out also the end branches like



(29)

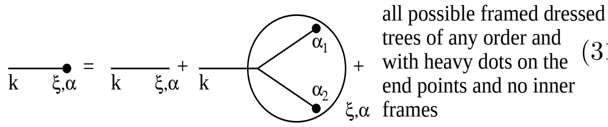
then the equation in Fig. 28 is replaced by



(30)

In the case of equations in Fig. (28) or (30) the Eq. (9.6) is replaced by a similar one in which the *r.h.s.* also contains cubic terms; their coefficients are still N dependent. However the N dependence is implicit through the fact that the frequencies are bound to vary between 0 and N. It should also be clear that, if the cut-off is N, only trees with at most $N+1$ vertices between any two successive frames are possible (and therefore can be considered in the resummations; for instance the resummation in Fig. 30 makes sense only if $N\ge 1$, while the other two are meningful even for $N=0$). The N dependence of the B's will not be explicitly marked except when necessary in Secs. 20 and22.

The ultimate resummation can be associated with the "total pruning" operation whereby all frames are pruned and oine is left just with dressed trees *without frames* in the formula corresponding to (9.1). The graphical representation of the latter resummation is



(31)

where again summations over α_j are understood. The equation in Fig. 31 becomes, explicitly, for $k\le N$,

$$\begin{aligned}\lambda^{(\alpha)}(k)=&\lambda^{(\alpha)}+\sum_{h=0}^{k}\sum_{n=2}^{\infty}\sum_{\substack{h_i\ge h\\ \alpha_1,\dots,\alpha_n}}^{N} B^{(\alpha)}_{\alpha_1,\dots,\alpha_n}(h;h_1,\dots,h_n)\cdot\\ &\cdot\lambda^{(\alpha_1)}(h_1)\cdots\lambda^{(\alpha_n)}(h_n)\end{aligned} \tag{9.9}$$

where the coefficients B must be computed according to the rules of Secs 8—see Fig. 19—and are expressed as sums of the coefficients of

$$\lambda^{(\alpha_1)}\dots\lambda^{(\alpha_r)}\int v^{(\alpha)}_{-1}(\varphi^{(-1)}_\xi,\partial\varphi^{(-1)}_\xi)\,d\xi$$

in $\mathcal{L}_{-1}V^{(-1)}(\sigma)/n(\sigma)$, σ being one of the trees with r end points in Fig. 31 deprived of the first frame and bearing no R superscript on the first vertex v_0 and with frequencies $h_1,\dots,h_r$ appeded to the vertices out of which emerge the r end branches of σ.

By the assumption of renormalizability and of existence of the operations $\mathcal{L}_k$ it follows that the coefficients in Eq. (9.9) will be such that if the form factors $\lambda^{(\alpha_i)}(h_i)$ are replaced by constants $\lambda^{(\alpha_i)}$ then the summation *at r fixed* converges uniformly in N.

The problem is that, as it will appear in the concrete case of φ^4 theory, the sum over r is not well controlled.

Introduce the functionals $\mathcal{B}_N,\mathcal{B}$ acting on the space of sequences $\underline{\boldsymbol{\lambda}}$ of functions $\boldsymbol{\lambda}^{(\alpha)}(k)$ and defined formally as

$$\begin{aligned}(\mathcal{B}_N\underline{\boldsymbol{\lambda}})^{(\alpha)}(k)=&\sum_{\substack{h_i\ge h\\ \alpha_1,\dots,\alpha_n}}^{N} B^{(\alpha)}_{\alpha_1,\dots,\alpha_n}(h;h_1,\dots,h_n)\cdot\\ &\cdot\lambda^{(\alpha_1)}(h_1)\cdots\lambda^{(\alpha_n)}(h_n)\end{aligned} \tag{9.10}$$

and

$$\begin{aligned}(\mathcal{B}\underline{\boldsymbol{\lambda}})^{(\alpha)}(k)=&\sum_{\substack{h_i\ge h\\ \alpha_1,\dots,\alpha_n}}^{\infty} B^{(\alpha)}_{\alpha_1,\dots,\alpha_n}(h;h_1,\dots,h_n)\cdot\\ &\cdot\lambda^{(\alpha_1)}(h_1)\cdots\lambda^{(\alpha_n)}(h_n)\end{aligned} \tag{9.11}$$

and rewrite (9.9) as

$$\boldsymbol{\lambda}(k)=\boldsymbol{\lambda}+(\mathcal{B}_N\underline{\boldsymbol{\lambda}})(k),\qquad 0\le k\le N \tag{9.12}$$

The difference with the preceding resummation schemes is that the *r.h.s.* of (9.12) does not really make sense other than as a formal power series in $\boldsymbol{\lambda}^{(\alpha)}(k)$ because, as mentionel above, there is no control over the sums over r in (9.10) or (9.11).

Therefore already very interesting use of (9.12) is that it can produce, by a formal solution by iteration, a well defined power series in $\boldsymbol{\lambda}$ leading to a a formal power series expresssion of the form factors associated with the resummation. Furthermore as $N\to\infty$ the coefficients of a given order in such power series for the form factors solution of (9.11) converge to the corresponding coefficients of the formal power series obtained by iterating

$$\boldsymbol{\lambda}(k)=\boldsymbol{\lambda}+(\mathcal{B}\underline{\boldsymbol{\lambda}})(k). \tag{9.13}$$

The result of the above iscussion is quite nontrivial: through the knowledge of all the coefficients $B^{(\alpha)}_{\boldsymbol{\alpha}}(h;\mathbf{h})$ one can compute the form factors $\boldsymbol{\lambda}(k)$ to any desired order in the renormalized couplins $\boldsymbol{\lambda}$ and then reduce the computation of $V^{(k)}$ to the computation of $V(\gamma)$ for all "trivially dressed trees", i.e. for the trees with only R supercripts on the vertices and *no frames* at all, provided their end points are interpreted as meaning the *r.h.s.* of (9.4) with $\boldsymbol{\lambda}(k)$ being now a form factor defined by (9.12) to any order of perturbation expansion.

In other words the knowledge of the coefficients in Eqs. (9.12) and (9.13) allows one to reduce the calculation of $V^{(k)}$ essentially to the same calculations that would be necessary in absence of renormalization. The economy of thought gained in using this approach in computing pertubation theory coefficients is obvious. However it is worth stressing that, as already remarked, in prctice the calculation of the B coeefficients is exactly equivalent to the evaluation of the trees with frames (i.e. to renormalization); it is perhaps better to regard Eqs. (9.12) and (9.13) as a convenient way to organize perturbative calculations by separating the "true calculations" (corresponding to the trees without frames) from the form factors calculations.

Unfortunately, unlike the simple cases of the "moderate" resummations described by Fig. (26),(28) or (30) or, more generally, involving a finite number of "pruned shapes", no rigorous use of (9.12) and (9.13) can be made to prescribe resummation rules because no information is vailable on the nonperturbative meaning to be attached to the *r.h.s.* of (9.12) and (9.13): one can only say that if on the *r.h.s.* the second order "dominates" then $\boldsymbol{\lambda}(k)$ should behave for $k\to\infty$ in the same way as the $\boldsymbol{\lambda}(k)$ that would be obtained from the most divergent resummations [i.e. from (9.6)], justifying the name given to them.

Many triviality arguments for φ^4 are based on this assumption (domination of the most divergent graphs), and this point will be discussed in more detail in Secs. 19,20 and 22.

It is customary to write (9.12) and (9.13) as difference equations obtained by "writing them for k and $k+1$ and subtracting"

$$\begin{aligned}\boldsymbol{\lambda}(k)&=\boldsymbol{\lambda}(k+1)+(B_N\underline{\boldsymbol{\lambda}})(k)\qquad 0\le k+1\le N\\ \boldsymbol{\lambda}(k)&=\boldsymbol{\lambda}(k+1)+(B\underline{\boldsymbol{\lambda}})(k)\qquad 0\le k\end{aligned} \tag{9.14}$$

where $\boldsymbol{\lambda}(-1)\equiv\boldsymbol{\lambda}$, and

$$\begin{aligned}(B\underline{\boldsymbol{\lambda}})^{(\alpha)}(k)=&\sum_{r=2}^{\infty}\sum_{\substack{h_i\ge k+1\\ \alpha_1,\ldots,\alpha_r}}^{\infty}B^{(\alpha)}_{\alpha_1,\ldots,\alpha_r}(k+1;h_1,\ldots,h_n)\cdot\\ &\cdot\lambda^{(\alpha_1)}(h_1)\cdots\lambda^{(\alpha_n)}(h_n)\end{aligned} \tag{9.15}$$

while B_N is defined in the same way but only for $k+1\le N$ and with the sum over the h_i's subject to the constraint $N\ge h_i\ge k+1$, furthermore the coefficients $B^{(\alpha)}_{\alpha_1,\ldots,\alpha_r}(k+1;h_1,\ldots,h_n)$ are replaced by N-dependent coefficients $B^{(\alpha)}_{N;\alpha_1,\ldots,\alpha_r}(k+1;h_1,\ldots,h_n)$which converge to $B^{(\alpha)}_{\alpha_1,\ldots,\alpha_r}(k{+}1;h_1,\ldots,h_n)$. The operator B is basically the *beta function* [see (Callan, 1970, 1976; Symanzik, 1970, 1973)] which therefore can be used to simplify conceptually the perturbation theory in the sense explained above.

Usually Eqs. (9.14) acquire a more homogeneous form if written for other form factors trivially related to the ones just discussed by

$$\lambda^{(\alpha)}(k)\stackrel{def}{=}\gamma^{\nu(\alpha)k}\lambda^{(\alpha)}_k \tag{9.16}$$

where $\nu(\alpha)$ is a suitable *dimension* (this will be discussed in detail in the treatment of the concrete φ^4_4–model)

To conclude this section it is useful to point out that the constants $\boldsymbol{\lambda}(k)$ verifying the first of (9.14) can be naturally called *effective couling constants at frequeny* γ^k because they represent the trivial trees k ——● ξ,α in the same sense in which the renormalized couplings represent the trivial trees k —— ξ,α.

By definition it is true that $\boldsymbol{\lambda}(N)$ is precisely the *bare coupling* (7.1) and they are formal power series in the *renormalized couplings* $\boldsymbol{\lambda}(N)=\boldsymbol{\lambda}_N(\boldsymbol{\lambda})$. Note that it is the formal power series (for $k=N$) generated by the recursive solution of the first of (9.14) starting from the zero-th order approximation $\boldsymbol{\lambda}(k)=\boldsymbol{\lambda}$.

It is convenient to label the formal power series solution of(9.12) and (9.13) [or (9.14)] by the symbols $\boldsymbol{\lambda}(k;N)$, $k\le N$ or, respectively, $\boldsymbol{\lambda}(k;\infty)$. The bare couplings are, in this notation $\boldsymbol{\lambda}(N;N)$ and they should not be confused with $\boldsymbol{\lambda}(N;\infty)$; note also that while $\boldsymbol{\lambda}(k;\infty)$ can be (as it will be in the cases treated later) regularization independent, the form factors $\boldsymbol{\lambda}(k;N)$ may be strongly dependent on the regularization used.

The latter statement requires some more detailed explanations , since the use of a different regularization seems to yield results which just are not comparable with the ones coming from another regularization. Therefore to illustrate the above statement it is convenient to "compare" the results of the Pauli–Villars regularization at a given order n and the corresponding results for a radically different regularization, e.g. the lattice regularization (see Secs 1,2). The comparison of the two approaches can be made by thinking that the lattice free fields are also decomposed into a sum of independent fields associated with a hierarchy of scales γ^k, $k = 0, 1, \ldots$ via the identities (3.4) of order n by setting $\varepsilon(p) = 2\sum_{i=1}^d \big(1-\cos ap_i\big)/a^2$ rather than $\varepsilon(p) = p^2$.

Then one proceeds, exactly as in the Pauli–Villars case, to study the effective potentials for the fields $\varphi^{(\le k)}$. Their effective potentials will be described by Eq. (9.13) with the B coefficients depending on the cut–off a (here $N = \infty$ from the beginning, because one does not need $N < \infty$ for regularization purposes when one is assuming $a > 0$): such coefficients converge to the coefficients in (9.14) as $a \to 0$ term by term, but for $a > 0$ they depend on a and for large r [see (9.11)] their dependence on a itself is strong.

It is even conceivable that $\boldsymbol{\lambda}(k;\infty)$ could be defined as a nonperturbative solution of (9.14) or ((9.13)] while $\boldsymbol{\lambda}(k;N)$ could admit interesting nonperturbative solutions only for suitably chosen regularizations [because the terms of (9.12) are regularization dependent in the sense above, while those of (9.13) are not]. This question will be discussed in more detail in Sec. 22.

x. Schwinger functions and effective potentials

If f is a smooth test function consider the expectation EE_{int} with respect to the *interaction measure* $e^{V(\varphi^{\le N})}\prod_{j=1}^N P(d\varphi^{(j)})$ and let $\mathcal{E}_{(0,k)} \stackrel{def}{=} \mathcal{E}_0\mathcal{E}_1\cdots\mathcal{E}_k$ and $\varphi(f) = \int \varphi_\xi\, f(\xi)\, d\xi$. Then the following formal chain of identities establishes an example of a relation between effective potentials and Schwinger functions

$$
\begin{aligned}
&S(f;p) \stackrel{def}{=} \mathcal{E}^T_{int}(\varphi(f);p) \equiv \frac{\partial^p}{\partial\theta^p}\log\mathcal{E}_{int}(e^{\theta\varphi(f)})\big|_{\theta=0} \equiv\\
&\equiv \sum_{k_1,\ldots,k_p} \mathcal{E}^T_{int}(\varphi^{(k_1)}(f),\ldots,\varphi^{(k_p)}(f);1,\ldots,1) \equiv\\
&\equiv \sum_{q=1}^p \sum^{0,\infty}_{\substack{m_1,\ldots,m_q\ge 1\\ m_1+\ldots+m_q=p}} \sum_{k_1<\ldots<k_q} \frac{p!}{m_1!\ldots m_q!}\cdot\\
&\cdot\,\mathcal{E}^T_{int}(\varphi^{(k_1)}(f),\ldots,\varphi^{(k_q)}(f);m_1,\ldots,m_q) \equiv\\
&\equiv \sum\nolimits^* \frac{\partial^p}{\partial\theta_1^{m_1}..\theta_q^{m_q}}\log\frac{\mathcal{E}_{\ge 0}\Big(e^{\theta_1\varphi^{(k_1)}+\ldots+\theta_q\varphi^{(k_q)}}e^V\Big)}{\mathcal{E}_{\ge 0}(e^V)}\Big|_{\theta_i=0} \equiv\\
&\equiv \sum\nolimits^* \frac{\partial^p}{\partial\theta_1^{m_1}\ldots\partial\theta_q^{m_q}}\\
&\log\frac{\mathcal{E}_0\ldots\mathcal{E}_{k_q}\Big(e^{[\theta_1\varphi^{(k_1)}(f)+\ldots+\theta_q\varphi^{(k_q)}(f)]}e^{V^{(k_q)}}\Big)}{\mathcal{E}_0\ldots\mathcal{E}_{k_q}\Big(e^{V^{(k_q)}}\Big)}\Big|_{\theta_i=0} \equiv\\
&\equiv \sum\nolimits^* \sum_{s=0}^\infty \frac{1}{s!} \qquad (10.1)\\
&\mathcal{E}^T_{(0,k_q)}(\varphi^{(k_1)},\ldots,\varphi^{(k_q)}(f),V^{(k_q)};m_1,\ldots,m_q,s)
\end{aligned}
$$

where $\sum^*$ denotes the three summations in the third line together with the combinatorial factor following them.

In some sense the key step in (10.1) is the identity preceding the last where V is replaced by the effective potential.

The functions $S(f;p)$ are called the *truncated Schwinger functions* of order p for the interacting measure: they are trivially related to the nontruncated Schwinger functions of Sec. 4. The relevance of (10.1) is to show that the Schwinger functions can be expressed in terms of the effective potentials [and as it can be seen from (10.1) viceversa, at least formally].

Even though (10.1) might *a priori* present convergence problems it will be possible to check that, in fact, the *r.h.s.* of the series (10.1) will converge order by order in the perturbation series in powers of the renormalized coupling constants $\boldsymbol{\lambda}$: this will be so provided convergence problems do not arise already in the perturbative definitions of the effective potentials themselves and it will be so in various cases which will be encountered in this paper (more precisely in the polynomial theories).

Sometimes one wishes to study more complex "observables" like

$$\rho(f) = \int_\Lambda :\cos\alpha\varphi_\xi:\, f(\xi)\,d\xi \qquad (10.2)$$

through their average values, and the average values of their powers, with respect to the interaction measure.

A way to analyze such quantities via the effective potential technique, which in particular can also be applied to the Schwinger functions, is to add $\rho(f)$ to the interaction potential and to try to show that if $V = V_1 + V_{2,N} + V_{3,N} + \ldots$ yields a well defined ultraviolet stable effective potential so does $\varphi(f) + V$.

Examples of how this could be done are provided by the theory of the sine-Gordon interaction.

However for reasons of space I shall not dedicate much time to questions of the above type.

It is worth stressing that the convergence of the Schwinger functions of a theory with cut-off N to their limit values as $N \to \infty$ needs not be pointwise but it might take place in the sense of distributions or worse, at least if one does not express the results in terms of $S(f;p)$, i.e. of smoothed expressions involving the truncated averages, but rather in terms of the non smoothed Schwinger functions

$$\mathcal{E}^T_{int}(\varphi_{\xi_1},\ldots,\varphi_{\xi_p};1,\ldots,1) \qquad (10.3)$$

It is probably important to avoid putting any specific convergence requirements on how the expectations (10.3) should approach their limits as $N \to \infty$; in absence of physical reasons to prefer one type of convergence to other types one should leave this question aside, allowing for any type of convergence which will *a posteriori* be subject to only one constraint, namely that of leading to a probability measure P_{int} on the space of fields in a sense suitable to infer the existence of, say, a Wightman field.

xi. The cosine interaction model in two dimensions, perturbation theoryand multipole expansion

The ideas and methods of the preceding sections can now be applied to the actual theory of the simplest fields. If $\varphi^{(\le N)} = \sum_{j=-1}^{N} \varphi^{(j)}$ denotes a regularized free field as defined in Sec. 3 via a first order Pauli-Villars regularization [see (3.3) and (3.7)] consider the interaction $\mathcal{I}_N$:

$$\begin{aligned} V_1(\varphi^{(\le N)}) = &\int_\Lambda \frac{\lambda}{2} \sum_{\sigma=\pm1} \big[\nu + : e^{i\sigma\alpha\varphi_\xi^{(\le N)}} : \big] \equiv \\ \equiv &\int_\Lambda \big[\nu + \lambda \,:\cos\alpha\,\varphi_\xi^{(\le N)}\big]\, d\xi \end{aligned} \tag{11.1}$$

where $\alpha > 0$ is, here, a real positive number and λ, ν are reals: this will be called the "cosine interaction" or the "massive sine-Gordon interaction" with "open boundary conditions:. The latter specification refers to the fact that in Secs. 11–15 the field $\varphi^{(k)}$ will be supposed to have covariance given by $\overline{C}^{(k)}$ in (3.7) ("nonperiodic boundary conditions") and not by its periodized version denoted, in (3.7), by $C^{(k)}$. Nevertheless, to avoid complicating the notations we shall denote simply by $C^{(k)}$ and not by $\overline{C}^{(k)}$ the covariance of $\varphi^{(k)}$, since there will be no possibility of confusion in Secs. 11–15.

It will turn out that the interaction $\mathcal{I}$ in (11.1) is renormalizable (actually trivially superrenormalizable, in the sense defined at the end of Sec. 8, for $\alpha^2 < 4\pi$ and slightly less trivially also for $\alpha^2 \in [4\pi, 8\pi)$).

By the general theory of Sec. 6 the effective interaction $V^{(k)}$, as given by (6.8), will be described in terms of trees with end points bearing a position index ξ and an index $\alpha = 0, \pm1$ (not to be confused with the constant α in (11.1)) representing respectively the three terms in the intermediate expression in (11.1). Since $\alpha = 0$ represents a constant and trees represent truncated expectations the index $\alpha = 0$ can only appear in the trivial tree k——— $\xi,0$. The indexes $\alpha = \pm1$ will be denoted σ and they will be called *charges* indices.

Using the cluster interpretation of trees (see Fig. 7) one can interpret each vertex v of a tree as a cluster and define the *charge* Q_v of a tree as the sum of the indices σ associated with the end points in the cluster defined by v.

Given a tree γ let v be one of its vertices with frequency label h_v which, if thought of as a cluster, contains the end points labels $\xi_{j_1}, \ldots, \xi_{j_s}$ with charge indices $\sigma_{j_1}, \ldots, \sigma_{j_s}$: then we set

$$\begin{aligned} \varphi_v^{(\le h_v)} &\stackrel{def}{=} \sum \sigma_{j_n} \varphi_{j_n}^{(\le h_v)}, \qquad \textit{cluster field} \\ Q_v &\stackrel{def}{=} \sum \sigma_{j_n}, \qquad \textit{cluster charge} \end{aligned} \tag{11.2}$$

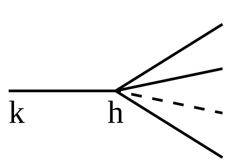
When $v = r =$ (root of the tree g) the φ_v, Q_v will also be denoted $\varphi(\gamma)$ and $Q(\gamma)$. Given any $h \ge -1$ it makes sense, naturally, to consider the fields $\varphi_v^{(\le h)}$ and $\varphi_v^{(h)}$.

To find the rules for the computation of $V(\gamma)$ one proceeds empirically trying to find an appropriate ansatz. After a while, as reasonable ansatz, emerges that the contribution to the effective potential of the tree γ is

$$\overline{V}(\gamma)\ : e^{i\,\alpha\,\varphi(\gamma)} : \tag{11.3}$$

where $\overline{V}(\gamma)$ is a suitable function of the tree γ.

Let $\gamma_1, \ldots, \gamma_s$ be s subtrees with root v_0 equal to the first nontrivial vertex of γ branching out of v_0 in $|g$: symbolically this is represented in Fig. 32 where $k = h_r =$ (frequency of the root of γ) and $h = h_{v_0}$:



(32)

Then combining (11.3) with the general recursion relation (6.4) one finds the following relation between the various coefficients $\overline{V}(\gamma)$:

$$\begin{aligned} \overline{V}(\gamma) = \overline{V}(\gamma_1)\cdots\overline{V}(\gamma_s)\,\mathcal{E}_{k+1}\cdots\mathcal{E}_{h-1}\cdot \\ \cdot\,\mathcal{E}_h^T(: e^{i\sigma\varphi^{(\le h)}(\gamma_1)} :, \ldots, : e^{i\sigma\varphi^{(\le h)}(\gamma_s)} :), \end{aligned} \tag{11.4}$$

which, by the rules on Wick monomials [see (A3.15) and (A3.16) in Appendix A3] yields for $k < h$

$$\begin{aligned} \overline{V}(\gamma) = &\overline{V}(\gamma_1)\cdots\overline{V}(\gamma_s)\,\frac{\prod_{j=1}^{s} : e^{i\,\alpha\,\varphi^{(\le k-1)}(\gamma_j)} :}{e^{i\,\alpha\,\varphi^{(\le k-1)}(\gamma)} :}\cdot \\ &\cdot \sum_{\tau\in\mathcal{G}^*} \prod_{\lambda\in\tau} (e^{-\alpha^2 C_\lambda^{(h)}} - 1) \end{aligned} \tag{11.5}$$

where one should remark that the ratio is *φ–independent* and $\mathcal{G}^*$ is the set of simple graphs connecting $\gamma_1, \ldots, \gamma_s$ (i.e. graphs with no repeated bonds and such that for any two "objects" γ_i, γ_j there is a path of bonds connecting them) regarding $\gamma_1, \ldots, \gamma_s$ as symbolic objects determined by the vertices $v_1, \ldots, v_s$ following v_0 in γ; furthermore if $\lambda = (\gamma_i m \gamma_j)$ we have set

$$C^{(h)}_{\lambda} \stackrel{def}{=} \mathcal{E}_h(\varphi^{(h)}(\gamma_i)\,\varphi^{(h)}(\gamma_j)) \stackrel{def}{=} C^{(h)}_{\gamma_i\gamma_j} \equiv \sum_{\substack{\xi\in\gamma_i\\ \eta\in\gamma_j}} \sigma_\xi\sigma_\eta C^{(h)}_{\xi\eta}$$

$$\begin{aligned} C^{(\le h)}_{\lambda} &\stackrel{def}{=} \mathcal{E}_0\cdots\mathcal{E}_h(\varphi^{(\le h)}(\gamma_i)\,\varphi^{(\le h)}(\gamma_j)) \stackrel{def}{=} C^{(\le h)}_{\gamma_i\gamma_j} \equiv \\ &\equiv \sum_{\substack{\xi\in\gamma_i\\ \eta\in\gamma_j}} \sigma_\xi\sigma_\eta C^{\le(h)}_{\xi\eta} \end{aligned} \tag{11.6}$$

Remark that he two relations in (11.6) have the interpretation of *electrostatic potential* between the charged clusters γ_i and γ_j relative to the *electric potential* $C^{(\cdot)}_{\xi\eta}$.

If we use the definition : $e^{i\alpha\sigma\varphi} := e^{\frac{1}{2}\alpha^2\mathcal{E}(\varphi^2)}\,e^{i\alpha\sigma\varphi}$ (see Appendix A3) Eq. (11.5) becomes

$$\begin{aligned} \overline{V}(\gamma) =&\overline{V}(\gamma_1)\cdots\overline{V}(\gamma_s)\,e^{-\alpha^2\sum_{i<j}C^{(h-1)}_{\gamma_i\gamma_j}}\cdot \\ &\cdot\sum_{\tau\in\mathcal{G}^*}\prod_{\lambda\in\tau}(e^{-\alpha^2 C^{(h)}_{\lambda}}-1) \end{aligned} \tag{11.7}$$

which considered together with

$$\overline{V}(\gamma) = \frac{1}{2}\lambda, \quad \text{or} \quad \overline{V}(\gamma) = \nu \tag{11.8}$$

for, respectively, k ——— ξ,σ or k ——— $\xi,0$.

provides a recursive definition of $\overline{V}(\gamma)$ and proves the ansatz (11.3). The effective potential has therefore the form

$$\begin{aligned} V^{(k)} =&\sum_{n=1}^{\infty}\sum_{\sigma_1,\ldots,\sigma_n}\int_\Lambda d\xi_1\cdots d\xi_n\cdot \\ &\cdot\sum_{\gamma\,:\,k(\gamma)=k}\frac{\overline{V}(\gamma)}{n(\gamma)}\ :\ e^{i\,\alpha\,\varphi^{(\le k)}(\gamma)}:, \end{aligned} \tag{11.9}$$

where the third sum runs over the trees γ with n end points (i.e. of degree n) carrying the end point labe;s $\xi_1,\sigma_1,\ldots,\xi_n,\sigma_n$ and with root frequency k.

Expression (11.9) will be called the *multipole expansion* for the effective interaction on scale γ^{-k}. The name comes from the following simple and interesting argument.

Consider the quantity Z below nd compute it by expanding the exponential in powers ad using the properties of the Gaussian integral (see Appendix A3):

$$\begin{aligned} Z \stackrel{def}{=}& \int e^{V^{(k)}(\varphi^{(\le k)})}P(d\varphi^{(\le k)}) = \\ =&\sum_{p=0}^{\infty}\int\frac{1}{p!}\big(V^{(k)}(\varphi^{(\le k)})\big)^p\,P(d\varphi^{(\le k)}) = \\ =&\sum_{p=0}^{\infty}\int d_{\boldsymbol{\sigma}_1}X_1\cdots d_{\boldsymbol{\sigma}_p}X_p\prod_{i=1}^{p}w(X_i,\boldsymbol{\sigma}_i)\cdot \\ &\cdot e^{-\alpha^2\sum_{i<j}V_{\boldsymbol{\sigma}_i\,\boldsymbol{\sigma}_j}(X_i,X_j)} \end{aligned} \tag{11.10}$$

where

$$\begin{aligned} &\int d_{\boldsymbol{\sigma}}X \equiv \sum_{n=0}^{\infty}\sum_{\sigma_1,\ldots,\sigma_n}\int d\xi_1\cdots d\xi_n, \\ &X=(\xi_1,\ldots,\xi_n),\ \boldsymbol{\sigma}=(\sigma_1,\ldots,\sigma_n), \\ &w(X,\boldsymbol{\sigma}) = \sum_{\substack{degree\ \ \gamma=n\\ \boldsymbol{\sigma}(\gamma)=\boldsymbol{\sigma}}}\frac{\overline{V}(\gamma)}{n(\gamma)}, \\ &V_{\boldsymbol{\sigma}\boldsymbol{\sigma}'}(X,X') = \sum_{\substack{\xi\in X\\ \xi'\in X'}}\sigma_\xi\,\sigma_{\xi'}\,C^{(\le k)}_{\xi\,\xi'} \end{aligned} \tag{11.11}$$

i.e. Z in (11.10) is indeed formally [i.e. modulo convergence problems in (11.10)], the partition function of a multipole gas in which the multipole with charges $\sigma_1,\ldots,\sigma_n$ located in the volume elements $d\xi_1\cdot d\xi_n$ has activity

$$w(\xi_1,\ldots,\xi_n;\sigma_1,\ldots,\sigma_n)\,d\xi_1\cdots d\xi_n \equiv w(X,\boldsymbol{\sigma})\,dX \tag{11.12}$$

To complete the analysis of perturbation theory for the cosine interaction one has to show that the theory is ultraviolet finite. This is indeed the case for $\alpha^2 < 4\pi$ but if $\alpha^2 \ge 4\pi$ this is so only for $\alpha^2 < 8\pi$ and, perhaps, for $\alpha^2 = 8\pi$.This problem is studied in Sec. 12 below.

xii. Ultraviolet stability for cosine inter- -action and renormalizability for $\alpha^2 < 8\pi$

Let $\varphi^{(\le k)} = \sum_{j=-1}^{k}\varphi^{(j)}$ be a sample field in which $\varphi^{(j)}$ verifies (3.15) and (3.16) and let the covariances $C^{(j)}$, $j=-1,0,\ldots,N$ verify (3.19) ($j_0=1$ in the present case) being defined by $\overline{C}^{(j)}$ in (3.7) [see comment following (11.1)].

To study the ultraviolet stability of the effective potentials $V^{(k)}(\varphi^{(\le k)})$ one bounds [see (11.3)] the quantity

$$\begin{aligned} M(\Delta_1,\ldots,\Delta_n;\overline{\gamma}) \stackrel{def}{=}& \sum_{\substack{\gamma\,:\,s(\gamma)=\overline{\gamma},\\ k(\gamma)=k\,;\,\boldsymbol{\sigma}(\gamma)=\boldsymbol{\sigma}}}\int_{\Delta_1\times\cdots\times\Delta_n}\cdot \\ &\cdot\frac{|\overline{V}(\gamma)|}{n(\gamma)}e^{\frac{1}{2}\alpha^2 C^{(\le k)}_{\gamma\gamma}}d\xi_1\cdots d\xi_n \end{aligned} \tag{12.1}$$

having estimated the Wick–ordered exponentials

$$:\ e^{i\,\alpha\,\varphi^{(\le k)}(\gamma)}\ := e^{\frac{1}{2}\alpha^2 C^{(\le k)}_{\gamma\gamma}}\,e^{i\,\alpha,\varphi^{(\le k)}(\gamma)}$$

by $e^{\frac{1}{2}\alpha^2 C^{(\le k)}_{\gamma\gamma}}$; and the sum runs over all trees with fixed shape $s(]g)=\overline{\gamma}$ [to avoid confusion the shape is here denoted $s(\gamma)$ rather than $\sigma(\gamma)$ as in Sec. 8], fixed root frequency index k and fixed charge labels $\boldsymbol{\sigma}=(\sigma_1,\ldots,\sigma_n)$ for the n end points. Hence the sum runs over the frequency labels h_v that can be assigned to the nontrivial vertices $v > r$ of the shape $\overline{\gamma}$. Finally $\Delta_1,\ldots,\Delta_n$ are

n cubes extracted from a pavement of Λ with cubes of side size γ^{-k}; we shall denote the latter pavement by $\mathcal{Q}_k$, assuming that the side of Λ is divisible by γ^{-k}.

Of course one looks for bounds uniform in N and uniformly summable over the choices of $\Delta_1, \ldots, \Delta_n$ in $\mathcal{Q}_k$. In fact this is motivated by the remark that the contribution to $V^{(k)}$ from the trees with shape $\overline{\gamma}$ and charges $\boldsymbol{\sigma}$ cab be written

$$\sum_{\Delta_1,\ldots,\Delta_n}\;\sum_{\substack{\gamma:\, s(\gamma)=\overline{\gamma},\\ k(\gamma)=k\,;\,\boldsymbol{\sigma}(\gamma)=\boldsymbol{\sigma}}}\int_{\Delta_1,\ldots,\Delta_n}\frac{V(\gamma)}{n(\gamma)}\cdot \qquad\qquad (12.2)$$
$$\cdot\, e^{\frac{1}{2}\alpha^2 C^{(\le k)}_{\gamma\gamma}}\, e^{i\alpha\varphi^{(\le k)}(\gamma)}\, d\xi_1\cdots d\gamma_n$$

so that a bound, valid for all N and all $k \le N$, like

$$M(\Delta_1,\ldots,\Delta_n;\overline{\gamma}) \le m(\overline{\gamma})\, e^{-\kappa\gamma^k d(\Delta_1,\ldots,\Delta_n)}\cdot \qquad\qquad (12.3)$$
$$\cdot\, \gamma^{\left((\frac{1}{4\pi}\alpha^2-2)(n-1)+\frac{1}{4\pi}\alpha^2\right)k},$$

where $m(\overline{\gamma})$ is a suitable constant depending only on the shape $\overline{\gamma}$, would be sufficient to show that the effective potentials are well defined order by order in perturbation theory, so that they converge to limits as $N \to \infty$ at least on subsequences; actually it will be clear that one could also easily prove plain convergence without need of subsequences.

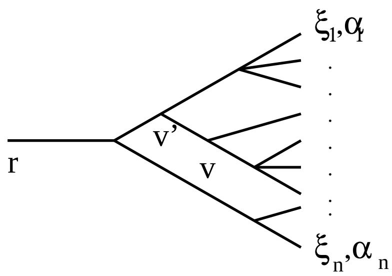
The estimate (12.3) shows more because it shows that the effective potential has a strong *short range* property on the scale γ^{-k} naturally associated with with the frequency k; the short range property is expected to play an important role in the infrared stability, but as it will become clear later, it also plays a role in ultraviolet stability.

In trying to prove (12.3) it is convenient to rewrite the recursive relation (11.7) as

$$\overline{V}(\gamma)\, e^{\frac{1}{2}\alpha^2 C^{(\le k-1)}_{\gamma\gamma}} = e^{-\frac{1}{2}\alpha^2\left(C^{(\le h-1)}_{\gamma\gamma}-C^{(\le k-1)}_{\gamma\gamma}\right)}\cdot$$
$$\cdot\Big(\prod_{i=1}^{s}\overline{V}(\gamma_i)\, e^{\frac{1}{2}\alpha^2 C^{(\le k-1)}_{\gamma_i\gamma_i}}\Big)\cdot \qquad\qquad (12.4)$$
$$\sum_{\tau\in\mathcal{G}^*}\prod_{\lambda\in\tau}\left(e^{-\alpha^2 C^{(h)}_\lambda}-1\right), \quad \text{for } h<k$$

where the relation $\sum_{i,j} C^{(\le h-1)}_{\gamma_i\gamma_j} \equiv C^{(\le h-1)}_{\gamma\gamma}$ is used and we set $C^{(-1)} \equiv 0$.

Let $v > r$ be any vertex of γ and denote v' the vertex of γ preceding v; denote γ_v the subtree of γ with root at v' and first vertex v; for instance in Fig. 33 γ_v is the tree consisting in all the branches of γ that can be reached by climbing the tree starting from v' and passing through v:



(33)

Call $\xi_1,\ldots,\xi_n,\sigma_1,\ldots,\sigma_n$ the end point labels for the positions and, respectively, the charges. Eq. (12.4) implies [see below]

$$|\bar{V}(\gamma)|\; e^{(\alpha^2/2)C^{(\le k-1)}_{\gamma\gamma}} \le \mathcal{N}_\delta\cdot\Big(\prod_{v>r} e^{-\kappa_0\gamma^{h_v} d^*(X_v)}\Big)\cdot$$
$$\cdot\Big(\prod_{v>r} e^{-(\alpha^2/2)\left(C^{(\le h_v-1)}_{\gamma_v\gamma_v}-C^{(\le h_{v'}-1)}_{\gamma_v\gamma_v}\right)}\Big)\cdot \qquad\qquad (12.5)$$
$$\cdot\Big(\prod_{i=1}^{n}\frac{\lambda}{2} e^{(\alpha^2/2)C^{(\le h_{v_i}-1)}_{\xi_i\xi_i}}\Big)$$

where v_i is the tree vertex directly connectd to the end point ξ_i and $\mathcal{N}_{\overline{\gamma}}, \kappa_0 > 0$ are constants and

$$d^*(X_v) = \text{graph distance of the points of } X_v \text{ modulo the clusters inside } v, \qquad (12.6)$$

i.e. $d^*(X_v)$ is obtained by drawing lines connecting points in the cluster X_v belonging to *distinct* maximal subclusters of X_v (which are the clusters corresponding to the vertices of γ following v immediately, see Fig. 7 for instance) in such a way that any subcluster can be reached from any other by walking on such lines and possibly jumping inside the subclusters: then $d^*(X_v)$is the minimum of the sum of the lengths of the above lines over all possible ways of drawing them.

The exponential factor in (12.5) requires an explanation: it arises from a bound on the last product in (12.4) and from the exponential decay of $C^{(k)}_{\xi\,\eta} = C^{(0)}_{\gamma^k\xi\,\gamma^k\eta}$ [see (3.19)] for some $\kappa > 0$

$$\Big|\prod_{\lambda\in r}(e^{-\alpha^2 C^{(h)}_\lambda}-1)\Big| \le \prod_{\lambda\in r}\left(e^{\alpha^2 C^{(h)}_\lambda}\alpha^2|C^{(h)}_\lambda|\right) \le \qquad (12.7)$$
$$\prod_{\lambda\in r}\Big[\alpha^2 e^{\alpha^2 n_v^2 C^{(0)}_{00}}\sum^{*}_{\xi,\eta} A e^{-\kappa\gamma^h|\xi-\eta|}\Big],$$

where n_v is the number o vertices in X_v ($n_v \le n$), and the sum runs over the pairs ξ, η in the subclusters joined by λ, whose number is bounded by $n_v \le n$. Since $|\xi-\eta|$ is larger or equal to the minimum distance between the two sub-clusters, (12.5) follows, with $\mathcal{N}_{\overline{\gamma}}$ being a coefficient depending only on the family of numbers n_v, i.e. on the shape $\overline{\gamma} = s(\gamma)$ of γ only.

To proceed one has to find a reasonable bound on the first product in (12.5). Let $C^{[\cdot]}_{\overline{\gamma}_v,\overline{\gamma}_v}$ denote the same expression as $C^{(\cdot)}_{\overline{\gamma}_v,\overline{\gamma}_v}$ when all the points in the cluster corresponding to v are collapsed in one of them; it is

$$C^{[\cdot]}_{\overline{\gamma}_v,\overline{\gamma}_v} \equiv C^{(\cdot)}_{00}\, Q_v^2 \tag{12.8}$$

where Q_v is the charge of the cluste v. Then the first product in (12.5) can be written

$$\begin{aligned}&\Big[\prod_v e^{-\frac12\alpha^2\left(C_{00}^{(\le h_v-1)}-C_{00}^{(\le h_{v'}-1)}\right)Q_v^2}\Big]\cdot\\ &\cdot\Big[\prod_v e^{\frac12\alpha^2\left(C^{(\le h_v-1)}_{\gamma_v\gamma_v}-C^{(\le h_v-1)}_{\overline{\gamma}_v\overline{\gamma}_v}\right)}\cdot\\ &\cdot e^{\frac12\alpha^2\left(C^{(\le h_v-1)}_{\gamma_v\gamma_v}-C^{(\le h_v-1)}_{\overline{\gamma}_v\,\overline{\gamma}_v}\right)}\Big]\end{aligned} \tag{12.9}$$

and the term in the last brackets can be bounded by using (3.19) and

$$\begin{aligned}&\sum_{\xi,\eta\in X_v}\Big(|C^{(\le h_v-1)}_{\xi\eta}-C^{(\le h_v-1)}_{00}|+\\ &+|C^{(\le h_{v'}-1)}_{\xi\eta}-C^{(\le h_{v'}-1)}_{00}|\Big)\le\\ &\le 2n_v^2 A_{\frac12}(1-\gamma^{-\frac12})\left(\gamma^{h_v}d(X_v)\right)^{\frac12}=\\ &=\widetilde{A}\left(\gamma^{h_v}d(X_v)\right)^{\frac12}\end{aligned} \tag{12.10}$$

where $d(X_v)$ is the length of the shortest path connecting all the points of the cluster X_v. In the last step of (12.10) use has been made of (3.19) via

$$\begin{aligned}&|C^{(\le h)}_{\xi\eta}-C^{(\le h)}_{00}|\le|C^{(0)}_{\gamma^p\xi\,\gamma^p\eta}-C^{(0)}_{00}|\le\\ &A_{\frac12}\sum_{p=0}^{h}\left(\gamma^h|\xi-\eta|\right)^{\frac12}\end{aligned} \tag{12.11}$$

Hence using the inequality

$$\sum_v \gamma^{h_v} d^*(X_v) \ge \sum_v \frac{1}{n_n^2}\gamma^{h_v} d(X_v). \tag{12.12}$$

[*Hint:* Eq. (12.12) does not hold "without the sums"; see (18.15) for a similar but deeper inequality], one finds

$$-\frac{\kappa}{2}\sum_v \eta^{h_v} d^*(X_v)+\widetilde{A}\sum_v[\gamma^{h_v}d(X_v)]^{\frac12}\le A(\overline{\gamma})<\infty \tag{12.13}$$

and one can bound (12.5) as

$$\begin{aligned}&|\overline{V}(\gamma)|\,e^{\frac12\alpha^2 C^{(\le k-1)}_{\overline{\gamma}\,\overline{\gamma}}}\le\\ &\le\mathcal{N}'_{\overline{\gamma}}\Big[\prod_{v>r}e^{-\frac12\alpha^2Q_v^2\left(C_{00}^{(\le h_v-1)}-C_{00}^{(\le h_{v'}-1)}\right)}\cdot\\ &\cdot e^{-\frac12\kappa\gamma^{h_v}d^*(X_v)}\Big]\cdot\prod_{i=1}^n e^{\frac12\alpha^2C^{(\le h_{v_i}-1)|}_{\xi_i\,\xi_i}}\end{aligned} \tag{12.14}$$

The integral (12.1) can now be estimates using (see Appendix A4)

$$\begin{aligned}&\int_{\mathbb{R}^2\times\ldots\times\mathbb{R}^2}d\xi_2\,d\xi_3\cdots d\xi_n\prod_v e^{-\frac14\kappa\gamma^{h_v}d^*(X_v)}\le\\ &\le B_n\prod_{v>r}\gamma^{-2h_v(s_v-1)},\end{aligned} \tag{12.15}$$

with s_v = number of branches emerging from the vertex v in γ and B_n is some constant.

Using also $C^{(\le h)}_{00}=(h+1)C^{(0)}_{00}$ we see that

$$\begin{aligned}&M(\Delta_1,\ldots,\Delta_n;\overline{\gamma})\le\mathcal{N}''_{\overline{\gamma}}e^{-\frac14\kappa d(\Delta_1,\ldots,\Delta_n)}\cdot\\ &\cdot\sum_{\mathbf{h}}\big[\prod_{i=1}^n e^{\frac12\alpha^2h_{v_i}C^{(0)}_{00}}\big]\cdot\\ &\cdot\prod_{v>r}\gamma^{-2h_v(h_v-1)}e^{-\frac12Q_v^2(h_v-h_{v'})C^{(0)}_{00}},\end{aligned} \tag{12.16}$$

where the sum runs over the frequency labelings of the shape $\overline{\gamma}$ such that $k(\gamma)=k$.

Taking into account the relation between the number s_v of branches emerging from v in γ and the number n_v of points in the cluster X_v corresponding to v

$$\sum_{v>w}(s_v-1)=n_w-1 \tag{12.17}$$

one easily checks, denoting $C=C^{(0)}_{00}\equiv\frac{1}{2\pi}\log\gamma$:

$$\begin{aligned}&\sum_i\frac{\alpha^2}{2}(h_{v_i}-k)\,C-\log\gamma\sum_{v>r}\cdot\\ &\cdot\big[2(h_v-k)(s_v-1)+\frac{\alpha^2}{2}C\,Q_v^2\,(h_h-h_{v'})\big]\equiv\\ &\equiv\log\gamma\sum_{v>r}\Big(\big(\frac{\alpha^2}{4\pi}-2\big)(n_v-1)+\frac{\alpha^2}{4\pi}-\\ &-\frac{\alpha^2}{4\pi}Q_v^2\Big)\,(h_v-h_{v'})\end{aligned} \tag{12.18}$$

so that, using again (12.17) and (12.16),

$$\begin{aligned}&M(\Delta_1,\ldots,\Delta_n;\overline{\gamma})\le\mathcal{N}'''_{\overline{\gamma}}e^{-\frac14\kappa d(\Delta_1,\ldots,\Delta_n)}\gamma^k\\ &\gamma^{\left((\frac{1}{4\pi}-2)(n-1)-\frac{1}{4\pi}\alpha^2\right)k}\sum_{\mathbf{h}}\sum_{v>r}\gamma^{-\rho_v\,(h_v-h_{v'})},\end{aligned} \tag{12.19}$$

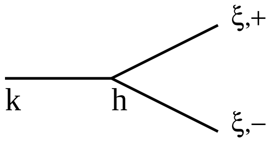
with

$$\rho_v\stackrel{def}{=}-\big(\frac{\alpha^2}{4\pi}-2\big)(n_v-1)-\frac{\alpha^2}{4\pi}+\frac{\alpha^2}{4\pi}Q_v^2. \tag{12.20}$$

The summation in (12.19) is over the frequency labelings of $\overline{\gamma}$ and, therefore, over the h's such that $N\ge h_v-h_{v'}\ge1$.

Eq. (12.3) follows, provided $\rho_v > 0$ for all v. In fact since $v > r$ implies $n_v \geq 2$ and $|Q_v v| \geq 0$ it is clear that $\rho_v \geq -\frac{1}{2\pi}\alpha^2 + 2$—i.e. $\rho_v > 0$ if $\alpha^2 < 4\pi$.

This proves (12.3) and the ultraviolet stability for $\alpha^2 < 4\pi$. Since one can easily check, as the bounds (12.19) and (12.20) hint, that for $\alpha^2 \geq 4\pi$ the contribution to $V^{(k)}$ from the trees



(34)

is in fact divergent as $N \to \infty$, the problem has to be reexamined for $\alpha^2 \geq 4\pi$—i.e. renormalization is necessary.

The key remark to study the case $\alpha^2 \geq 4\pi$ is that the bounds (12.19) and (12.20) can be improved.

In fact let v be a "zero charge" or "neutral" vertex of γ: $Q_v = 0$. Let v' be the vertex preceding v and let $h_v, h_{v'}$ be their frequency labels. Then in the evaluation of $V(\gamma)$ the subtree γ_v of γ with root v' and containing v and all the following vertices has the meaning, according to the general theory of the tree expansion in Secs. 4 and 5,

$$\mathcal{E}^T_{h_{v'}}\big(: e^{i\,\alpha\,\varphi^{(h_{v'})}(\gamma_{v_1})} :, \ldots, : e^{i\,\alpha\,\varphi^{(h_{v'})}(\gamma_{v_s})} :\big) \tag{12.21}$$

if $v_1, \ldots, v_s$ are the vertices immediately following v.

However, when all the points of the cluster X_v coincide it is $\varphi(\gamma_v) = 0$ because $Q_v = 0$ and since (12.21) is a truncated expectation it must vanish [in fact the first argument becomes identically 1 and $\mathcal{E}^T(1, \ldots) \equiv 0$].

Therefore (12.21) will be equal to $e^{i\,\alpha\,\varphi^{(h_{v'}-1)}(\gamma_{v'})}$ times a factor which will be proportional, given $\varepsilon \in (0,1)$ arbitrarily (see (3.6)), to

$$\Big(\gamma^{h_{v'}} \sum_{\xi,\eta \in X_v} |\xi - \eta|\Big)^{1-\varepsilon}. \tag{12.22}$$

If one collects together the contributions to the $V^{(k)}$ from the trees having the same shape up to the charge indices and having fixed clusters of zero charge then one realizes that this improves the estimate producing a result which is a finite sum of terms which can all be bounded by the same bound that can be put on the "worst" among them, namely the one obtained by replacing $: e^{i\,\alpha\,\varphi^{(h_{v'}-1)}(\gamma_{v'})} :$ by $: \cos\big(\alpha\,\varphi^{(h_{v'}-1)}(\gamma_{v'})\big) :$. The latter will, in turn, introduce in the evaluation of the expressions analogous to (12.21) a factor proportional to

$$\big(\gamma^{h_{v'}} |\xi - \eta|\big)^{2(1-\varepsilon)} \tag{12.23}$$

if $Q_{v'} = 0$ because the cosine differs from 1 by a second order infinitesimal. The details will not be discussed here as a much more complicated similar analysis will be presented in Sec. 18. Via some simple algebra this leads to replacing (12.3) by a bound on

$$\begin{aligned}\overline{M}(\Delta_1, \ldots, \Delta_n) = \sum_{\boldsymbol{\sigma}} \int_{\Delta_1 \times \ldots \times \Delta_n} d\xi_1 \cdots d\xi_n \cdot \\ \cdot \sum_{\substack{s(\gamma) = \overline{\gamma} \\ \boldsymbol{\sigma}(\gamma) = \boldsymbol{\sigma}}} V(\gamma) e^{\frac{1}{2}\alpha^2 C^{(\leq k)}_{\gamma\gamma}}\end{aligned} \tag{12.24}$$

which, if v_0 denotes the first vertex of $\overline{\gamma}$ following the root r, is estimated by

$$\begin{aligned}\big|\overline{M}(\Delta_1, \ldots, \Delta_n)\big| \leq \overline{\mathcal{N}}_\gamma e^{-\frac{\kappa}{4} d(\Delta_1, \ldots, \Delta_n)} \gamma^{\left((\frac{\alpha^2}{4\pi} - 2)n + \frac{\alpha^2}{4\pi}\right)k} \cdot \\ \cdot \sum_{\mathbf{h}} \Big(\prod_{v > v_0} \gamma^{-(\rho_v + 2 - \varepsilon)(h_v - h_{v'})}\Big) \gamma^{-\rho_{v_0}(h_{v_0} - h)}\end{aligned} \tag{12.25}$$

because in the intermediate steps the integral (12.15) will be replaced by

$$\begin{aligned}\int d\xi_2 \cdots d\xi_n \Big(\prod_{v > r} e^{-\frac{\kappa}{4}\gamma^{k_v} d^*(X_v)}\Big) \cdot \\ \cdot \prod_{\substack{v:\,\rho_v = 0 \\ v > v_0}} \Big(\gamma^{h_{v'}} \sum_{\xi,\eta \in X_v} |\xi - \eta|\Big)^{2-\varepsilon}\end{aligned} \tag{12.26}$$

by using the remarks leading to (12.23): the first nontrivial vertex v_0 of $\overline{\gamma}$ plays a special role, because if $Q_{v_0} = 0$ the expresion $: \cos(\alpha\,\varphi^{(\leq k)}(\gamma_{V_0}))$ :: will be proportional to the result obtained after the last truncation and no further truncation will be done at frequency k. Therefore no factors like (12.23) can be contributed by the first vertex v_0.

The itegral (12.26) obviously leads to an extra factor in (12.15) of the form $\overline{B}^n \prod_{v > v_0} \gamma^{-2(1-\varepsilon)(h_v - h_{v'})}$. In fact the product of exponentials in (12.26) forces the points in the cluster v to be within a distance γ^{-h_v}; hence (12.23) can be replaced in the integral (12.26) by $(\gamma^{h_{v'}}\gamma^{-h_v})^{2(1-\varepsilon)}\, h_v^2\, (B')^{n_v}$, provided $\frac{\kappa}{4}$ is replaced by $\frac{\kappa}{8}$ and B' is conveniently chosen (see Appendix A4).

The bound (12.25) proves that if one collects together several trees of the same type and if use is made of the charge symmetry then all trees with nonzero charge $Q_{v_0} \neq 0$ (hence $|Q_{v_0}| \geq 1$) yield $\rho_{v_0} > 0$ and $\rho_v + 2 - 2\varepsilon > 0$ if ε is taken small enough, for all $\alpha^2 < 8\pi$. Hence (12.25) proves that the ultraviolet stability can be violated, if $\alpha^2 < 8\pi$, only by trees which have zero charge: $Q_{v_0} = 0$.

For $\alpha^2 < 8\pi$ not all the neutral trees have ultraviolet stability problems: onlu the neutral ones with $n = n_{v_0}$ end points suche that [see (12.20)]

$$\big(\frac{\alpha^2}{4\pi} - 2\big)(n-1) + \frac{\alpha^2}{4\pi} < 0 \tag{12.27}$$

So, for $\alpha^2 < 8\pi$ there is a sequence of *tresholds* obtained by setting the *l.h.s.* of (12.27) equal to 0:

$$\frac{\alpha_n^2}{4\pi} = 2\frac{2n-1}{2n} = 1, \frac{3}{2}, \frac{5}{3}, \dots \tag{12.28}$$

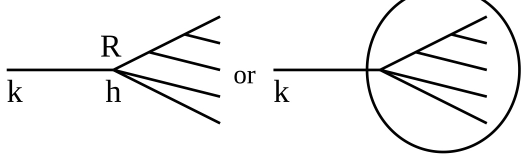
As α^2 reaches α_n^2 and beyond it the trees with n vertices and zero charge "become ultraviolet unstable"—i.e. their contribution to the effective potentials are not convergent as $N \to \infty$.

However the reason for the instability is somewhat trivial and it is manifestly due to the fact that the first non-trivial vertex v_0 of $|g$, when γ is neutral, gives a contribution to $V(\gamma)$ of the form $: e^{i\,\alpha\,\varphi^{(\le k)}(\gamma)} :$, which does not vanish when the positions of the end points labels $\xi_1 1, \dots, \xi_n$ become identical.

But id one defines $\mathcal{L}_\gamma^{(k)} = 0$ unless $Q_{v_0} = 0$ or $\alpha^2 < \alpha_n^2$ and

$$\mathcal{L}_\gamma^{(k)} \int : e^{i\,\alpha\,\varphi^{(\le k)}(\gamma)} : \overline{V}(\gamma)\, d\boldsymbol{\xi} \stackrel{def}{=} \int \overline{V}(\gamma)\, d\boldsymbol{\xi}, \tag{12.29}$$

if $Q_{v_0} = 0$ and $\alpha^2 < \alpha_n^2$ and if one collects together the trees γ of the same shape up to the charge indices and with the same frequency indices and the same vertices of zero charge one sees that the operaors $\mathcal{L}_h^{(\gamma)}$ define trnormalizations operations, according to the general theory of Secs. 69, such that the dressed graphs have the form in Fig. 35:



(35)

Either they contain an index R as a superscript on the first vertex v_0 after the root r or they are entirely contained in a single frame with an index $\sigma = 0$ appended to the frame [meaning that they contribute a constant to the efective potential because $\mathcal{L}_k^{(\gamma)}$ takes values in the space of the constants by (12.9).

A tree with an R over the first vertex will mean a contribution to the to the effective potential which is equal to the one that would be given by the tree without the R but with $: e^{i\,\alpha\,\varphi^{(\le k)}(\gamma)} :$ replaced by $: e^{i\,\alpha\,\varphi^{(\le k)}(\gamma)} - 1 :$.

Collecting again the contributions to the effective potential from all the trees with given shape up to the charge indices and summing their contributions over all the possible frequency and charge labels at fixed neutral vertices one sees that the contribution to the effective potential sums up to the same quantity (12.4) with the replacement

$$: e^{i\,\alpha\,\varphi^{(\le k)}(\gamma)} : \qquad \to \qquad : \cos \alpha\, \varphi^{(\le k)}(\gamma) - 1 : \tag{12.30}$$

and the latter expression vanishes when all the points $\xi_1, \dots, \xi_n$ collapse into a single point and the order of zero is of the order of the square of the zero of $\varphi^{(\le k)}(\gamma)$. The latter can be evaluated by recalling the basic smoothness properties of $\varphi^{(\le k)}$ described by (3.16) (recall that the space dimension is here $d = 2$): it is of the order of

$$B^2 \big(\gamma \sum_{\xi,\eta} |\xi - \eta|\big)^{2(1-\varepsilon)}, \tag{12.31}$$

if $B = \sup B_\Delta$ and $\varepsilon > 0$ is prefixed arbitrarily.

This improves the bound (12.25) by replacing also ρ_{v_0} by $\overline{\rho}_{v_0} = \rho_{v_0} + 2 - 2\varepsilon$.

The arbitrariness of ε implies that, if $\alpha^2 < 8\pi$, then ε can be chosen so that $\overline{\rho}_{v_0} > 0$ and, therefore, all the unframed dressed trees are ultrviolet finite in the sens that, collecting together the contributions from the trees with given shape, up to the charge indices, one obtains a total contribution to the effective potential which is ultraviolet finite.

The framed trees contribute only to the constant part of the effective potential and therefore need not be studied. However their theory would also be simple and they turn out to be untraviolet finite: in fact the sum of the contributions to the effective potential coming from the neutral trees of a given degree is a constant which can be written as $\int \nu_k\, d\boldsymbol{\xi}$ and, from (12.25) and the general theory one can find

$$|\nu_k| \le \widetilde{\mathcal{N}}(\overline{\gamma})\, \big(\gamma^{(\frac{\alpha^2}{4\pi}-2)(n-1)+\frac{\alpha^2}{4\pi}}\big)^k\, \lambda^n \tag{12.32}$$

Since, given $\alpha^2 < 8\pi$, it is $\mathcal{L}_k^{(\gamma)} = 0$ if n, the number of end points ("degree") of γ is large enough it follows that the cosine interaction is super-renormalizabl in the sense of Sec. 8 (see the final comments of Sec. 8).

Exercise: study the exponential interaction (5.5) and show that it is ultraviolet finite up to $\alpha^2 < 4\pi$. Show that it is not renormalizable for $\alpha^2 \ge 4\pi$ (*hint:* just imitate the same steps and estimates used for the cosine case).

xiii. Beyond perturbation theory in the cosine interaction case: asymptotic freedom and scale invariance

Having completed the perturbative analysis for the cosine field theory in terms of formal power series with no control on convergence one wonders what it really means to study an interacting field theory.

The simplest type of result that one can think to try to prove for the interacting measures P_{int} is the following.

There exist (infinitely many inequivalent) one parameter families P_λ of measures on the space $\mathcal{S}'(\mathbb{R}^2)$, of the distributions on $\mathbb{R}^2$, whose Schwinger functions admit asymptotic expansion in the parameter λ near $\lambda = 0$ coinciding with the formal perturbatio expansion of the cosine interaction discussed in Secs. 11 and 12 (with $\nu = 0$).

Super-renormalizability is the deep property behind the methods, so far known, to obtain a proof of the above proposition in the cosine interaction case as well as in the corresponding proof for many other supr-renormalizable field theories (e.g. $-\lambda\ :\ \varphi^4\ :$ in two dimensions or $-\lambda\ :\ \varphi^4\ :\ -\mu\ :\ \varphi^2\ :\ \nu$ in three dimensions; in fact the ideas and methods involved do not distinguish between the mentioned theories).

The first idea is to try to build P_λ as a limit of measures of the form

$$Z^{-1}\Big(\prod_{j=0}^{N}\chi_j(\varphi^{(j)})\Big)\, e^{V(\varphi^{(\le N)})}\prod_{j=0}^{N}P(d\varphi^{(j)}) \tag{13.1}$$

where χ_j are characteristic functions selecting fields having so large a probability that

$$1\ge\int\prod_{j=0}^{\infty}\Big(\prod_{j=0}^{N}\chi_j(\varphi^{(j)})P(d\varphi^{(j)})\Big)\ge e^{-\varepsilon(\lambda)|\Lambda|} \tag{13.2}$$

with $\varepsilon(\lambda)\xrightarrow[\lambda\to 0]{}0$ faster than any power and (see (11.1) with $\nu=0$)

$$V(\varphi^{(\le N)})=\int_\Lambda\big(\lambda\ :\ \cos\alpha\varphi^{(\le N)}_\xi\ :+\nu_N(\lambda)\big)d\xi \tag{13.3}$$

where $\nu_N(\lambda)$ is the sum of the finitely many counterterms due to the renormalization described in Sec. 12 [see (12.31)] if any, i.e. if $4\pi\le\alpha^2<8\pi$.

The characteristic functions χ_j will be so chosen to allow one to treat "naively" the fields $\varphi^{(j)}$ when $\chi_j(\varphi^{(j)})=1$: i.e. χ_j will be the characteristic functions of the set

$$\Big\{\varphi\,\big|\,|\sin\big(\frac{\alpha}{2}(\varphi^{(j)}_\xi-\varphi^{(j)}_\eta)\big)|\Big\}<B_j\,(\gamma^j|\xi-\eta|)^{1-\varepsilon},\forall\xi,\eta\in\Lambda \tag{13.4}$$

where $B_j=B\,(1+j)^a\,\log(e+j+\lambda^{-1})$ for some $B,\varepsilon>0, a>\frac12$ (to be chosen). The probability of the above event with respect to $P(d\varphi^{(j)})$ is bounded below, by using (3.17), for all $a\ge\frac12$ by

$$\begin{aligned}&\prod_{\Delta\subset\Lambda}\big(1-\overline{A}e^{-\overline{\alpha}B^2(1+j)}\big)\big(\log(e+j+\lambda)\big)^2\ge\\&\ge\big(1-\overline{A}e^{-\overline{\alpha}B^2(1+j)}\cdot(\log(e+j+\lambda))^2\big)^{\gamma^{2j}|\Lambda|},\end{aligned} \tag{13.5}$$

In (13.2) one can take, with χ_j as in (13.4),

$$\begin{aligned}\varepsilon(\lambda)=&\sum_{j=0}^{\infty}\gamma^{2j}\log\Big(1-\overline{A}\cdot\\&\cdot(1+j+\lambda^{-1})^{-\overline{\alpha}B^2(1+j)\,\log(1-\overline{A}(1+j+\lambda^{-1}))}\Big)=\\&=O(\lambda^\infty)\end{aligned} \tag{13.6}$$

i.e. $\varepsilon(\lambda)\xrightarrow[\lambda\to 0]{}0$ faster tan any power of λ.

Since the amount of phase space thrown away by the insertion of the characteristic functions in (13.1) is, if measured with the free-field measure, very negligible [see (13.2) and (13.4)] it is quite clear that the perturbation theory expansion for the Schwinger functions of the measure (13.1) and those of the measure obtained by taking away from (13.1) the characteristic functions are identical uniformly in N.

Therefore if one succeeds in showing that the measure (13.1) has a limit as $N\to\infty$ (possibly only on subsequences) the one parameter family claimed to exist in the above proposition is constructed.

This is in fact true and it is the way which we will be followed in proving the proposition stated at the beginning of this section. Since the construction clearly depends on the arbitrary parameter B in (13.4) one must expect that the family P_λ of measures obtained as limits of (13.1) is B–dependent.

The measure (13.1) will be called a "restricted cosine field": it is an object of limited interest even in the limit $N\to\infty$. Its importance lies only in the fact that its understanding is preliminary to the understanding of the interesting case essentially consisting in letting $B\to\infty$.

It is important the following remark: the restrictions (13.4) *do not imply* that the field $\varphi^{(\le N)}$ is constrained to be smooth for large N. Actually a simple computation shows that the cut-off on rough or large fields imposed by the inequalities (13.4) is such that $\varphi^{(\le N)}_\xi-\varphi^{(\le N)}_\eta$ have essentially the same covariance, hence the same average size, with respect to the free Gaussian measure or with respect to the free restricted Gaussian measure [i.e. the Gaussian measure restricted to the ensemble of fields described by (13.4)]. This means that the problem of taking the limit as $N\to\infty$ of (13.1) is still nontrivial and that some new idea is necessary for its solution.

Arguments on field theory are often given, in the literature, which treat the the fields as if they verified (13.4) and the problem of controlling what happens when the field violates the conditions imposed in (13.4), i.e. the problem of controlling the *large fluctuations*, is often solved by handwaving saying that the large fluctuations are "depressed" by the positivity of the action.

It will be clear that, instead, many real problems arise in trying to give a rigorous meaning to such arguments; in my understanding the situation is, in general, very subtle and I cannot see the actual solution of the above large fluctuations problem (even in the cases in which it is known how to handle it on a mathematically rigorous basis) as just a refined way of rephrasing the mentioned positivity argument. Furthermore this is a case in which it makes no sense to appeal to "physical arguments" because the issue is [precisely whether field theory has anything to do with physics.

The problem of the relevance of the large fluctuations seems to have been clearly perceived as a deep one, even in field theories with a formally positive action in the context of constructive field theory and it should be regarded as one of its a conceptual contributions, see (Benfatto *et al.*, 1978, 1980a,b; Glimm, 1968a,b; Glimm and Jaffe, 1968, 1970a,b; Magnen and Seneor, 1976; Nelson, 1966, 1973a,b,c; Nicolò, 1983).

The new ideas needed to deal with the problem of proving the existence of the limit (13.1) as $N \to \infty$ at fixed B are two:
(i) the *asymptotic freedom* and
(ii) *scale invariance.*
Their role and interplay in field theory seems to have been clearly realized as early as 1959 by Wilson, see (Wilson, 1971). It turns out that they are best illustrated in the theory of the cosine field.

Suppose that one wishes to study the distribution of the low frequency fields $\varphi^{(0)}, \ldots, \varphi^{(p)}$ in the restricted ensemble. Then the function

$$\begin{aligned} F^{(N)}(\varphi^{(0)}, \ldots, \varphi^{(p)}) &\stackrel{def}{=} [\prod_{j=0}^{p} \chi_j(\varphi^{(j)})] \cdot \\ &\cdot \int [\prod_{j=p+1}^{N} \chi_j(\varphi^{(j)})] e^{V(\varphi^{(\le N)})} \prod_{j=p+1}^{N} P(d\varphi^{(j)}) \end{aligned} \tag{13.7}$$

is the density of the distribution of $\varphi^{(0)}, \ldots, \varphi^{(p)}$ with respect to the measure $P(d\varphi^{(0)}) \cdots P(d\varphi^{(p)})$ and the first step is to show that it is an integrable function with respect to the measure $P(d\varphi^{(0)}) \cdots P(d\varphi^{(p)})$

From the above discussion on the relevance of the phase space "neglected" [see (13.2) and (13.6)], it is natural to think that the result of the integration in (13.7) should simply be

$$e^{V^{(k)}(\varphi^{(\le k)})} \tag{13.8}$$

up to corrections negligible as $\lambda \to 0$ and due to the presence of the characteristic functions in (13.1).

However this does not really make sense, because the theory of the preceding sections provides an asymptotic expansion in λ for $V^{(k)}$ which has little chance of being convergent.

The next best guess is that instead of (13.8) one gets, for a prefixed integer $t \ge 0$

$$e^{\left[V^{(k)}(\varphi^{(\le k)})\right]^{[t]} + \lambda^{t+1} R_t(\varphi^{(\le k)};\lambda)} \tag{13.9}$$

where $\left[\,\cdot\,\right]^{[t]}$ denotes the truncation of a power series in λ to order t and R_t represents a "remainder".

Therefore $\left[V^{(k)}(\varphi^{(\le k)})\right]^{[t]}$ will be given just by the perturbation theory developed in the preceding sections counting only trees with at most t end points; the choice of t in (13.9) is arbitrary provided that the remainder can be well estimated for the chosen t.

The validity of a result like (13.9) means that the integral of $e^{V(\varphi^{(\le N)})}$ over $\varphi^{(N)}, \ldots, \varphi^{(p+1)}$ can be performed successively by using perturbation theory; therefore in order to have any hope of proving (13.9) with reasonable bounds on the remainder it is necessary that $V^{(q)}(\varphi^{(\le q)}$ regarded as a "potential" on $\varphi^{(q)}$ at fixed $\varphi^{(q-1)}, \ldots, \varphi^{(0)}$ has a very small small size, at least on the restricted ensemble (13.4); actually, not only should its size be small, but it should even go to 0 as $q \to \infty$, *asymptotic freedom*, if $N = \infty$.

In order that the above property holds for all $q \le N$ it must of course hold for $q = N$. Hence the check of the property of asymptotic freedom starts with a check of its validity for $q = N$.

To explain what the above words concretely mean one considers the field $\varphi^{(N)}$ and remarks that, as discussed in Sec. 3, it can be regarded as smooth and essentially constant on cubes Δ of size γ^{-N}, which will be thought as extracted from a pavement Q_N of Λ with cubic tesserae of side length γ^{-N}. Furthermore the values of $\varphi^{(N)}$ on different tesserae are almost independent because of the exponential decay on scale γ^{-N} of the covariance of $\varphi^{N)}$.

This suggests writing the nonconstant (i.e. nontrivial) part of the interaction as a sum of contributions each coming from a given $\Delta \in Q_N$, i.e. as

$$\begin{aligned} &\sum_{\Delta \in Q_N} \lambda \int_\Delta :\, \cos\alpha\,(\varphi^{(\le N-1)}_\xi + \varphi^{(N)}_\xi)\, : d\xi \;=\; \sum_{\Delta \in Q_N} \cdot \\ &\cdot (\lambda\, e^{\frac{\alpha^2}{2} C^{(\le N)}_{00}} |\Delta|) \cdot \left| \int_\Delta \cos\alpha\,(\varphi^{(\le N-1)}_\xi + \varphi^{(N)}_\xi)\, \frac{d\xi}{|\Delta|} \right| \equiv \\ &\equiv \sum_{\Delta \in Q_N} \lambda \gamma^{(\frac{\alpha^2}{4\pi} - 2)N} S_\Delta, \end{aligned} \tag{13.10}$$

where use has been made of $C^{(\le N)}_{00} = (N+1)\frac{\log\gamma}{2\pi}$, $|\Delta| = \gamma^{-2N}$, and by the preceding arguments one regards the variables S_Δ, which are defined here and have "size of $O(1)$" because they are averages of a cosine, as random variables functions of the field $\varphi^{(N)}$ parameterized by the field $\varphi^{(\le N-1)}$ and $\varphi^{(j)}$, $j \le N-1$, are supposed to be in the set defined by (13.4).

The variables S_Δ can be thought of as continuous spins sitting on the lattice Q_N and the calculation of the integral

$$\int \chi_N(\varphi^{(N)}) e^{\sum_\Delta \lambda \gamma^{(\frac{\alpha^2}{4\pi} - 2)N} S_\Delta} P(d\varphi^{(N)}) \tag{13.11}$$

can be thought of as the problem of evaluating the partition function of a spin system, on the lattice Q_N, which is a perturbation by an energy $\lambda W = \sum_\Delta \lambda \gamma^{(\frac{\alpha^2}{4\pi} - 2)N} \overline{S}_\Delta$ of the "free measure"

$$P(\prod_\Delta d\overline{S}_\Delta) = \int \prod_\Delta \delta(S_\Delta - \overline{S}_\delta) \chi_N(\varphi^{(N)}) P(d\varphi^{(N)}), \tag{13.12}$$

which, intuitively, can be thought of an almost factorized measure with respect to the variables $\overline{S}_\Delta$.

So the problem of computing the integral (13.11) in terms of its value for $\lambda = 0$ can be interpreted as a statistical mechanics problem for a spin system of bounded, uncorrelated spins with a local perturbation whose size is

$$\lambda\gamma^{(\frac{\alpha^2}{4\pi}-2)N} \tag{13.13}$$

If $\alpha^2 < 8\pi$ one sees that the "effective coupling on the fields with frequency N" is (13.13) " and it goes to zero as $N \to \infty$: which means that the spin system is a "very high temperature" for large N, and one can very reasonably hope to use the high temperature expansion techniques of statistical mechanics to estimate perturbatively the integral (13.11). The result of such estimates is in general that

$$\int e^{\lambda W} dP = e^{\sum_{p=1}^{t} \frac{\lambda^p}{p!}\mathcal{E}^T(W;p)+R_t} \tag{13.14}$$

and

$$|R_t| \le \lambda^{t+1} \times \frac{\text{system volume in}}{\text{lattice spacing units}} \times \mathit{const} \tag{13.15}$$

It is therefore clear that the result of the integral (13.14) gives rise to a very complex new function of $\varphi^{(\le N-1)}$.

For this reason one does not say that a theory is asymptotically free just if the computation of the effective coupling constant for $\varphi^{(N)}$ gives a result tending to zero with $N \to \infty$ as in (13.13).

The correct definition of asymptotic freedom is set up by considering the main term of (13.9) and by interpreting it as a potential for $\varphi^{(k)}$ parameterized by $\varphi^{(lek-1)}$; then one computes the "effective coupling constant $\lambda_N(k)$" and says that the interaction is asymptotically free if

$$\lim_{k\to\infty}\lim_{N\to\infty}\lambda_N(k) = 0 \tag{13.16}$$

The self consistent nature of this condition being clear, one can hope to be really able to check (13.16) and use it to obtain good estimates on R_t.

Although the calculation, or estimates, of $\lambda_N(k)$ looks *a priori* much harder than the evaluation of $\lambda_N(N)$ performed above [see (13.13)], it turns out that one can easily estimate $\lambda_N(k)$ by the general theory of perturbations developed in the previous sections.

To obtain an estimate of $\lambda_N(k)$ one first needs its precise definition: in fact $[V^{(k)}(\varphi^{(\le k)})]^{[t]}$ no longer depends on a single constant which, as above for $k = N$, can be naturally related to N but it is rather a "many body" nonlocal interaction: is is a sum over the trees with $\le t$ end points of terms like [see (11.9),(12.31)] like

$$\begin{aligned}&\int \sum_{\boldsymbol\sigma} \sum_{\overline{\gamma},\,\boldsymbol\sigma(\overline{\gamma})=\boldsymbol\sigma} V(\xi_1,\ldots,\xi_n;\overline{\gamma})\cdot\\ &\cdot\big(:\cos(\alpha\,\varphi^{(\le k)}(\gamma)) - \delta_{Q_{\overline{\gamma}},0}\ :\big)\,d\xi_1\ldots d\xi_n\end{aligned} \tag{13.17}$$

where $\overline{\gamma}$ denotes a tree shape of degree n (i.e. with n end points), $|Bs$ are the charges at the end points of $\overline{\gamma}$ and $Q_{\overline{\gamma}} = \sum_i \sigma_i$ is the total charge of $\overline{\gamma}$.

To interpret (13.17) as a spin-spin interaction for a lattice spin system one has to recall the main property of $\varphi^{(k)}$ of being approximately constant and smooth on the scale γ^{-k} and of being independently distributed on the same scale (approximately, of course).

Therefore following the same philosophical principles already used above one splits (13.17) into a sum over all possible n–tuples of tesserae $\Delta_1,\ldots,\Delta_n \in Q_k$ of terms like

$$\begin{aligned}&\int_{\Delta_1,\ldots,\Delta_n} V(\xi_1,\ldots,\xi_n;\overline{\gamma})\cdot\\ &\cdot\big(:\cos(\alpha\,\varphi^{(\le k)}(\gamma)) - \delta_{Q_{\overline{\gamma}},0}\ :\big)\,d\xi_1\ldots d\xi_n\end{aligned} \tag{13.18}$$

Then one will interpret (13.8) as a many body interaction between the spins $(S_{\Delta_1},\ldots,S_{\Delta_n}) \simeq (\varphi^{(k)}_{\xi_1},\ldots,\varphi^{(k)}_{\xi_n}$ and check that (13.18) is bounded by

$$C_t\lambda_N(k)^n e^{-\kappa\gamma^k\, d(\Delta_1,\ldots,\Delta_n)} \tag{13.19}$$

uniformly in N and with $\lambda_N(k)$ and κ independent on the particular term like (13.18) contributing to the effective potential, and also independent of the considered expansion order t; here $C_t, \kappa > 0$ are suitable constants.

Then the constant $\lambda_N(k)$ will be naturally called the *effective coupling constant for the field* $\varphi^{(k)}$: the interaction (13.18) then susceptible to the very same interpretation as (13.10) in terms of continuous lattice spin systems.

All the technical work necessary to study the bounds (13.19) in the cosine field case has already been done in the proof of its renormalizability: in fact (12.25) with ρ_{v_0} replaced by $\rho_{v_0} + 2 - 2\varepsilon$, as explained after (12.31) immediately yields a bound on (13.18) of the form (13.10) with

$$\begin{aligned}C_t &= \overline{C}_t B^2, \\ \lambda_N(k) &= \lambda\,\gamma^{(\frac{\alpha^2}{4\pi}-2)\,k}(1+k)^{2a}(\log(e+k+\lambda^{-1})^4)\end{aligned} \tag{13.20}$$

and $\overline{C} + t$ is B–independent and $\kappa = \frac{\kappa_o}{4} > 0$, for all N provided ε in (13.4) is chosen so that $\rho_{v_0}+2-2\varepsilon > 0$—i.e. $\varepsilon \ll 2 - \frac{\alpha^2}{4\pi}$.

Therefore the cosine interaction is asymptotically free for $|a^2 < 8\pi$ provided it is correctly renormalized for $\alpha^2 \in [4\pi, 8\pi)$.

One should remark the deep difference between the cases $\alpha^2 < 4\pi$ and $\alpha^2 \in [4\pi, 8\pi)$: in the first case conditions (13.4) are not necessary to obtain (13.20) because the bounds (12.19) and (12.25) can be used and because they had been obtained without using the smoothness property of $\varphi^{(\le k)}$.

Such properties are necessary to obtain the improvement on (12.25) (i.e. $\rho_{v_0} \to\sim \rho_{v_0}+2$) needed to achieve ultraviolet stability for $\alpha^2 < 8\pi$. Recall that the improvement follows, after renormalization, only if $| \cos\alpha\,\varphi^{(\le k)}(\gamma) - 1$: is bounded by (12.31) and this is possible only if the smoothness condition in (13.4) holds [the boundedness of the fields is not really necessary and one could proceed without it].

Actually the latter remark shows that (13.20) can be improved by replacing C_t by a B–independent constant if $\alpha^2 < 4\pi$. One can also remark that while the proof of the proposition at the beginning of Sec. 13 on the existence of the P's easily implies, by the arbitrariness of B in (13.4) the complete construction of the cosine theory in the case $\alpha^2 < 4\pi$ this is no longer so for $\alpha^2 \ge 4\pi$, when the presence of the field cut-off introduced by the characteristic functions in (13.1) is really essential to have asymptotic freedom. In the latter case, $\alpha^2 \in [4\pi, 8\pi)$, new ideas are necessary to control the $B\to\infty$ limit.

The discussion of the latter limit will be postponed to Sec. 14 and in the present section no more differences will arise between the cases $\alpha^2 < 4\pi$ and $\alpha^2 \in [4\pi, 8\pi)$.

Having checked the asymptotic freedom for the cosine interaction one realizes that the partial solution of the ultraviolet problem provided by the proposition stated at the beginning of this section still requires the analysis of the many statistical mechanics problems (one per scale) of weakly coupled continuous lattice spin systems.

One can, in fact, regard as such the problem of performing the successive integrations over $\varphi^{(p)}$ of

$$\chi(\varphi^{(\le p)})\, e^{[V^{(p)}(\varphi^{(\le p)})]^{[}t]} \tag{13.21}$$

for $p = N, N-1, \ldots, k+1$.

The reason behind the feasibility of the above feat is the second important idea on the problem: the free fields $\varphi^{(0)}, \ldots, \varphi^{(N)}$ are identically distributed up to trivial scaling (see Sec. 3).

This means that, whatever p is, the integral (13.21) can be regarded as the computation of the partition function of the *same spin system* on a fixed lattice affected by a perturbation which is p dependent and which, by the asymptotic freedom property, has a p dependence becoming weaker as p becomes larger.

Therefore, as a matter of fact, one can perform the integral of (13.21) over $\varphi^{(p)}$ by trying to use the naive formula

$$\begin{aligned}&\mathcal{E}_p(\chi_p\, e^{[V^{(p)}(\varphi^{(\le p)})]^{[}t]}) = \\ &= e^{\left(\left[\sum_{j=1}^t \frac{1}{j!}\mathcal{E}_p^T([V^{(p)}(\varphi^{(\le p)})]^{[}t],j)\right]^{[t]} + \theta\,\lambda_N(p)^{t+1}\overline{R}_t\gamma^{2p}|\Lambda|\right)}\end{aligned} \tag{13.22}$$

[see (13.4)], where$|\theta| \le 1$ and $\overline{R}_t$ is a positive constant depending on C_t and κ [see (3.19)]l the factor γ^{2p} in front of the volume $|\Lambda|$ comes from the fact that the volume has to be measured on the scale on which the field $\varphi^{(p)}$ "lives" (see below).

The validity of (13.22) rests on the following lemma:

Lemma 1: *Formula (13.22) is valid for $p = 0$ id one replaces $[V^{(0)}]^{[t]}$ by a finite linear combination of expressions like (13.17) with $k = 0$ which are such that the integrals (13.18) are bounded by (13.19) with $p = 0$ and $\lambda_N(0)$ replaced by a free parameter λ.*

By the scale invariance of the multiscale decomposition such a lemma would then imply (13.22) for arbitrary p.

Accepting lemma 1, hence (13.22), remark that

$$\Big[\sum_{j=1}^t \frac{1}{j!}\mathcal{E}_p^T([V^{(p)}(\varphi^{(\le p)})]^{[}t],j)\Big]^{[t]} \equiv \big[V^{(p-1)}\big]^{[t]} \tag{13.23}$$

which is evident if one recalls the definition of the formal power series in λ for $V^{(p-1)}$ [see (5.13), (5.14) and the relations following them in Sec. 5].

Then since $\big[V^{(p-1)}\big]^{[t]}$ verifies the bound (13.19) with $p-1$ replacing p, provided $\varphi^{(0)}, \ldots, \varphi^{(p-1)}$ verify (13.4), the integral (13.7) is, recursively, estimated by

$$\begin{aligned}F^{(k)}(\varphi^{(0)}, \ldots, \varphi^{(k)}) = \big(\prod_{j=0}^k \chi_j(\varphi^{(j)})\big)\cdot \\ \cdot\, e^{\left([V^{(k)}]^{[t]} + \sum_{p=k+1}^N \theta\overline{R}_t\gamma^{2p}|\Lambda|\lambda_n(p)|^{t+1}\right)}\end{aligned} \tag{13.24}$$

where $|\theta| \le 1$ and the remainder is simply the sum of the remainders produced by successively integrating the fields $\varphi^{(N)}, \ldots, \varphi^{(k+1)}$ using (13.22), i.e. lemma 1 above.

So the remainder in (13.24) is bounded by

$$\begin{aligned}\overline{\overline{R}}_k|\Lambda| = |\Lambda|\overline{R}_t\sum_{p=k}^\infty \cdot \\ \cdot\big(\lambda\gamma^{(\frac{\alpha^2}{4\pi}-2)\,p}\,(1+p)^{2a}\,(\log(e+k+\lambda^{-1}))^4\big)\end{aligned} \tag{13.25}$$

This proves that $F^{(k)}$ [see (13.7)] is well defined and bounded uniformly in N if $\alpha^2 < 8\pi$: in fact it is enough to choose in (13.24) and (13.25) the arbitrary integer $t \ge 0$ to be not smaller than t_0 where t_0 is the first integer such that $(\frac{\alpha^2}{4\pi}-2)(t_0+1)+2 < 0$, so that $t_0 = 1$ if $\alpha^2 < 4\pi$, $t_0 = 2$ if $\alpha^2 \in [4\pi, \frac{16\pi}{3})$, $t_0 = 3$ if $\alpha^2 \in [\frac{16\pi}{3}, 6\pi)$, $t_0 = 4$ if $\alpha^2 \in [6\pi, \frac{32\pi}{5})$, *etc.*

If $F^{(k)}$ is well defined and bounded as N varies it follows from abstract analysis that there is a subsequence of the sequence of measures (13.1) which converges "weakly" to a limit P_λ as $N\to\infty$ for all values of $\lambda\in\mathbb{R}$; any such one parameter family will verify the properties in the proposition stated at the beginning of the section [there are many many sequences of measures (13.1) since we can change the parameter B in (13.4) or, more generally, since one can modify the choice of the characteristic functions]. I shall not discuss the details of such an analysis, since I consider it not too relevant to the heart of the matter treated here.

So the discussion of the proposition of the proposition at the beginning of this section is complete for the cosine interaction and rests on the above technical lemma; the lemma will not be proved here (although to do so is not particularly difficult since it is a "mean field theory bound" in its statistical mechanics interpretation, as the reader familiar with statistical mechanics can convince himself). The relevance of Lemma 1 for the ultraviolet problem from the constructive field theory point of view has been pointed out in (Benfatto *et al.*, 1978, 1980a,b; Gallavotti, 1978, 1979a,b) and then used by many workers who have often built it in as an important ingredient necessary in the development of more daring and deep ideas [see (Balaban, 1982a,b, 1983; Gawedski and Kupiainen, 1980, 1983, 1984), see (Westwater, 1980) for related ideas]. Some of the methods in (Gallavotti, 1978) in the brilliant papers on the hierarchical model in statistical mechanics [see (Bleher and Sinai, 1973, 1975) and (Collet and Eckmann, 1978) [these are the methods used to attack the model called the "hierarchical field" in (Gallavotti, 1978, 1979a)]; in some sense the role of the application of such methods to field theory was to point out the path to follow to apply renormalization group in constructive field theory using techniques already developed in statistical mechanics and taking almost literally the ideas introduced in statistical mechanics by (Wilson, 1970).

The proof of Lemma 1 can be found in a rudimentary form in (Gallavotti, 1978, 1979a) and in a complete form in (Benfatto *et al.*, 1978, 1980a,b) where a much stronger version (see Lemma 2 of Sec. 14 of this paper) is derived; in (Gallavotti, 1979a) lemma 1 is obtained by literally reducing it to a classical statistical mechanics problem of high temperature expansions for a system of weakly coupled spins, using the techniques of (Kunz, 1978; Sylvester, 1979) later improved in (Cammarota, 1982) [see (Seiler, 1982) for a review].

The proof in (Benfatto *et al.*, 1978, 1980a,b) has been criticized as unnecessarily too complex being based on "delicate" properties of higher order elliptic boundary value problems; I do not think that this criticism is justified. While it is true that one relies on properties of PDE's, interesting in themselves but technically involved, it should be stressed that the proof proposed in the above reference is conceptually very simple and intuitive and it also provides a general technique for the theory of Markov fields. The basic ideas behind the proof are explained in simple cases in (Gallavotti, 1981). A simpler account of the other earlier ideas can be found in (Gallavotti, 1979a,b). The detailed proofs of Lemma 1 presented in the above quoted papers should not mislead the reader into believing that they are much more than technical developments of a very simple probabilistic idea. I also believe that the so-called simpler proofs are either weaker or equivalently difficult, not surprisingly so by the well-known law of conservation of difficulties.

Field theory is a technical domain and I believe that all proofs there are equivalently hard and equivalent to the first proofs ever given on the same subjects; it is useful to devise new ones because they can lead to the more efficient organization of the proofs and to the intuition behind them, which seems an essential step for further progress.

xiv. Large deviations: their control and the complete construction of the cosine field beyond $\alpha^2 = 4\pi$

The work done in Sec. 13 solves in some sense the problem of the ultraviolet stability when the random fields into which one decomposes the free field are constrained to fluctuate by a finite amount. The amount of the allowed fluctuations is determined by the parameter B in (13.4).

One cannot easily take the limit $B \to \infty$ because (see Sec. 13) diverge with B diverge in general ($R_t \xrightarrow[B\to\infty]{} \infty$).

Actually this is the case for $\alpha^2 \in [4\pi, 8\pi)$ while for $\alpha^2 < 4\pi$, as already mentioned in Sec. 13, the properties (13.4) are not necessary to obtain bounds on the effective potentials and the error term in (13.24) is uniform in B [because in (13.20) the constant C_t can be taken independent of B; see the remark after (13.20)].

For $\alpha^2 < 4\pi$ it is therefore easy to let $B \to \infty$ and build a family P_λ, $\lambda \in \mathbb{R}$, of probability measures on the fields on $\mathbb{R}^2$, which verifies the properties of the proposition at the beginning of Sec. 13 but which is not concentrated on an ensemble of fields restricted by (13.4); this is a family of measures that can naturally be taken as defining the interacting cosine field for $\alpha^2 < 4\pi$; with some extra work it could also be proved that the limit as $N \to \infty$ of the interaction measure with $B = +\infty$ exists without any need of passing to subsequences and, hence, no nonuniqueness problems arise.

A complete theory for the cosine interaction for $\alpha^2 < 4\pi$ has been first worked out in (Frölich, 1976), where the infrared limit is also studied.

Much more interesting, as a field theory problem, is the case $\alpha^2 \in [4\pi, 8\pi)$. So far the possibility of removing the "field cut-off" B when $\alpha^2 \geq 4\pi$ has been really proved only in the interval; $\alpha^2 \in [4\pi, \frac{32\pi}{5}) \subset [4\pi, 8\pi)$; the values $\alpha^2 \in [\frac{32\pi}{5}, 8\pi)$ have not yet been reached because, as it will become clear soon, one has to find some suitable positivity property of the effective potential, and in (Benfatto *et al.*, 1982) and in (Nicolò, 1983) the positivity has been checked "by hands" rather than on the basis of a general algorithm; since the positivity requirements become more and more stringent as $\alpha^2 \to 8\pi$ it is impossible to take α^2 too close to 8π unless one understands in a simpler way why things seem to adjust to produce the right signs at the right moments.

I shall first discuss in some detail the mechanism which allows one to remove the field cut-off ($B \to \infty$) for $\alpha^2 \in [4\pi, \frac{16\pi}{3})$: this is the case in which the minimum value that can be given to t i n (13.24) is $t = 2$, as discussed in

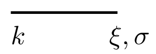
Sec. 13.

Since t is so small it is easy to write explicitly $[V^{(k)}(\varphi^{(\le k)})]^{[t]}$ in terms of the graphically eloquent tree language or even as a plain old-fashioned formula.

As an example one has, in the trees picture:

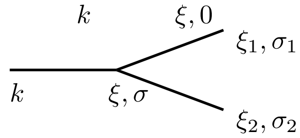
$$[V^{(k)}(\varphi^{(\le k)})]^{[2]} = \int \sum_{\substack{\gamma:\, k(\gamma)\le 2\\ degree\ \gamma\ge 2}} \frac{V(\gamma)}{n(\gamma)}\, d\boldsymbol{\xi} \tag{14.1}$$

and the $V(\gamma)$ are represented, if $\sigma = \pm 1$, by the following tree graphs



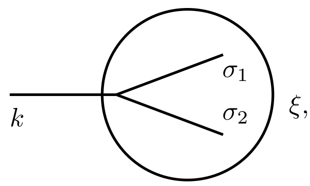
$$= \frac{\lambda}{2} \, : \, e^{i\alpha\sigma\varphi_\xi^{(\le k)}} : \tag{14.2}$$

$$= \nu \tag{14.3}$$



$$= \left(\frac{\lambda}{2}\right)^2 \left(e^{-\alpha^2 C^{(h)}_{\xi_1\xi_2}\sigma_1\sigma_2} - 1\right) e^{-\alpha^2 C^{(\le h-1)}_{\xi_1\xi_2}\sigma_1\sigma_2} \cdot \tag{14.4}$$

$$\cdot \, : \, \left(e^{i\,\alpha\,\sigma_2\varphi^{(\le k)}_{\xi_1} + \sigma_2\varphi^{(\le k)}_{\xi_2}} - \delta_{\sigma_1+\sigma_2,0}\right) \, :$$



$$= \left(\frac{\lambda}{2}\right)^2 \sum_{h=0}^{k} \int \left(e^{-\alpha^2 C^{(h)}_{\xi_1\xi_2}\sigma_1\sigma_2} - 1\right) e^{-\alpha^2 C^{(\le h-1)}_{\xi_1\xi_2}\sigma_1\sigma_2} d\xi_1 \tag{14.5}$$

and in (14.4), (14.5) the subtraction affects only the zero-charge trees ($\sigma_1 + \sigma_2 = 0$) as expressed by $\delta_{\sigma_1+\sigma_2=0}$; the combinatorial factor is 1 for (14.2) and (14.3) and 2! for (14.4) and (14.5).

If we sum over frequencies and charges the following analytic representation for $[V^{(k)}]^{[2]}$ emerges:

$$\begin{aligned}
[V^{(k)}]^{[2]} \equiv & \int_\Lambda \left(\lambda \, : \, \cos\alpha\varphi^{(\le k)}_\xi \, : + \nu\right) d\xi + \left(\frac{\lambda}{2}\right)^2 \int d\xi d\eta \cdot \\
& \cdot \left(e^{-\alpha^2 C^{(\le N)}_{\xi\eta}} - e^{-\alpha^2 C^{(\le k)}_{\xi\eta}}\right) \, : \cos\alpha(\varphi^{(\le k)}_\xi + \varphi^{(\le k)}_\eta) \, : + \\
& + \left(\frac{\lambda}{2}\right)^2 \int_\Lambda d\xi\, d\eta \left(e^{\alpha^2 C^{(\le N)}_{\xi\eta}} - e^{\alpha^2 C^{(\le k)}_{\xi\eta}}\right) \cdot \\
& \cdot \, : \cos\alpha(\varphi^{(\le k)}_\xi - \varphi^{(\le k)}_\eta) - 1 \, : - \\
& - \left(\frac{\lambda}{2}\right)^2 \int_\Lambda \left(e^{\alpha^2 C^{(\le k)}} - 1\right) d\xi d\eta
\end{aligned} \tag{14.6}$$

In this special case one represents the features of the general case discussed in Sec. 13: the only "dangerous term" is the third, big because of the $+\alpha^2$ in the exponential. However using the ideas of the preceding section one can check (as already done in general in Sec. 13) that its contribution to the effective coupling is

$$-\frac{\lambda^2}{4} \int_{\Delta^2} |\xi - \eta|^{\frac{\alpha^2}{2\pi}} \alpha^2 B_k^2 \,(\gamma^k |\xi - \eta|)^{2-2\varepsilon} d\xi d\eta = \lambda_N(k)^2 \tag{14.7}$$

where $e^{\alpha^2 C^{(\le N)}_{\xi\eta}}$ has been bounded, uniformly in N, by $C^{(\le N)}_{\xi\eta} \le \frac{1}{2\pi} \log|\xi - \eta|^{-1}$ and it has been assumed [see (3.16)] that for $\xi, \eta \in \Delta$

$$|\sin\frac{\alpha}{2}(\varphi^{(\le k)}_\xi - \varphi^{(\le k)}_\eta)| \le B_k\,(\gamma^k|\xi - \eta|)^{1-\varepsilon} \tag{14.8}$$

for some B_k; Δ is a cube of the pavement Q_k of Λ by cubes of side length γ^{-k}. Then the integral (14.7) is easily evaluated by a scale transformation of Δ to a unit box and, in conformity with the general bounds of Sec. 13, yields

$$\lambda_N(k)^2 \equiv \lambda^2 \gamma^{-4k} \gamma^{\frac{\alpha^2}{2\pi}k} B_k^2 \cdot const \simeq \left(\lambda\gamma^{(\frac{\alpha^2}{4\pi} - 2)k}\right)^2 B_k^2 \tag{14.9}$$

expressing the asymptotic freedom of the second order contribution to $V^{(k)}$, for $\alpha^2 < 8\pi$.

The problem of going beyond the formal perturbation theory is that one cannot neglect the region where (14.8) does not hold with B_k given by

$$B_k = B\,(\log(e + k + \lambda^{-1}))\,(1 + k)^a \tag{14.10}$$

as one would like to do on the grounds that, for $a \ge \frac{1}{2}$ the probability of field fluctuations violating (14.8) is exceedingly small, as described by the phase-space estimates (13.6).

In fact although such fluctuations are irrelevant in the description of the free field they might be enhanced in the interacting field case, because the potential $[V^{(k)}(\varphi^{(\le k)})]^{[2]}$ becomes very large (and, worse, its size is even N dependent, even for k small, in the region $(\xi, \eta) \in \Lambda^2$ where (14.8) is violated.

At this point one is often confronted with the statement "*well, the free field $\varphi^{(\le k)}$ will have a distribution which depresses the phase-space region where the free field distribution contains, among other things, a term $e^{-\frac{1}{2}\int_\Lambda (\partial\varphi_\xi)^2 d\xi}$*".

More precisely one refer here to the possibility of bounding the third term in (14.6) by using the inequalities $(1 - \cos x) \le \frac{x^2}{2}$ and

$$\left|e^{\alpha^2 C^{(\le N)}_{\xi\eta}} - e^{\alpha^2 C^{(\le k)}_{\xi\eta}}\right| \le const\, \frac{e^{-\kappa\gamma^k|\xi-\eta|}}{|\xi - \eta|^{\frac{\alpha^2}{2\pi}}} \tag{14.11}$$

which follows from the properties of $C^{(0)}_{\xi\eta}$. Denoting $\omega^k_{\xi\eta} \overset{def}{=} 1 - \cos\alpha(\varphi^{(\le k)}_\xi - \varphi^{(\le k)}_\eta)$, $\zeta^k_{\xi\eta} \overset{def}{=} (\varphi^{(\le k)}_\xi - \varphi^{(\le k)}_\eta)$, and using Lagrange's interpolation, one finds the bound

$$\begin{aligned}&\big|\lambda^2\int|\,\omega^k_{\xi\eta}\,|\big(e^{\alpha^2(C^{(\le N)}_{\xi\eta}-C^{(\le k)}_{\xi\eta})}\big)\,d\xi d\eta\,\big|\le\\&\le const\,\lambda^2\alpha^2\int|\zeta^k_{\xi\eta}\,|\,|\xi-\eta|^{-\frac{\alpha^2}{2\pi}}e^{-\gamma^k\kappa|\xi-\eta|}d\xi d\eta\le\\&\le const,\lambda^2\alpha^2\gamma^{2(\frac{\alpha^2}{4\pi}-2)k}\int|\,\partial\varphi^{(\le k)}_\xi|^2d\xi\end{aligned}\tag{14.12}$$

[the wick ordering in the third term in (14.6) has been neglected, as it is not very important, since the term comes from a zero charge tree], expressing the notion that the "bad term" in (14.6) is dominated by $\int|\,\partial\varphi^{(\le k)}_\xi|^2d\xi$ times a small constant, if k is large.

However the proof that follows controls the large part of (14.6) by a method *not reducible* just to the inequality (14.12) and making use of more detailed properties of the expression (14.6); this seems to be the reason why the proof below cannot be extended to cover the whole range $\alpha^2\in[4\pi,8\pi)$; this does not mean that a proof based just on the validity of the inequality (14.12) is not possible—and, indeed, one should look for it.

One checks that the region of the φ fields where (13.4) fails gives only a very small correction in the estimate of the error terms via the following argument.

Fix $B>1$ in (13.4) once and for all [see below] and $a>\frac12$ large (say $a=\frac32$; this parameter could probably be taken even equal to $\frac12$ by suitably refining the estimates below).

Given $\varphi^{(0)},\ldots,\varphi^{(k-1)},\varphi^{(k)}$, define

$$\begin{aligned}\mathcal{D}_k\;=\big\{\,\xi,\eta\in\Lambda\,|\,\sin\frac{\alpha}{2}(\varphi^{(\le k)}_\xi-\varphi^{(\le k)}_\eta)|>\\>\;B_k\,(\gamma^k|\xi-\eta|)^{1-\varepsilon}\big\}\end{aligned}\tag{14.13}$$

where $\varepsilon>0$ is the number in (13.4) fixed so that (13.20) holds [i.e. $\varepsilon\ll 2-\frac{\alpha^2}{4\pi}$]. Let $\mathcal{D}_{-1}=\emptyset$.

Define also the set

$$\begin{aligned}\mathcal{R}_k=\big\{\,\Delta\,|\,\Delta\in Q_k,\exists\,\xi\in\Delta,\eta\ \text{ with }\gamma^k|\xi-\eta|<1\text{ and}\\|\sin\frac{\alpha}{2}(\varphi^{(\le k)}_\xi-\varphi^{(\le k)}_\eta)|>\frac{B_k}{\sigma}(\gamma^k|\xi-\eta|)^{1-\varepsilon}\big\}\end{aligned}\tag{14.14}$$

where $\sigma>1$ will be conveniently chosen later. Then for $k\ge0$

$$\mathcal{D}_k\subset\mathcal{D}_{k-1}\cup(\mathcal{R}_k\times\mathcal{R}_k)\tag{14.15}$$

In fact let $(\xi,\eta)\in\mathcal{D}_k$ and $\xi\in\Delta,\eta\in\Delta'$. Suppose that $(\xi,\eta)\notin\mathcal{D}_{k-1}\cup(\mathcal{R}_k\times\mathcal{R}_k)$; then $\gamma^k|\xi-\eta|<B_k^{-1/(1-\varepsilon)}<B_k<1$ and

$$\begin{aligned}&|\sin\frac{\alpha}{2}(\varphi^{(\le k)}_\xi-\varphi^{(\le k)}_\eta)|<\frac{B_k}{\sigma}(\gamma^k|\xi-\eta|)^{1-\varepsilon},\\&|\sin\frac{\alpha}{2}(\varphi^{(k-1)}_\xi-\varphi^{(k-1)}_\eta)|<B_{k-1}(\gamma^{k-1}|\xi-\eta|)^{1-\varepsilon},\end{aligned}\tag{14.16}$$

otherwise $\Delta\in\mathcal{R}_k$ and $\Delta'\in\mathcal{R}_k$ so that either $(\xi,\eta)\in\mathcal{R}_k\times\mathcal{R}_k$ or $(\xi,\eta)\in\mathcal{D}_{k-1}$. But (14.16) implies, for $k\ge1$, the contradiction with $(\xi,\eta)\in\mathcal{D}_k$:

$$\begin{aligned}&|\sin\frac{\alpha}{2}(\varphi^{(\le k)}_\xi-\varphi^{(\le k)}_\eta)|\equiv\\&|\sin\frac{\alpha}{2}(\varphi^{(k-1)}_\xi-\varphi^{(k-1)}_\eta)|\cos\frac{\alpha}{2}(\varphi^{(k)}_\xi-\varphi^{(k)}_\eta)+\\&+\cos\frac{\alpha}{2}(\varphi^{(k-1)}_\xi-\varphi^{(k-1)}_\eta)\,\sin\frac{\alpha}{2}(\varphi^{(k)}_\xi-\varphi^{(k)}_\eta)|\le\\&\le\Big(B_{k-1}\gamma^{-(1-\varepsilon)}+\frac{B_k}{\sigma}\Big)(\gamma^k|\xi-\eta|)^{1-\varepsilon}\le\\&B_k(\gamma^k|\xi-\eta|)^{1-\varepsilon}\,(\gamma^{-(1-\varepsilon)}+\frac{1}{\sigma})\le B_k\,(\gamma^k|\xi-\eta|)^{1-\varepsilon}\end{aligned}\tag{14.17}$$

provided σ is chosen, as it can and will, so large that

$$\gamma^{-(1-\varepsilon)}+\frac{1}{\sigma}\le\theta<1,\qquad\forall k\ge1\tag{14.18}$$

The case $k=0$ is analogous, if $\varphi^{(-1)}\equiv0$.

Coming back to (14.15) assume, inductively, that it has been possible to prove that

$$\int e^{V(\varphi^{(\le N)})}P(d\varphi^{(\le N)})\cdots P(d\varphi^{(k+1)})\le e^{\hat V^{(k)}_\Lambda+R_+(k)|\Lambda|},\tag{14.19}$$

where denoting $\omega^{k,\pm}_{\xi\eta}\stackrel{def}{=}\alpha(\varphi^{(\le k)}_\xi\pm\varphi^{(\le k)}_\eta)$,

$$\begin{aligned}&\hat V^{(k)}_\Lambda\stackrel{def}{=}\lambda\int_\Lambda:\,\cos\alpha\varphi^{(\le k)}_\xi\,:\,d\xi+\\&+\frac{\lambda^2}{4}\int_{\Lambda^2}\big(e^{-\alpha^2C^{(\le N)}_{\xi\eta}}-e^{-\alpha^2C^{(\le k)}_{\xi\eta}}\big)\,:\cos\alpha\omega^{k,+}_{\xi\eta}\,:\,d\xi d\eta+\\&+\frac{\lambda^2}{4}\int_{\Lambda^2}\big(e^{\alpha^2C^{(\le N)}_{\xi\eta}}-e^{\alpha^2C^{(\le k)}_{\xi\eta}}\big):\cos\alpha\omega^{k,-}_{\xi\eta}-1:\,d\xi d\eta-\\&-\frac{\lambda^2}{4}\int_{\Lambda^2}\big(e^{\alpha^2C^{(\le k)}_{\xi\eta}}-1\big)\,d\xi\,d\eta\end{aligned}\tag{14.20}$$

i.e. one assumes that the part of the interaction which caused the worst problem in (14.6) is actually missing in (14.20) [and of course one will also assume, see below, a good bound on $R_+(k)$].

The reason this is not a terrible approximation is related to a special property of $\overline{V}^{(k)}\equiv[V^{(k)}(\varphi^{(\le k)})]^{[2]}$, whereby such bad terms, if present, would be very negative and therefore they could be really thrown out of the integration of the exponential of (14.20) because one is interested only in upper bounds (the lower bounds having been discussed in Sec. 13). The negativity of $\overline{V}^{(k)}-\hat V^{(k)}$ i.e. , if $\omega^{k,\pm}_{\xi\eta}\stackrel{def}{=}\alpha(\varphi^{(\le k)}_\xi\pm\varphi^{(\le k)}_\eta)$, of

$$\int_{\mathcal{D}_k}\big(e^{\alpha^2C^{(\le N)}_{\xi\eta}}-e^{\alpha^2C^{(\le k)}_{\xi\eta}}\big)\;:\cos\alpha\omega^{k,-}_{\xi\eta}-1:\,d\xi d\eta\equiv$$

$$\equiv \int_{\mathcal{D}_k} \Big(\big(e^{\alpha^2 C^{(\le N)}_{\xi\eta}} - e^{\alpha^2 C^{(\le k)}_{\xi\eta}} \big)\, e^{\alpha^2\,(C^{(\le k)}_{00}-C^{(\le k)}_{\xi\eta})} \cdot (\cos\omega^{k,-}_{\xi\eta} - 1) + (1 - e^{-\alpha^2\,(C^{(\le k)}_{00}-C^{(\le k)}_{\xi\eta})}) \Big)\, d\xi d\eta \tag{14.21}$$

holds, because in $\mathcal{D}_k$ it is

$$\begin{aligned} &-2\sin^2\frac{\alpha}{2}(\varphi^{(\le k)}_\xi - \varphi^{(\le k)}_\eta) + \alpha^2(C^{(\le k)}_{00} - C^{(\le k)}_{\xi\eta}) \le \\ &\le \big(-2B_k^2 + \alpha^2\overline{C}\big)(\gamma^k|\xi-\eta|)^{2-2\varepsilon} \end{aligned}\tag{14.22}$$

where we used, with $\overline{C}_\varepsilon, \overline{C}$ being suitable constants,

$$\begin{aligned} &C^{(\le k)}_{00} - C^{(\le k)}_{\xi\eta} = \sum_{j=0}^{k}(C^{(j)}_{00} - C^{(j)}_{\xi\eta}) \le \\ &\le \sum_{j=0}^{k}\overline{C}_\varepsilon(\gamma^j|\xi-\eta|)^{2-2\varepsilon} \le \overline{C}(\gamma^k|\xi-\eta|)^{2-2\varepsilon}. \end{aligned}\tag{14.23}$$

If B is supposed to be chosen so that $B_0^2 > \alpha^2\overline{C}$ it follows that the *r.h.s.* of (14.22) is bounded by

$$-B_k^2(\gamma^k|\xi-\eta|)^{2-2\varepsilon} < 0. \tag{14.24}$$

This remark makes it possible to neglect the interaction, or at least its bad part, in the regions where the field is rough and one can therefore use the free-field properties to prove this via rigorous bounds.

The precise way to make use of the above ideas to study the integral

$$e^{\hat V^{(k)}_\Lambda} P(d\varphi^{(k)}) \tag{14.25}$$

is the following. The first step in estimating (14.25) is to replace $\widehat{V}^{(k)\Lambda}$ by a simpler function, at least as far as functional dependence on $\varphi^{(k)}$ is concerned. Note that the $\varphi^{(k)}$ dependence of (14.20) is neither polynomial nor trigonometrical since $\varphi^{(k)}$ enters in a most complex way into the integration domains.

To find the simpler form that is sought imagine that Λ in (14.20) is replaced by an arbitrary set J and call $\widehat{V}^{(k)}_J$ the resulting expression. Then for suitably chosen $A, \overline{A}(\lambda)$:

$$\begin{aligned} \widehat{V}^{(k)}_J \le& \widehat{V}^{(k)}_{J/\mathcal{R}_k} + \mathcal{N}(\mathcal{R}_k)\cdot \\ &\cdot (\lambda\gamma^{(\frac{\alpha^2}{2\pi}-2)k} + \lambda^2\gamma^{2(\frac{\alpha^2}{2\pi}-2)k}B_k^2) = \\ =& \widehat{V}^{(k)}_{J/\mathcal{R}_k} + \mathcal{N}(\mathcal{R}_k)\,\overline{A}(\lambda)\gamma^{2(\frac{\alpha^2}{2\pi}-2)k}B^2 \end{aligned}\tag{14.26}$$

which follows immediately from the asymptotic freedom bounds (13.20), which in turn hold because $\varphi^{(\le k)}_\xi - \varphi^{(\le k)}_\eta$ is considered only in the region $(J\times J)/\mathcal{D}_k$: $\mathcal{N}(\mathcal{R}_k)$ is just the number of boxes composing $\mathcal{R}_k$.

Therefore (14.25) can be bounded above by

$$\sum_{\mathcal{R}_k}\int e^{\widehat{V}^{(k)}_{\Lambda/\mathcal{R}_k}}\chi(\mathcal{R}_k)P(d\varphi^{(\le k)})e^{\overline{A}(\lambda)B^2\gamma^{(\frac{\alpha^2}{4\pi}-2)k}\mathcal{N}(\mathcal{R}_k)} \tag{14.27}$$

where χ recalls that $\varphi^{(k)}$ is constrained to be such that the *r.h.s.* of (14.14) is precisely $\mathcal{R}_k$.

Then call H_J the expression obtained from $\widehat{V}^{(k)}_J$ by replacing $\mathcal{D}_k$ by $\mathcal{D}_{k-1}$; denoting $\omega^{k,\pm}_{\xi\eta} \stackrel{def}{=} \alpha(\varphi^{(\le k)}_\xi \pm \varphi^{(\le k)}_\eta)$ it is, therefore, by definition

$$\begin{aligned} H_{\Lambda/\mathcal{R}_k} =& \lambda\int_{\Lambda/\mathcal{R}_k} :\cos\alpha\varphi^{(\le k)}_\xi:\, d\xi + \frac{\lambda^2}{4}\int_{(\Lambda/\mathcal{R}_k)^2} d\xi d\eta\cdot \\ &\cdot (e^{-\alpha^2 C^{(\le N)}_{\xi\eta}} - e^{-\alpha^2 C^{(\le k)}_{\xi\eta}})\, :\cos\alpha\omega^{k,+}: + \frac{\lambda^2}{4}\cdot \\ &\cdot \int_{(\Lambda/\mathcal{R}_k)^2/\mathcal{D}_{k-1}} d\xi d\eta\,(e^{\alpha^2 C^{(\le N)}_{\xi\eta}} - e^{\alpha^2 C^{(\le k)}_{\xi\eta}})\cdot \\ &:\cos\alpha\omega^{k,+} - 1: - \frac{\lambda^2}{4}\int_{(\Lambda/\mathcal{R}_k)^2}(e^{\alpha^2 C^{(\le k)}_{\xi\eta}} - 1)\,d\xi d\eta \end{aligned}\tag{14.28}$$

And one checks that

$$\begin{aligned} \widehat{V}^{(k)}_{\Lambda/\mathcal{R}_k} \equiv& H_{\Lambda/\mathcal{R}_k} + \frac{\lambda^2}{4}\int_{\mathcal{S}_k}\big(e^{\alpha^2 C^{(\le N)}_{\xi\eta}} - e^{\alpha^2 C^{(\le k)}_{\xi\eta}}\big)\cdot \\ &\cdot :\cos\alpha(\varphi^{(\le k)}_\xi - \varphi^{(\le k)}_\eta) - 1:\, d\xi d\eta \end{aligned}\tag{14.29}$$

where $\mathcal{S}_k = (\Lambda/\mathcal{R}_k)^2 \cap (\mathcal{D}_{k-1}/\mathcal{D}_k)$ so that

$$\begin{aligned} &(\Lambda/\mathcal{R}_k)^2/\mathcal{D}_k = \\ &= \big\{\big[(\Lambda/\mathcal{R}_k)^2/\mathcal{D}_{k-1}\big] \cup \mathcal{S}_k\big\}/\big[(\Lambda/\mathcal{R}_k)^2 \cap (\mathcal{D}_k/\mathcal{D}_{k-1})\big] \end{aligned}\tag{14.30}$$

and the set $(\Lambda/\mathcal{R}_k)^2 \cap (\mathcal{D}_k/\mathcal{D}_{k-1})$ is empty because of $\mathcal{D}_k \subset \mathcal{D}_{k-1} \cup (\mathcal{R}_k \times \mathcal{R}_k)$ [see (14.15)].

Let $(\xi,\eta) \in \mathcal{S}_k \subset \mathcal{D}_{k-1} \cap (\Lambda/\mathcal{R}_k)^2$, $k \ge 1$; then it is $(\gamma^{k-1}|\xi-\eta|)^{1-\varepsilon}B_{k-1} < 1$, i.e.

$$(\gamma^{k-1}|\xi-\eta|)^{1-\varepsilon} < \gamma^{1-\varepsilon}B^{-1}_{k-1} \tag{14.31}$$

because the sine is bounded by 1. Hence for all $k \ge 1$ and $(\xi,\eta) \in \mathcal{S}_k$

$$\begin{aligned} &|\sin\frac{\alpha}{2}(\varphi^{(\le k)}_\xi - \varphi^{(\le k)}_\eta)| \ge |\sin\frac{\alpha}{2}(\varphi^{(k-1)}_\xi - \varphi^{(k-1)}_\eta)| - \\ &- |\cos\frac{\alpha}{2}(\varphi^{(k)}_\xi - \varphi^{(k)}_\eta)| - |\sin\frac{\alpha}{2}(\varphi^{(k)}_\xi - \varphi^{(k)}_\eta)| \ge \\ &\ge B_{k-1}\Big[1 - \big(\frac{B_k}{\sigma}(\gamma^{k-1}|\xi-\eta|)^{1-\varepsilon}\big)^2\Big]^{\frac12}\cdot \\ &\cdot (\gamma^{k-1}|\xi-\eta|)^{1-\varepsilon} - \frac{B_k}{\sigma}\gamma^{1-\varepsilon}(\gamma^{k-1}|\xi-\eta|)^{1-\varepsilon} \ge \\ &\ge B_k\Big[\frac{B_k-1}{B_k}\big(1 - \frac{B_k^2}{B_{k-1}^2\sigma^2}\big)^{\frac12} - \frac{\gamma^{1-\varepsilon}}{\sigma}\Big]\cdot \\ &\cdot (\gamma^{k-1}|\xi-\eta|)^{1-\varepsilon} \ge B_k\,\theta\,(\gamma^{k-1}|\xi-\eta|)^{1-\varepsilon}, \end{aligned}\tag{14.32}$$

where if we suppose (as we can) that σ is large enough, and use (14.11), θ is

$$\theta = \min_{k\ge1}\Big[\frac{B_k-1}{B_k}\big(1-\frac{B_k^2}{B_{k-1}^2\sigma^2}\big)^{\frac12}-\frac{\gamma^{1-\varepsilon}}{\sigma}\Big]>0 \quad (14.33)$$

The inequality between the first and the last terms can be checked directly also for $k=0$.

Therefore the integral in (14.29) is for all $k\ge0$ non-positive, provided [see also (14.22),(14.23)]

$$\begin{aligned}&-2\sin^2\frac{\alpha}{2}(\varphi^{(\le k)}_\xi-\varphi^{(\le k)}_\eta)+\alpha^2(C^{(\le k)}_{00}-C^{(\le k)}_{\xi\eta})\le\\ &\le\big(-2\theta^2B_k^2\gamma^{-2(1-\varepsilon)}+\alpha^2\overline{C}\big)(\gamma^k|\xi-\eta|)^{2-2\varepsilon}<0\end{aligned} \quad (14.34)$$

i.e. if B is supposed large enough, as is possible. Hence for all $k\ge0$

$$\widetilde V^{(k)}_{\Lambda/\mathcal{R}_k}\le H_{\Lambda/\mathcal{R}_k} \quad (14.35)$$

which implies, if we go back to (14.27),

$$\begin{aligned}&\int e^{\widetilde V^{(k)}_\Lambda}P(d\varphi^{(k)})\le\sum_{\mathcal{R}_k}\Big[\int\chi(\mathcal{R}_k)e^{H_{\Lambda/\mathcal{R}_k}}P(d\varphi^{(k)})\Big]\cdot\\ &\cdot e^{\mathcal{N}(\mathcal{R}_k)\overline{A}(\lambda)\gamma^{(\frac{\alpha^2}{4}-2)k}}\end{aligned} \quad (14.36)$$

The advantage of replacing $\widetilde V^{(k)}_{\Lambda\mathcal{R}_k}$ by $H_{\Lambda/\mathcal{R}_k}$ in (14.27) is that the function $H_{\Lambda/\mathcal{R}_k}$ is a "simple trigonometrical expression" in the fields $\varphi^{(k)}$ [see (14.28)], and no dependence is any more present on the very complicated set $\mathcal{D}_k$; of course there is a dependence on $\mathcal{R}_k$, but $\mathcal{R}_k$ is a union of cubes and therefore this dependence is not so bad; besides, one wishes to keep it fixed, as the integral (14.36) is performed at fixed $\mathcal{R}_k$ (because of the presence of the χ functions).

At this point one needs a way of estimating integrals like the one in (14.36). What is known about the integrand is that H_J can be written as

$$\begin{aligned}H_j=&\sum_{\Delta\in Q_k}\lambda\int_{\Delta\cap J}\cos\alpha\varphi^{(\le k)}_{\xi_1}\frac{d\xi_1}{|\Delta|}+\\ &+\sum_{\substack{\Delta_1,\Delta_2\in Q_k\\ \sigma_1,\sigma_1=\pm1}}\lambda^2\int_{(\Delta_1\times\Delta_2)\cap J^2}\frac{d\xi_1}{|\Delta_1}\frac{d\xi_2}{|\Delta_2|}\cdot\\ &\cdot h^{(2)}_{\sigma_1\sigma_2}(\xi_1,\xi_2)\cos\alpha(\sigma_1\varphi^{(\le k)}_{\xi_1}+\sigma_2\varphi^{(\le k)}_{\xi_2})+\\ &+\sum_{\Delta_1,\Delta_2\in Q_k}\lambda^2\int_{(\Delta_1\times\Delta_2)\cap J^2}\frac{d\xi_1}{|\Delta_1}\frac{d\xi_2}{|\Delta_2|}\cdot\\ &\cdot h^{(2)}(\xi_1,\xi_2)\frac{1-\cos\alpha(\varphi^{(\le k)}_{\xi_1}-\varphi^{(\le k)}_{\xi_2})}{(\gamma^k|\xi_1-\xi_2|)^{1-\varepsilon}}+\\ &+\sum_{\Delta_1,\Delta_2\in Q_k}\int_{(\Delta_1\times\Delta_2)\cap J^2}\frac{d\xi_1}{|\Delta_1}\frac{d\xi_2}{|\Delta_2|}h^{(0)}(\xi_1,\xi_2)\end{aligned} \quad (14.37)$$

where the $h^{(\cdot)}$ functions are λ independent and where the $\mathcal{D}_{k-1}$ dependence can be thought as included in the h functions. Furthermore the theory of the preceding sections or the explicit expressions for the h functions [see (14.28)] imply that

$$\int_\Delta|\lambda|\,|h(\xi_1)|\,d\xi_1\le\widetilde A\,\lambda\gamma^{(\frac{\alpha^2}{4\pi}-2)\,k}\le\overline{H}_k, \quad (14.38)$$

$$\begin{aligned}&\int_{\Delta_1\times\Delta_2}\lambda^2\,|h^{(2)}(\xi_1,\xi_2)|\,d\xi_1\,d\xi_2\le\\ &\le\big(\lambda\gamma^{(\frac{\alpha^2}{4\pi}-2)\,k}\big)\,\overline{A}\,B_k^2e^{-\kappa\gamma^kd(\Delta_1,\Delta_2)}\le\overline{H}_k\end{aligned} \quad (14.39)$$

where $\widetilde A,\overline{A},\overline{H}_k$ are suitably chosen constants.

In other words at fixed $\mathcal{R}_k$ the integral in (14.36) looks like the partition function of a classical spin system on the lattice Q_k.

The reason the estimates (14.38), (14.39) do not depend on $\varphi^{(k-1)}$ is that in the "bad terms" of H_Λ *no pair* $(\xi,\eta)\in\mathcal{D}_{k-1}$ appears, so that (14.39) is obtained by the same estimates leading to the proof of asymptotic freedom (and actually follows from them) is Sec. 13.

It is possible to formulate a rather general version of the Mayer expansion allowing one to estimate naively the integral (14.36).

Given $\Delta\in Q_k$ let $\chi^b_\Delta,\overset{\circ}{\chi}{}^b_\Delta\overset{def}{=}1-\chi^b_\Delta$ be the characteristic functions of the events on $\varphi^{(k)}$

$$\begin{aligned}\{\varphi^{(k)}\,|\,&\exists\,\xi,\eta\in\Delta\text{ such that}\\ &|\varphi(k)_\xi-\varphi(k)_\eta|<b\,(\gamma^k|\xi-\eta|)^{1-\varepsilon}\}\end{aligned} \quad (14.40)$$

and of its complement, respectively.

Let R be a subset of Λ pavable by Q_k, i.e. union of Δ's in Q_k, and let R^c be its complement; denote $\chi_R\overset{def}{=}\prod_{\Delta\subset R}\chi_\Delta$ if R is the disjoint union of cubes $\Delta\subset R$; then the following lemma, closely related to Lemma 1, Sect. 13, holds.

Lemma 2. *Given an integer $t\ge0$ there exist constants G,g,g',b^*, depending only on t and the parameters $\gamma,\varepsilon,\kappa$, such that if H_Λ verifies (14.38)(14.40) then*

$$\begin{aligned}&\int P(d\varphi^{(k)})\,\chi^b_{R^c}\,\overset{\circ}{\chi}{}^b_R\,e^{H_{\Lambda/R}}\le\Big[\int\overset{\circ}{\chi}{}^b_RP(d\varphi^{(k)})\Big]^{\frac12}\cdot\\ &\qquad\cdot e^{\left(\left[\sum_{p=1}^t\mathcal{E}^T_k(H_\Lambda;p)\,p!^{-1}\right]^{[t]}\right)}\cdot\\ &\qquad\cdot e^{\left(\delta(b,\overline{H}_k)\,\gamma^{2k}|\Lambda|+\delta'(b,\overline{H}_k)\,\mathcal{N}(R)\right)}\end{aligned} \quad (14.41)$$

where the errors have a value close to the one which would be naively expected from the point of view of statistical mechanics:

$$\begin{aligned}&\delta(b,\overline{H}_k)\le G\left((\overline{H}_k\,b^g\,e^{\overline{H}_kb^g})^{t+1}+e^{-g'\,b^2+g\,\overline{H}_k\,b^g}\right)\\ &\delta'(b,\overline{H}_k)\le G\,\overline{H}_kb^g,\end{aligned} \quad (14.42)$$

and $\mathcal{N}(R)$ is the number of cubes Δ in R. Furthermore if b is large enough, $b > b^*$:

$$\int P(d\varphi^{(k)})\,\chi_\Lambda^b\,e^{H_\Lambda} \geq \geq e^{\left(\left[\sum_{p=1}^t \mathcal{E}_k^T(H_\Lambda;p)\,p!^{-1}\right]^{[t]}\right)-\delta(b,\overline{H}_k)\,\gamma^{2k}|\Lambda|}, \tag{14.43}$$

and finally, for suitably chosen and suitably chosen constants α_0, β_0:

$$\int P(d\varphi^{(k)})\,\mathring{\chi}_R^b \leq \left(\alpha_0 e^{-\beta_0\,b^2}\right)^{\mathcal{N}(R)} \tag{14.44}$$

The k dependence of the constants is trivially due to the scaling properties of the field. The first bound in (14.42) could be easily improved: here it is given in the form in which it had been found in (Benfatto *et al.*, 1978, 1980a,b, 1982) where Lemma 2 is proved under the extra assumption that γ is close to 1 (an assumption which can be easily released but which is sufficient for our purposes since γ is restricted only by $\gamma > 1$).

Clearly (14.43) implies as a special case Lemma 1 of 13. Lemmas 1 and 2 will not be proved here, because their statistical mechanical character makes them somewhat foreigners to field theory; also a detailed proof would be very long in spite of its conceptual simplicity; the reader can find it in the references given above.

At this point it is not difficult to conclude the estimates. First remark that in the present case

$$\Big[\sum_{p=1}^{2}\mathcal{E}_k^T(H_\Lambda;p)\,p!^{-1}\Big]^{[2]} \leq \widehat{V}^{(k-1)}(\varphi^{(k-1)}) \tag{14.45}$$

because the *l.h.s.* , being the result of a Gaussian integral of simple trigonometric functions, can be explicitly computed; after the simple calculation one finds that the difference between the *r.h.s.* and the *l.h.s.* is given exactly by

$$\int_{\mathcal{D}_{k-1}} d\xi\,d\eta\,\left(e^{\alpha^2 C_{\xi\eta}^{(\leq k)}} - e^{\alpha^2 C_{\xi\eta}^{(\leq k-1)}}\right)\cdot \cdot\,:\,\cos\,\left[\alpha\,(\varphi_\xi^{(\leq k-1)} - \varphi_\eta^{(\leq k-1)})\,-1\right]: \tag{14.46}$$

which is not positive for the same reasons the (14.21) and (14.22) were not positive.

Therefore one applies Lemma 2 to evaluate the integral (14.36), choosing $t = 2$ and $b = B_k$; the result, by using also (14.45) and (14.46), is

$$\int e^{\widehat{V}^{(k)}}P(d\varphi^{(k)}) \leq \sum_{R_k} e^{\widehat{V}^{(k-1)}+\gamma^{2k}\delta(B_k,\overline{R}_k)}\cdot \cdot\left(\alpha_0 e^{-\beta_0 B_k^2}\right)^{\mathcal{N}(R_k)/2}\,e^{\mathcal{N}(R_k)\delta'(B_k,\overline{R}_k)} \equiv \tag{14.47}$$

$$\equiv e^{\widehat{V}^{(k-1)}}e^{\gamma^{2k}\delta(B_k,\overline{R}_k)}\left(1+\alpha_0^{\frac12}e^{-\frac12\beta_0 B_k^2}e^{\delta'(B_k,\overline{R}_k)}\right)^{|\Lambda|\gamma^{2k}} \equiv$$
$$\equiv e^{\widehat{V}^{(k-1)+\varepsilon(k)|\Lambda|}}$$

and, by (14.42),(14.39), and (14.38), $\sum_{k=0}^\infty \varepsilon(k) = O(\lambda^t)$.

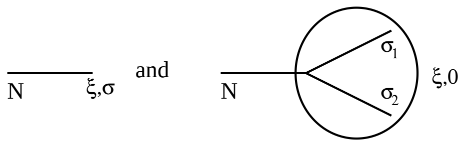
This means that if one assumes (14.19) for $k = N-1$ and (14.29) holds then for all $k \geq K(\lambda)$, the (14.19) holds with

$$R_+(k) = R_+(k+1) + \varepsilon(k) \tag{14.48}$$

Hence the ultraviolet stability will be proved as soon as one shall have been able to check (14.29) for $k = N-1$— i.e. to obtain an estimate of

$$\int e^{V(\varphi^{(\leq N)})}\,P(\varphi^{(N)}). \tag{14.49}$$

In the latter case the $V(\varphi^{(\leq N)})$ is just the sum of the trees



(36)

i.e.

$$\lambda\int_\Lambda\,:\,\cos(\alpha\,\varphi_\xi^{(\leq N)})\,d\xi + \nu\int_\Lambda d\xi - \left(\frac{\lambda}{2}\right)^2\cdot \cdot\int_\Lambda d\xi d\eta\left(e^{\alpha^2 C_{\xi\eta}^{(\leq N)}}-1\right) \equiv \sum_{\Delta\in Q_N}\lambda\,\gamma^{(\frac{\alpha^2}{4\pi}-2)N}\cdot \cdot\int_\Lambda \cos(\alpha\varphi_\xi^{(\leq N)})\,\frac{d\xi}{|\Delta|} + \nu\sum_\Delta \gamma^{-2N} - -\sum_\Delta(\lambda\gamma^{(\frac{\alpha^2}{4\pi}-2)N})^2\,h \tag{14.50}$$

where h is

$$\int_{\Delta\times\Delta}\gamma^{-\frac{\alpha^2}{2\pi}N}e^{\alpha^2 C_{\xi\eta}^{(\leq N)}}\left(1-e^{-\alpha^2 C_{\xi\eta}^{(\leq N)}}\right)\frac{d\xi d\eta}{|\Delta|} \tag{14.51}$$

Hence in the first step one does not have to worry about $\mathcal{D}_N/\mathcal{D}_{N-1}$, because there is no obstacle in using Lemma 2 to evaluate the integral (14.49): assumptions (14.38) and (14.39) are satisfied with $H_\Lambda \equiv V^{(N)} \equiv V$, by (14.50) and (14.51); i.e. in the first step there is no need to worry about the smoothness of $\varphi^{(N)}$ in order to get the asymptotic freedom bounds, as explicitly remarked in Sec. 13 (see comments following (13.13)).

The validity of the inductive hypothesis for $k = N-1$ is completed by checking that

$$\Big[\sum_{p=1}^{2}\frac{1}{p!}\mathcal{E}_N^T(V^{(N)};p)\Big]^{[2]} \leq \widehat{V}^{(N-1)} \tag{14.52}$$

which is proved as (14.45) by (14.21) and (14.52) written for $k = N-1$.

This completes the proof for $\alpha^2 < \frac{16}{3}\pi$.

There would be no problem in applying the above techniques to evaluate the integral of $e^{V^{(N)}}$ to an arbitrarily fixed order $t > 2$.

If $\alpha^2 \in [\frac{16}{3}, 6\pi)$ still nothing , basically, changes in the above scheme of proof except that the series of the errors, both in the upper ad lower bounds, will converge only if $t \geq 3$; the positivity of the "bad terms" was used in an essential way in the above proof in two steps and now it can be used in the same way. In fact the two steps were, first, to remove the region $\mathcal{D}_{k-1}$ from $\widehat{V}_{\Lambda/R_k}$ (done in (14.35)) and, second, to "rebuild" $\widehat{V}^{(k-1)}$ (done in (14.45) and (14.52)).

The just mentioned two "positivity steps" are now, for $\alpha^2 \in [\frac{16}{3}, 6\pi)$, carried through in the same way, because it turns out that no new positivity property is needed on $\widehat{V}^{(k)}$ besides the one, already pointed out and amply used, present in the second order part of $\widehat{V}^{(k)}$: the second order dominates in the inequalities necessary to control the third order terms and its positivity properties are enough for the estimates.

The situation changes for $\alpha^2 \geq 6\pi$: now the second order dominates only in the inequalities necessary to carry out the first of the two steps of the proof where the positivity is needed. In the second step it is not known whether it dominates; in fact the proof has been carried through in the interval $[6\pi, (\sqrt{17}-1)\pi)$ and, later, up to $\frac{32}{5}\pi$ by using other ideas slightly improving on the above ones, based on detailed properties of the effective interaction to fourth order in (Benfatto *et al.*, 1982) and in (Nicolò, 1983).

In order to obtain ultraviolet stability up to $\alpha^2 < 8\pi$ some new idea seems necessary, and the paper (Nicolò, 1983) seems to go in the right direction; see also the comment after (14.12) above

The above difficulties are also an indirect consequence of the fact that the large–fluctuations problem has not been solved in a naive way, by free field domination (see comments after (14.21)), and a better understanding of this point seems important and desirable.

The techniques used for the sine–Gordon equation can be used also to treat the exponential interaction (5.5) for $\alpha^2 < 4\pi$ (see (Frölich, 1976)); the exponential interaction can be treated also for $\alpha^2 \gg 4\pi$, for $d = 2$, or for $d \geq 3$, which are cases in which it can be proved to be trivial (see (Albeverio *et al.*, 1979)).

XV. The cosine field and the screening phenomena in the 2-dimensional Coulomb gas and in related Statistical Mechanical systems

Before studying the φ^4 fields it is appropriate to conclude the theory of the cosine fields by pointing out their "surprising" connection with the two–dimensional classical statistical mechanics of Coulomb systems and Yukawa gases.

The "neutral Coulomb gas" and the "charged Yukawa gas" describe charged particles of charge ± 1 presenting, for some values of temperature and density very interesting and non trivial "screening phenomena".

In general a system of charged particles interacting via a potential C_{xy} will be defined by the grand canonical partition function $Z^1(\Lambda, b, \lambda)$

$$\sum_{n=0}^{\infty} \big(\frac{\lambda}{2}\big)^n \frac{1}{n!} \sum_{\sigma_1,\ldots,\sigma_n} \int_{\Lambda^n} e^{-\beta \sum_{i<j} \sigma_i \sigma_j C_{x_i,x_j}} \, dx_1 \ldots dx_n \tag{15.1}$$

with $\sigma_j = \pm 1$, or $Z^2(\Lambda, \beta, \lambda)$, for *a priori* neutral systems:

$$\sum_{n=0}^{\infty} \big(\frac{\lambda}{2}\big)^n \frac{1}{n!} \sum_{\substack{\sigma_1,\ldots,\sigma_n \\ \sum_i \sigma_i = 0}} \int_{\Lambda^n} e^{-\beta \sum_{i<j} \sigma_i \sigma_j C_{x_i,x_j}} \, dx_1 \ldots dx_n \tag{15.2}$$

The cases which can be studied in terms of the cosine interaction are

(a) The "regularized Yukawa gas", with C_{xy} given by

$$C^{m_0,M}_{xy} = \int e^{i\,p\,(x-y)} \big(\frac{1}{m_0^2 + p^2} - \frac{1}{M^2 + p^2}\big) \frac{dp}{(2\pi)^2} \tag{15.3}$$

(b) The "regularized Coulomb gas" given by

$$V^{(m_0,M)}_{xy} = C^{(0,M)}_{xy} - C^{(0,m_0)}_{xy} \tag{15.4}$$

where the *r.h.s.* has to be interpreted as the limit of $C^{(m,M)}_{xy} - C^{(m,m_0)}_{xy}$ as $m \to 0$; i.e. as

$$\int \frac{dp}{(2\pi)^2} \big(\frac{M^2}{M^2 + p^2} \cos p\,(x-y) - \frac{m_0^2}{m_0^2 + p^2}\big) \tag{15.5}$$

Note that when the regularization parameter M is let to $+\infty$ it is:

$$V^{(m_0,M)}_{xy} \xrightarrow[M\to\infty]{} \frac{1}{2\pi} \log(a m_0 |x-y|)^{-2} \tag{15.6}$$

where $a > 0$ is a suitable constant ($a = \log 2 - g$, g being the Euler-Mascheroni constant). We set $\overline{m}_0 \stackrel{def}{=} a m_0$.

The partition function for the above systems can be easily written in terms of a Gaussian random field $\psi^{(-R,N)}$, sum of $N + R + 1$ independent fields:

$$\psi^{(-R,N)}_{\xi} = \sum_{j=-R}^{N} \psi^{(j)}_{\xi} \tag{15.7}$$

and in terms of the functions

$$: \cos \alpha \psi^{(-R,N)}_{\xi} :_{UV} \equiv e^{\frac{1}{4}\alpha^2 C^{(m_0,m_N)}_{00}} \cos \alpha \psi^{(-R,N)}_{\xi} \tag{15.8}$$

where $m_N \stackrel{def}{=} m_0\gamma^{N+1}$.

The covariance $C^{(j)}$ of the random field $\psi^{(j)}$ will have Fourier transform (of a free field with open boundary conditions, see Sec. 3)

$$\frac{1}{m_0^2\gamma^{2j}+p^2}-\frac{1}{m_0^2\gamma^{2j+2}+p^2}, \tag{15.9}$$

Then one checks that the regularized Yukawa gas partition function $Z^1_{Y,N}(\Lambda.\beta.\lambda)$ is, if we set $\alpha=\sqrt{\beta}$, $M=m_0\gamma^{N+1}$,

$$\int e^{\lambda\int :\cos\alpha\psi^{(-R,N)}_\xi:_{UV}\,d\xi} P(d\psi^{(-R)})\ldots P(d\psi^{(N)}) \tag{15.10}$$

while the regularized Coulomb gas partition function in the neutral grand canonical ensemble and with potential (15.4) with $M=m_0\gamma^{N+1}$ and with m_0 replaced, for notational convenience, by $m_0\gamma^{-1}$, is $Z^0_{C,N}(\Lambda,\beta,\lambda)$ given by

$$\lim_{R\to\infty}\int e^{\lambda\int :\cos\alpha\psi^{(-R,N)}_\xi:_{UV}\,d\xi} P(d\psi^{(-R)})\ldots P(d\psi^{(N)}) \tag{15.11}$$

It is convenient to introduce also the auxiliary partition function $Z^1_{C,R,N}(\Lambda,\beta,\lambda)$

$$\int e^{\lambda\int :\cos\alpha\psi^{(-R,N)}_\xi:_{UV}\,d\xi} P(d\psi^{(-R)})\ldots P(d\psi^{(N)}) \tag{15.12}$$

which will be called "infrared regularized (non-neutral) Coulomb gas partition function, and (15.11) can then be written

$$Z^0_{C,N}(\Lambda,\beta,\lambda)=\lim_{R\to\infty} Z^1_{C,R,N}(\Lambda,\beta,\lambda) \tag{15.13}$$

Finally we remark the following relation between the Coulomb gas and the Yukawa gas (see (15.10)):

$$Z^1_{Y,N}\equiv Z^1_{C,0,N} \tag{15.14}$$

The proof of (15.10)–(15.15) has essentially already been explained in Sec. 11 (and called there the "multipole expansion"); however the interpretation work necessary to derive the present statement from Sec. 11 is such that it is simpler to derive the above relations from scratch.

Consider the integral in (15.10) and expand the exponential in powers: calling, for simplicity, $C^{(-R,N)}\equiv C^{(m_0\gamma^{-R},m_0\gamma^{N+1})}$ the covariance of $\psi^{(-R,N)}$, we see that

$$\begin{aligned}&\sum_{p=0}^\infty\frac{\lambda^p}{2^p\,p!}\sum_{\sigma_1,\ldots,\sigma_p} e^{\frac12\alpha^2\,p\,C^{[0,N]}_{00}}\int dx_1\ldots dx_p\cdot\\ &\cdot\mathcal{E}\Big(\prod_{j=1}^p e^{i\alpha\sigma_j\psi^{(0,N)}_{x_j}}\Big)\equiv\sum_{p\ge0,\,\sigma_1,\ldots,\sigma_p}\frac{\lambda^p}{2^p\,p!}\int dx_1\ldots\cdot\\ &\cdot e^{-\frac12\alpha^2\sum^{1,n}_{i,j}\sigma_i\sigma_jC^{[0,N]}_{00}}=Z^1_{Y,N}(\Lambda,\beta,\lambda)\end{aligned} \tag{15.15}$$

because the diagonal terms ("self energy"terms) in the sum $\sum_{i,j=1}^p$ are canceled by the exponential factor outside the integral; in the first step of (15.15) the formulae for the Wick ordering of the cosine and for the expectation $\mathcal{E}$ of the exponentials have been used (see Appendix A3 and Sec. 11).

Recalling that the cancellation was due to the exponential factor due to the Wick ordering, we see that the evaluation of the integral (15.12) by the same technique will lead to expressions of the *r.h.s.* of (15.12) as

$$\begin{aligned}&\sum_{p=0}^\infty\frac{\lambda^p}{2^p\,p!}\sum_{\sigma_1,\ldots,\sigma_p}\int dx_1\ldots dx_p\cdot\\ &e^{-\alpha^2\sum_{i<j}C^{[0,N]}_{x_i,x_j}-\frac12\alpha^2\sum_{i,j=1^p}C^{[-R,-1]}_{x_i,x_j}}\end{aligned} \tag{15.16}$$

because the $:\cdot:_{UV}$ in (15.8) is a "partial Wick ordering" and therefore it can produce the cancellation of the diagonal terms of only the "ultraviolet part of the potential", i.e. $C^{(0,N)}_{xy}$.

Expression (15.16) can be rewritten as

$$\begin{aligned}&\sum_{p=0}^\infty\frac{\lambda^p}{2^p\,p!}\sum_{\sigma_1,\ldots,\sigma_p}\int dx_1..dx_p e^{-\alpha^2\sum_{i<j}\sigma_i\sigma_jC^{(0,N)}_{x_ix_j}}\cdot\\ &\cdot e^{-\frac12\alpha^2\sum_{i,j=1}^p(C^{(-R,-1)}_{x_ix_j}-C^{(-R,-1)}_{00})}\cdot\\ &e^{-\frac12\alpha^2(\sum_{i=1}^p)^2C^{(-R,-1)}_{00}}\equiv\sum_{p=0}^\infty\frac{\lambda^p}{2^p\,p!}\sum_{\sigma_1,\ldots,\sigma_p}\int dx_1..dx_p\cdot\\ &\cdot e^{-\alpha^2\sum_{i<j}(C^{(R,N)}_{x_ix_j}-{}^{(R,N)}_{00})\sigma i\sigma_j}\cdot e^{-\frac12\alpha^2Q^2_{\boldsymbol\sigma}C^{(-R,-1)}_{00}}\end{aligned} \tag{15.17}$$

with $Q_{\boldsymbol\sigma}=\sum_i\sigma_i$. In this way one obtains an expression for $Z^1_{C(-R,N)}$, Eq. (15.12), implying (15.11) with Coulomb potential $V^{\overline{m}_0,M)}$ with $M=m_0\gamma^{N+1}$, $\overline{m}_0=m_0\gamma^{-1}$ (the later choice is a matter of notational convenience, γ being fixed).

It is expected that the "neutral Coulomb gas" described by

$$Z^0_C(\Lambda,\beta,\lambda)=\lim_{N\to\infty} Z^0_{C(N)}(\Lambda,\beta,\lambda) \tag{15.18}$$

is a well defined thermodynamical system, exhibiting some kind of screening phenomena, for $\alpha^2<4\pi$; basically it should behave as a neutral Yukawa gas with m_0 determined by λ,α (and $M=+\infty$) at least for α^2 small (see (Brydges, 1978), (Frölich and Spencer, 1981)).

For $\alpha^2\in[4\pi,8\pi)$ one expects that the Coulomb as "collapses in the ultraviolet", remaining nontrivial in the sense that the collapse produces just a background of multipoles on which free charges move and interact through nontrivial screening phenomena (note that in two dimensions the Coulomb gas interaction does not go to zero at infinity and, for $\alpha^2 \in [4\pi, 8\pi)$ it even diverges too fast near zero, making the partition function infinite because it involves integrating the non summable factor $|x-y|^{-\alpha^2/2\pi}$).

The same collapse is expected to happen to the Yukawa gas in the same region of α^2 except that no screening in the infrared is necessary in order for the system to exhibit well defined thermodynamic behavior (in fact the potential decays exponentially at infinity as a consequence of the choice $m_0 > 0$, which gives a meaning to m_0^{-1} as a natural screening length); however screening phenomena are expected to occur in the ultraviolet region where the Yukawa gas should collapse in the same way as the Coulomb gas, i.e. by producing an infinite density background of multipoles on which free charges move.

In other words the conjecture is that the Coulomb gas (nonregularized in the infrared and in the ultraviolet) and the Yukawa gas (nonregularized in the in the ultraviolet) with parameters λ, α, m_0 describe the same physical phenomena, or at least have partially overlapping physical interpretations, if the Yukawa range m_0^{-1} is suitably chosen as a function of λ, α for $\alpha^2 \in [0, 8\pi)$.

For $\alpha^2 > 8\pi$ it is believed that the nonregularized Yukawa gas is trivial (i.e. it collapses "without hope") and the Coulomb gas no longer exhibits infrared phenomena of any kind; at least not so strong to produce exponentially decaying effective interactions or correlations.

The work done in Secs. 11 and 12 on the cosine interaction allows one to make rigorous some of the above conjectures, though much work remains to be done towards the complete understanding of the whole theory.

In the case of the Yukawa gas the above mentioned connection (the sine-Gordon transformation) between the Yukawa gas and the cosine field allows one to translate the properties of stability of the Yukawa gas in the region $\alpha^2 \in [0, 32\pi/5)$, where the cosine field stability id under control. And for $\alpha \in [3\pi, 32\pi/5)$ the above mentioned interpretation of the Yukawa gas as a gas with infinite density of collapsed dipoles (for $\alpha^2 \in [4\pi, 6\pi)$) and of dipoles and quadrupoles (for $\alpha^2 \in [6\pi, 32\pi/5)$), with zero total charge, emerges quite clearly. I do not enter here into the details of this interpretation of the results of the theory of the cosine interaction: the work begun in (Benfatto *et al.*, 1982) and (Nicolò, 1983).

In the case of the Coulomb gas some of the above conjectures also follow as corollaries of the theory of stability of the cosine field, but the connection requires some explanations.

The first remark is that the problem of studying the Coulomb systems with "no infrared cut-off", i.e. with $R = +\infty$ can be reduced to the theory of the cosine interaction in the *ultraviolet* regime by the following chains of identities and arguments.

Rewrite (15.12), using the factorization $' : e^{x+y} :=$ "$e^x :: e^y$: for x, y independent Gaussian variables:

$$\begin{aligned}
&Z^1_{C(R,N)}(\Lambda, \beta, \lambda) = \int P(d\psi^{(0)}) \dots P(d\psi^{(N)}) \cdot \\
&\Big[\int P(d\psi^{(-R)}) \cdots P(d\psi^{(-1)}) \exp\big[\sum_\sigma \int_\Lambda d\xi' \\
&\big(\frac{\lambda}{2} e^{-\frac12 \alpha^2 C_{00}^{(-R,-1)}} : e^{i\sigma\alpha\psi_\xi^{(0,N)}} :\big) : e^{i\sigma\alpha\psi_\xi^{(-R,-1)}} :\big]\Big] \\
&\equiv \int P(d\psi^{(0)}) \dots P(d\psi^{(N)}) \cdot \qquad\qquad (15.19)\\
&\cdot \Big[\int P(d\varphi^{(0)}) \cdots P(d\varphi^{(R-1)}) \exp\big[\sum_\sigma \int_{\Lambda\gamma^{-R}} d\xi' \\
&\lambda_{\sigma,R}(\xi') : e^{i\sigma\alpha\varphi_{\xi'}^{(0,R-1)}} :\big]\Big]
\end{aligned}$$

where $\xi = \xi'\gamma^R$ and $\varphi_{\xi'}^{(0,R-1)} \equiv \psi^{(-R,-1)}$, so that $\varphi^{(0,R-1)}$ has the same distribution as a sum of independent fields $\varphi^{(j)}$:

$$\varphi_{\xi'}^{(0,R-1)} = \sum_{j=0}^{R-1} \varphi_{\xi'}^{(j)} \tag{15.20}$$

with covariance $C^{(0)}_{\gamma^j\xi,\gamma^j\eta}$. This follows immediately from the definitions by computing and comparing covariances: actually one could put $\varphi_\xi^{(j)} = \psi^{(R-j)}_{\gamma^R\xi'}$. Furthermore, in (15.19) $\lambda_{\sigma,N}(\xi')$ means

$$\lambda_{\sigma,N}(\xi') = \frac12 \lambda\gamma^{2R} e^{-\frac12\alpha^2 C_{00}^{(-R,-1)}} e^{i\alpha\sigma\psi^{(0,N)}_{\gamma^R\xi'}} \tag{15.21}$$

The interpretation of

$$\begin{aligned}
&e^{V_c(\psi^{(0,N)})} \equiv \int P(d\varphi^{(0)}) \cdots P(d\varphi^{(R=1)}) \cdot \\
&\exp\big[\sum_\sigma \int_{\Lambda\gamma^{-R}} d\xi'\, \lambda_{\sigma,R}(\xi') : e^{i\sigma\alpha\varphi_{\xi'}^{(0,R-1)}} :\big]
\end{aligned} \tag{15.22}$$

is, clearly, that of an effective interaction in the sense used in the preceding section on field theory; it should describe the Coulomb gas on scales m_0^{-1} (through an equivalent gas of multipoles; see Section 13 for this interpretation; see also below).

To describe (15.22) one can try to find an expansion for $V_C(\psi^{(0,N)})$ in powers of λ.

The work for such an expansion has already been done in Sections 11 and 12, because the integral in (15.22) can be interpreted as an integral of the type studied here.

Using the results and the notations of Sections 11 and 12 one expresses it in terms of trees:

$$V_c = \int d\boldsymbol{\xi} \sum_{\boldsymbol{\sigma}} \sum_{\substack{\gamma:\, \boldsymbol{\sigma}(\gamma)=\boldsymbol{\sigma} \\ \boldsymbol{\xi}(\gamma)=\boldsymbol{\xi}}} \frac{V(\gamma)}{n(\gamma)} \tag{15.23}$$

where the $V(\gamma)$ are computed with exactly the same rules of Sect. 11 provided that we interpret the elementary trees

$$\overline{\mathbf{k} \qquad \xi,\sigma} \tag{37}$$

as $\lambda_{\sigma,R}(\xi)$ defined by (15.21), rather than $\frac{1}{2}\lambda$; the index σ is ± 1, while the index 0 is not allowed, because in the exponential in (15.22) there is no cosntant term.

All the results and bound in Sec. 12 carry through with essentially no change, besides the mentioned change of interpretation of fig. (37).

One therefore finds that $V(\gamma)$ can be expressed, to a given order in λ, as

$$\Big(\prod_{j=1}^{n}\lambda_{\sigma_j,R}(\xi_j)\Big)\,W_\gamma(\xi_1,\ldots,\xi_n) \tag{15.24}$$

and W_γ will satisfy (see (12.4),(12.8),(12.9), and (12.14))

$$\begin{aligned}|W_\gamma(\xi_1,\ldots,\xi_n)| &\le \mathcal{N}_{\widehat{\gamma}}\Big(\prod_{j=1}^{n} e^{\frac{1}{2}\alpha^2 C_{00}^{(\le h_{v_j}-1)}}\Big)\cdot\\ &\cdot\Big(\prod_{v>r} e^{-\frac{1}{2}\alpha^2 Q_v^2(h_v-h_{v'})C_{00}^{(0)}}\, e^{-\frac{1}{4}\kappa\gamma^{h_v} d^*(\xi_v)},\Big)\end{aligned} \tag{15.25}$$

where $\mathcal{N}_{\widehat{\gamma}}$ depends only on the shape of γ. Actually, one will be interested only in expressions like (15.24) summed on the indices of γ: in particular one is interested in the summations of (15.24) ove the different indices $\boldsymbol{\sigma},\boldsymbol{\xi}$ that can be appended to the endpoints of otherwise identical trees . In this way the charge symmetry is used and some modifications appear, as explaines in Sect. 12, which allow one to to improve the boumd (15.25) by replacing $\frac{Q_v^2}{4\pi}$ by $\big(\frac{Q_v^2}{4\pi}+2(1-\varepsilon)\delta_{Q_v,0}\big)$ if $v>v_0=$ (first nontrivial vertex of the tree); this cancellation, in fact, takes place already when one sums only over the charge coinfigurations which attribute the same absolute charge to each vertex v and integrate ove —$\boldsymbol{\xi}$.

According to the discussion of Sect 11 the *l.h.s.* of (15.22) can be interpreted, via (15.23), as the Boltzmann--Gibbs factor in a gas of multipoles, each represented by the trees with the same shape up to the charge indices which vary subject to the restriction that the absolute charge $|Q_v|$ of each vertex is fixed. The activity of the multipole will be defined, quite arbitrarily [see (Gallavotti and Nicolò, 1985c) for a deeper discussion]:

$$\begin{aligned}&\sum_{\boldsymbol{\sigma}}^{*}\sum_{\mathbf{h}}\int_{\gamma^{-R}\Delta\times(\gamma^{-R}\Lambda)^{n-1}} d\boldsymbol{\xi}_1\cdots d\boldsymbol{\xi}_n\cdot\\ &\cdot\frac{\Big(\prod_{j=1}^{n}\lambda_{\sigma_j,R}\Big)W_\gamma(\boldsymbol{\xi}_1,\cdots,\boldsymbol{\xi}_n)}{:\,e^{i\alpha\sum_j\sigma_j\psi^{(0,N)}_{\gamma^R\boldsymbol{\xi}_j}}\,:}\end{aligned} \tag{15.26}$$

where $\sum^*$ runs over all charge configurations $\boldsymbol{\sigma}$ which attribute gives absolute value to the total charge Q_v to each of the clusters associated with the vertices v of γ (called above, simply, vertex charges); the sum $\sum_{\mathbf{h}}$ runs over all the possible frequency labels that can be appended on the shape of γ, and Δ is a fixed unit cube.

The collection of the terms with the same vertex charge is natural for physical reasons (charge symmetry), and mathematically it produces the just-mentioned cancellations.

If we reexpress (15.26), by "going back to scale 1", it becomes

$$\begin{aligned}&\sum\int_{\gamma^{-R}\Delta\times(\gamma^{-R}\Lambda)^{n-1}} d\boldsymbol{\xi}_1\cdots d\boldsymbol{\xi}_n\,(\lambda\gamma^{2R-\frac{\alpha^2}{4\pi}R})\cdot\\ &\cdot\prod_{j=1}^{n}\,:\,e^{i\alpha\sum_j\sigma_j\psi^{(0,N)}_{\gamma^R\boldsymbol{\xi}_j}}\,:\,W_\gamma(\boldsymbol{\xi}_1,\cdots,\boldsymbol{\xi}_n)\,:\equiv\\ &\equiv\sum\gamma^{-2Rn}\int_{\Delta\times\Lambda^{n-1}} d\mathbf{x}\,(\lambda\gamma^{2R-\frac{\alpha^2}{4\pi}R})^n\cdot\\ &\quad\cdot:\,W_\gamma(\gamma^{-R}\boldsymbol{\xi}_1,\cdots,\gamma^{-R}\boldsymbol{\xi}_n)\,:\,e^{-\alpha^2 Y_N(\mathbf{x},\boldsymbol{\sigma})},\end{aligned} \tag{15.27}$$

where $\sum\equiv\sum_{\boldsymbol{\sigma}}^*\sum_{\mathbf{h}}$, see (15.26), and

$$Y_N(x_1,\ldots,x_n;\sigma_1,\ldots,\xi_n)$$

is the Yukawa potential, with ultraviolet cut-off N, of the charges $s_1,\ldots,\sigma_n$ at positions $x_1,\ldots,x_n$.

Therefore the activity of the multipole will be bounded (by using (15.24),(15.25) and the cancellation remarked after (15.25) and recalling that $\varepsilon>0$ is an arbitrary parameter which can be chosen as small as necessary)

$$\begin{aligned}&\sum_{\mathbf{h}}\gamma^{-2Rn}\int_{\Lambda^{n-1}} dx_2\cdots dx_n\,(\lambda\gamma^{2R-\frac{\alpha^2}{4\pi}R})^n\cdot\\ &\cdot\Big|\sum^{*}\,:\,W_\gamma(\gamma^{-R}\boldsymbol{\xi}_1,\cdots,\gamma^{-R}\boldsymbol{\xi}_n)\,:\,\Big|\le\\ &\le\mathcal{N}_{\widehat{\gamma}}\gamma^{-2R}\gamma^{-(\frac{\alpha^2}{4\pi}-2)Rn}\sum_{\mathbf{h}}\Big(\prod_{v>v_0}\cdot\\ &\gamma^{((\frac{\alpha^2}{4\pi}-2)(n_v-1)+\frac{\alpha^2}{4\pi}-\frac{\alpha^2}{4\pi}Q_v^2+2(1-\varepsilon)\,\delta_{Q_v,0})(h_v-h_{v'})}\Big)\cdot\\ &\cdot\gamma^{\big((\frac{\alpha^2}{4\pi}-2)(n-1)+\frac{\alpha^2}{4\pi}-\frac{\alpha^2}{4\pi}Q_{v_0}^2\big)(h_{v_0})}\,U_{N,n}\end{aligned} \tag{15.28}$$

where the sum over the frequency indices is of course bounded by the infrared cut–off: $h_v\le R$. For more details see (Gallavotti and Nicolò, 1985c).

Before discussing formula (15.28), let us note that if $N=-1$, i.e. if the Coulomb potential has no ultraviolet part (which is the case usually considered in the literature), the effective potential becomes a constant and i is no longer a random variable and it has the interpretation of (grand canonical) pressure of the gas. Therefore (15.28) becomes a bound on the Mayer coefficients of the gas (see below for some consequences of this remark).

To examine the remarkable formula (15.28) one distingushes two cases: either $\alpha^2\ge 8\pi$ or $\alpha^2<8\pi$. Below one uses the arbitrariness of ε by taking it conveniently small.

In the first case the *r.h.s.* of (15.28) goes to zero as $R \to \infty$, as can be checked elementarily, for $Q_{V_0} \neq 0$: the gas is a gas of "neutral multipoles" (i.e. in the infrared limit one is in a multipole phase; see (Frölich and Spencer, 1981). If $Q_{V_0} = 0$ then one can check that the *r.h.s.* of (15.28) is uniformly bounded in R.

On the second case let

$$\rho = (\frac{\alpha^2}{4\pi} - 2)n + 2 \equiv \frac{\alpha^2}{4\pi} - 2)(n-1) + \frac{\alpha^2}{4\pi}; \qquad (15.29)$$

then either $\rho \leq 0$ abd the *r.h.s.* of (15.28) *diverges* in general as $R \to \infty$ or $\rho > 0$ and in this case the bound (15.28) is uniformly bounded in R, and it tends to 0 if $Q_{v_0} \neq 0$.

The conclusions from the above estimates are

(1) If $\alpha^2 > 8\pi$ (hence $\rho > 0$), the picture od the Coulomb gas consisting, as far as its properties on scale m_0^{-1} are concerned, of neutral multipoles is consistent, because the activity of such multipoles is finite. This will be called the "multipole theorem" (see also (Frölich and Spencer, 1981)).

(2) If $\alpha^2 > \alpha_n^2$, where α_n^2 are the tresholds defined by setting $\rho = 0$ in (15.29), then thinking that the gas contains several multipoles pf charge p with $p \leq n$ but that no multipoles with charge higher than n can be well defined entities ("molecules", of course, should be their name) becomes consistent.

(3) It is remarkable that the above tresholds α_n^2, above which the Coulomb gas (with ultraviolet cut–off) generates molecules of p bound atoms precisely coincide with the tresholds α_n^2 where the Yukawa gas charges colapse into clusters of $p \leq n$ particles (in the ultraviolet limit), see (Frölich, 1976).

This is a confirmation of the above implicitly conjectured "duality" between the infrared properties of the Coulomb gas and the ultraviolet properties of the Yukawa gas for α^2 in $[0, 8\pi)$.

If we call $p_C(\lambda, \beta)$ the pressure of the Coulomb gas with ultraviolet cut–off, as a function of the charge activity λ and of the inverse temperature $\beta = \alpha^2$, the above analysis proves, as is easily checked, that if

$$p_C(\lambda, \beta) = \sum_{p \leq n} \lambda^p f_C^{(p)}(\beta) + \lambda^n R^{(n)}(\lambda, \beta), \qquad (15.30)$$

with $R^{(n)}(\lambda, \beta) \xrightarrow[\lambda \to 0]{} 0$, then the coefficients $f_C^{(p)}(\beta)$ can be shown to be uniformly bounded in the infrared limit $R \to \infty$ for $\alpha^2 > \alpha_n^2$, and that only the even ones have a nonzero limit.

In other words the Mayer series coefficients of order $\leq n$ are formally well defined by convergent integrals for $\alpha^2 > \alpha_n^2$. The latter property follows immediately by considering the case $N = -1$ in which, as remarked above, the effective potential coincides with the grand canonical pressure. It should be obvious that the general case $N \geq -1$ (but finite) can be reduced always, and in atrivial way, to the $N = -11$ case.

This leads to the natural conjecture that $p_C(\lambda, \beta)$ is smoother and smoother in λ at $\lambda = 0$ as $\beta = \alpha^2$ grows.

For $\alpha^2 < 4\pi$ not much can be said about smoothness; for $\alpha^2 \in (4\pi, 6\pi)$ the function should have two derivatives (actually three if $\alpha^2 \in (\frac{16}{3}\pi, 6\pi)$); for $\alpha^2 \in (6\pi, \frac{40}{6}\pi)$ it should have four derivatives (actually five for $\alpha^2 \in (\frac{32}{5}\pi, \frac{40}{6}\pi)$, *etc*); for $\alpha^2 > 8\pi$ the pressure should be infinitely smooth at $\lambda = 0$.

By *derivative* one means here that (15.30) holds as an asymptotic formula with $R^{(n)}(\lambda, \beta)$ tending to zero faster than λ^n as $\lambda \to 0$.

The conjecture suggests that while α^2 crows (i.e. as the temperature decreases), the Coulmb gas presents an inifinite sequence of phase transitions in which it passes from the "plasma phase", small α^2, with Debye screening phenomena, to the "multipole phase", α^2 large, with no screening in the infrared: the "Kosterlitz–Thouless" regime would be the last stage in a sequence of increasingly complex phase transitions in which bound states ("neutral molecules") of increasing size become possible in thermal equilibrium.

So far the basis for the above conjecture are the estimates of this section (15.28) which imply the finiteness of the coefficients of the Mayer expansion; such estimates have been pointed out in (Gallavotti and Nicolò, 1985c). Further work towards a full proof of (15.30), i.e. with estimates on the remainder in (15.30), is in progress (Benfatto *et al.*, 1986).

I think that the beautiful properties of the cosine interaction exhibited in this section justify its i inclusion in this work, althought they are not strictly an example of a problem of field theory: they show that field theory is not just a theory of quantum relativistic systems but that it can be relevant to very different matters, Coulomb gases are onle one example out of many more, in solid state physics and in physics of fluids, for instance.

xvi. Nature and classification of the divergences for φ^4 fields

In order to see how to build the operators $\mathcal{L}_k^{(\sigma)}$ realizing the renormalization of the φ^4 field theory $\mathcal{I}$ defined by

$$V_1 = \int \big(-\lambda : {\varphi_\xi^{(\leq N)}}^4 : -\mu : {\varphi_\xi^{(\leq N)}}^2 : - \\ - \alpha : (\partial\, \varphi_\xi^{(\leq N)})^2 : -\nu \big)\, d\xi \qquad (16.1)$$

it is useful, albeit not strictly necessary, to have a clear idea of how divergences arise in it and how strong they are.

I shall consider in detail only the foour parameter interaction (16.1) in four dimensions, calling $\lambda^{(\alpha)}$, $\alpha = 4, 2, 2', 0$, the parameters $-\lambda, -\mu, -\alpha, -\nu$ respectively.

If $d = 2, 3$ one could consider theories simpler than (16.1) which in some cases can be constructed as true field theories, going beyond the formal theory of perturbations, by literally repeating the arguments of Secs. 13,14 [e.g. if $d=2$ one could consider the interaction(5.3), or if $d=3$ one could consider the interaction (5.6)]. Some more details on these simple ("super-renormalizable") cases will be presented in Sec. 21.

Suppose that the field with ultraviolet cut–off γ^N, $\gamma>1$, is decomposed as a sum of independent fields living on scales γ^{-k}, $k=0,1,\ldots,N$ and satisfying (3.17)–(3.20) with $n=3$, (say):

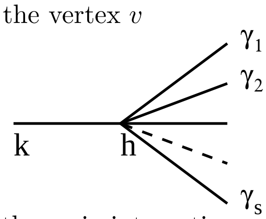
$$\varphi_\xi^{(\le N)}=\varphi_\xi^{(-1)}+\varphi_\xi^{(0)}+\ldots+\varphi_\xi^{(N)} \tag{16.2}$$

where $\varphi^{(-1)}$ is a degenrate field with covariance $C^{(-1)}$ which will be eventually put equal to zero, so that $\varphi^{(-1)}=0$, and which is introduced only for the purpose of unifying certain notations.

The use of a Pauli-Villars regularization of order $n\ge 2$ is necessary to give a meaning to (16.1) if $d=4$; actually the third term in (16.1) already requires $n\ge 2$ even for $d=2$, while the first two lose meaning only if $d\ge 4$ when $n=1$. Here the choice $n=3$ is motivated by the fact that in the subtraction algorithm built to renormalize the divergences it will be convenient to be able to say that the fields have two derivatives and therefore can be developed in Taylor series to second order included. It is not impossible that one could perform the work with $n=2$ but it would be certainly be harder : in any case this question acquires importance when one tries to go beyond perturbation theory; when one is dealing only with perturbation expansions it would be even better to have fields so regular to have derivatives of any order, e.g. the ones arising from the regularization (3.21).

Fixed $\boldsymbol{\lambda}=(-\lambda,-\mu,-\alpha,-\nu)$ in (16.1) the effective potential $V_1^{(k)}$ "on scale k" will be expressed in terms of simple trees as explained in Sec. 6: the end points will be marked by a pair (ξ,α) with $\xi\in\mathbb{R}^d$ and $\alpha=0,2,2',4$ expressing which of the four terms in (16.1) is represented by the end point under consideration.

An expression for $V(\gamma)$ can be found by the same technique used in the cosine field case in Secs. 11 and 12: namely let a tree γ bifurcate at the first nontrivial vertex v_0 after the root r into s subtrees $\gamma_1,\ldots,\gamma_s$, and let h be the "frequency" of the vertex v



(38)

As in the case of the cosin interactioc, one has to guess first the formof $V(\gamma)$, and an obvious guess is the following very general one:

$$V(\gamma)=\sum_P V^{(k)}(\xi_1,\ldots,\xi_n;\gamma,P)\,P(\varphi^{(\le k)},\partial\,\varphi^{(\le k)}), \tag{16.3}$$

where the summation runs over all the possible Wick monomials of the form $P=:(\partial\,\varphi_\xi)^2\,:$ if γ is a trivial tree, or

$$\begin{aligned}P=\,:\varphi_{\xi_1}^{n_1}\cdots\varphi_{\xi_{i_q}}^{n_s}\,\partial\varphi_{\xi_{i_q+1}}\ldots\partial\varphi_{\xi_{i_m}}\,:,\\ 0\le q\le n, 1\le n_i\le 4\end{aligned} \tag{16.4}$$

where the derivative ∂ means a derivative with respect to one of the coordinates of the field arguments; the above assumption is made only for the trees having nontrivial vertices. In fact (16.4) is not true for the trivial tree representing $-\alpha\,:(\partial\,\varphi_\xi)^2\,:$, k———— $\xi,2'$; this will be the only (natural) exception.

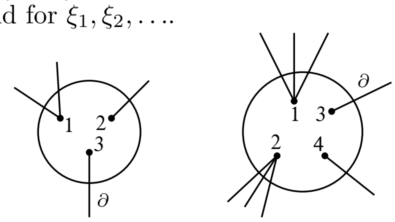
Assuming (16.3) and (16.4), with the above mentioned exception, we find the rules for the evaluation of the truncated expectations of products of Wick monomials, see Appendix C, yield the following recursion relation, deduced from diagram (38) after recalling that a tree vertex has the meaning of a truncated expectation (see Secs. 6,7):

$$\begin{aligned}V^{(k)}(\xi_1,\ldots,\xi_n;\gamma,P)=\sum_{P_1,\ldots,P_s}\Big[\prod_{j=1}^s V^{(k)}(\boldsymbol{\eta}_j;\gamma_j,P_j)\Big]\cdot\\ \cdot\sum_{v\in\gamma_P}\sum_{\substack{\tau\in\pi\\ connected}}\Big[\prod_{\substack{\lambda\in\tau\\ \lambda=(a,b)}}C_{ab}^{(k)}\Big]\Big[\prod_{\substack{\lambda\in\pi/\tau\\ \lambda=(a,b)}}C_{ab}^{(\le k-1)}\Big]\end{aligned} \tag{16.5}$$

where $\boldsymbol{\eta}_1,\ldots,\boldsymbol{\eta}_s$ are the s clusters into which the points $\boldsymbol{\xi}$ are decomposed by $\gamma_1,\ldots,\gamma_s$, i.e. the s clusters corresponding to the vertices $v_1,\ldots,v_s$ of the tree γ immediately following v_0 and such that $v_j\in\gamma_j$.

Let $\mathcal{T}_P$ be the set of graphs obtained as follows. Represent a Wick monomial P like (16.4) by drawing q points $\xi_{i_1},\ldots,\xi_{i_q}$ in $\mathbb{R}^d$ and $n_1,n_2,\ldots,n_q$ pairwise distinct lines, respectively, emerging from each of them, and $m-q$ points $\xi_{i_q+1},\ldots,\xi_{i_m}$ a line, labeled ∂ and emerging from each of them.

It is convenient to think of the points $\xi_{i_1},\ldots,\xi_{i_m}$ as enclosed in a box out of which emerge the lines just defined. For instance the monomials : $\varphi_{\xi_1}^2\,\varphi_{\xi_2}\,\partial\varphi_{\xi_3}$: and : $\varphi_{\xi_1}^3\,\varphi_{\xi_2}^3\,\partial\varphi_{\xi_3}\,\partial\varphi_{\xi_4}$: are represented as in Fig. 39, where $1,2,\ldots$ stand for $\xi_1,\xi_2,\ldots$.



(39)

and each of the above objects will be called a "Wick monomial". Then, given s Wick monomials $P_1,\ldots,P_s$, the symbol $\mathcal{T}_P$ will denote the set of graphs obtained by joining pairwise some of the lines associated with the Wick clusters representing $P_1,\ldots,P_s$ in such a way that (explicit examples are worked out in Figures 40–43 below):

(i) two lines emerging from the same cluster cannot be joined together;
(ii) there should be enough lines paired so that the lines plus the sets inside the boxes associated with each P_j form a connected set;
(iii) the set of the points associated with $P_1, \ldots, P_s$ together with the lines emerging from them and still "free" (i.e. not paired with the lines) represent, once the points from which they emerge are enclosed into a single box, the monomial P.

In the above definitions and constructions, as well as in the upcoming ones, one has to bear always in mind that the lines emerging from each point are regarded as pairwise distinct (and this will eventually give rise to a combinatorial problem).

Furthermore $\tau \subset \pi$ with the subscript "connected" (see (16.5)) means a subset of the lines of π which still keeps the connection between the boxes. A line λ obtained by pairing ("joining" or "contracting" will be synonymous of "pairing") two lines is identified by its two extreme points together with the field indices (∂ or nothing) which will be kept and appended to the line near the end point from which they emerge (so that it might happen that a line carries two, one or no indices ∂; if it carries only one it will be appended near the appropriate end point).

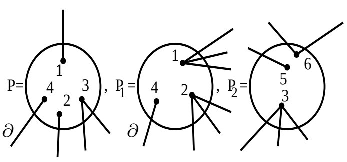
Therefore $\lambda = (a, b)$ with $a = \xi, b = \xi'$ represents a line obtained by joining two lines emerging from ξ and ξ'; similarly if $a = (\xi, \partial)$, $b = \xi'$ then $\lambda = (a, b)$ represents the line obtained by joining together a nonlabeled line emerging from ξ' and a labeled one emerging from ξ. The resulting line will be represented by a segment joining ξ to ξ' carrying a label ∂ near ξ (or equivalently carrying a label ∂_ξ); and similar interpretations are given to the cases $a = \xi$, $b = (\xi', \partial)$ or $a = (\xi, \partial)$, $b = (\xi', \partial)$.

The symbols $C^{(\cdot)}_\lambda \equiv C^{(\cdot)}_{ab}$ denote the appropriate covariances (propagators)

$$\begin{aligned} C^{(\cdot)}_{ab} &= \mathcal{E}(\varphi^{(\cdot)}_\xi \varphi^{(\cdot)}_\eta), \quad C^{(\cdot)}_{ab} = \mathcal{E}(\varphi^{(\cdot)}_\xi \partial_\eta \varphi^{(\cdot)}_\eta), \\ C^{(\cdot)}_{ab} &= \mathcal{E}(\partial_\xi \varphi^{(\cdot)}_\xi \varphi^{(\cdot)}_\eta), \quad C^{(\cdot)}_{ab} = \mathcal{E}(\partial_x \varphi^{(\cdot)}_\xi \partial_\eta \varphi^{(\cdot)}_\eta), \end{aligned} \tag{16.6}$$

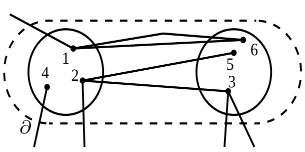
if $(a, b) = (\xi, \eta), (\xi, (\eta, \partial_\eta)), ((x, \partial_\xi), \eta), ((x, \partial_\xi), (\eta, \partial_\eta))$, respectively. Recall that here ∂ or ∂_ξ means a derivative with respect to some component of ξ whose index is omitted for simplicity of notation.

For instance consider Fig.40, where the integer j stands for j



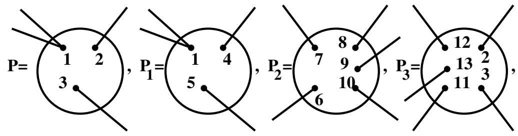
(40)

Then one possible element $\pi \in \mathcal{T}_P$ is drawn in Fig.41



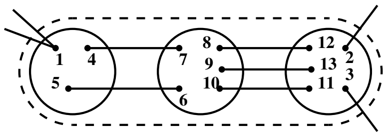
(41)

where the dotted box represents the box corresponding to P. A possible $\tau \subset \pi$ is any nonempty subset of the inner lines, inside the dotted box in Fig.41. Similarly if



(42)

a simple possible π is



(43)

and τ is any subset of the five inner lines which contains at least one of the first two and one of the last three.

Relation (16.5) defines recursively and completely the coefficients $V^{(k)}(\gamma; P)$ once one specifies the meaning of $V^{(k)}(\gamma; P)$ for the elementary trees γ_0 ${}_k$———${}_{\xi,\alpha}$.

Of course $V^{(k)}(\gamma_0; P) \equiv 0$ unless P is : ${\varphi^{(\le k)}}^4$:, : ${\varphi^{(\le k)}}^2$:, : $\partial\, {\varphi^{(\le k)}}^2$:, or 1; and in such cases $V^{(k)}(\gamma_0; P)$ is just $-\lambda, -\mu, -\alpha, -\nu$, respectively, for $\alpha = 4, 2, 2', 0$ (no confusion should arise between the renormalized coupling constant α and the endpoint index carrying the same name).

To find bounds on $V^{(k)}(\gamma; P)$ one can proceed as follows: first by using (16.5) one decomposes this quantity into a (very large) sum: each term of the sum will correspond to a fixed selection S of one index for every possible summation arising by applying recursively (16.5). One has to imagine that one such special selection S has been fixed (say one special choice of P, of $P_1, \ldots, P_s$, of π, τ *etc*, with similar choices made for each of the successive vertices of γ which arise while disassembling $\gamma_1, \ldots, \gamma_s$, *etc*).

the bases for the bound that will be derived shortly are the estimates (3.19).(3.20) and (3.17), and the following notions which have been already introduced in Secs. 4–sec(9) above and in the preceding lines of this section, but which it will be convenient to collect again and organize in the form in which they will be used below.

(1) Each vertex v of a tree γ is associated with a cluster of end points of γ; this cluster will be denoted $\boldsymbol{\xi}_v$.

(2) The selection S of the summation indices just introduced permits one to associate with each vertex v a monomial P_v which can be thought of as graphically represented by a box containing the points $\boldsymbol{\xi}_v$, with lines emerging from them and out of the box itself.: some of the lines may bear an index ∂; the lines emerging from the box represent the graphical image of the monomial P_v.

(3) The number n^e_v of lines emerging from the box enclosing the cluster $\boldsymbol{\xi}_v$ will be the sum $n^e_{1,v} + n^e_{2,v}$ of the number of labeled lines, $n^e_{1,v}$, and of the number, $n^e_{2,v}$ of unlabeled lines (e: external; 1: labeled 0; 0: unlabeled).

(4) A selection S of the summation indices leads to a graphical representation of the corresponding contribu-

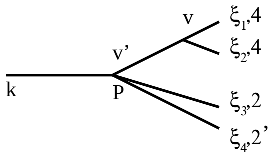
tions to $V^{(k)}(\gamma;S)$.

Out of each endpoints ξ of $|g$ emerge either four unlabeled lines or two labeled lines or no lines at all, depending upon the value of the appended type indices $\alpha = 4, 2, 2', 0$. The case $\alpha = 0$ can appear only in the trivial trees $k \overline{\qquad\qquad} \xi, 0$, which will be disregarded for the time being; in fact $\alpha = 0$ corresponds to a constant $P = 1$ and the truncation of the expectations eliminates it unless the tree is trivial, i.e. it indicates no truncations.

The structure of γ encloses the end points into a hierarchically arranged sequence of boxes, each corresponding to a tree vertex v, and it is possible to make the convention hat pairings of the lines are drawn in the graphical representation so that the lines contracted between the clusters $v_1, v_2, \ldots, v_s$, representing $P_{V_1}, \ldots, P_{v_s}$, to build the monomial P_v (v being the vertex immediately followed by $v_1, \ldots, v_s$) are all contained inside the box corresponding to v, as in Fig.4043 above.

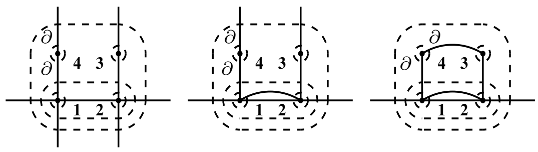
For uniformity of notation it is convenient to imagine that the end points of γ also represent clusters of a single point and that they generate little boxes around it (recall that, however, the end points of a tree are conventionally not regarded as tree vertices).

For instance, three possible selections corresponding to the tree



(44)

(where v, v' are vertex names, p, h, k are frequency labels) are represented by



(45)

(if $1, 2, \ldots$ stand for $\xi_1, \xi_2, \ldots$).

(5) If in S there is a line paired to another, there will be a smallest box containing the contracted line, i.e. the two endpoints (this is enough by the above drawing convention); if v is the corresponding tree vertex and h_v is its frequency index, then one says that thecontracted line has frequency h_v and one attributes the index h_v to each of the two lines giving rise to the contracted line of frequency h_v; the uncontracted lines will be given the frequency index $k = k(\gamma) =$ (frequency of the root of the tree); they are called "external".

So with each box one can associate a frequency index qhich is the frequency index h_v of the vertex v corresponding to the box. As a consequence one can associate with each line in S a frequency which is the frequency index of the line. Note that the association of a frequency index with a line depends on S and not just on the tree γ. By convention the box associated with the root r of the tree γ is the whole plane containing the tree.

(6) The above set of indices still does not secify completely the selection S: one has to mark, for this purpose, each line which belongs to the set called τ in (16.5) by a label, say θ, recalling its origin (as a lineof the set $|t$): we call it a "character label"; lines with the label θ will be called "hard" or "high-frequency" lines.

(7) It is important to stress, again, to avoid combinatorial errors, that in the above construction two lines emerging from the same vertex still have to be regarded as different and distinguishabe. To keep track of the combinatorics it is convenient to imagine that the lines emerging from the innermost vertices (i.e. from the end points) are numbered: from 1 to 4 if the vertex represents $-\lambda : \varphi^4 :$, from 1 to 2 if it represents $-\mu : \varphi^2 :$ or $-\alpha : (\partial\varphi)^2 :$. Such labels will be called "identity labels".

It is also clear that the number of selections differing just by the identity labels is bounded vy 4^n if the tree has n end points.

Before continuing it is important to stress that, by our definitions, a selection S of summation indices yields a connected graph joining all the endpoints $\boldsymbol{\xi}$ of γ with lines marked by

(a) a frequency index,
(b) a character index or no index per internal line: if the index is missing the line is "soft"; if it is present the line is "hard",
(c) an index ∂ or no index per each extreme of a line, and
(d) an identity index per each end point og the line (internal or not).

the frequency index and the character indices cannot be randomly assigned: they are organized by the tree γ in such a way that, if we draw the boxes corresponding to each vertex of γ, the lines internal to each box form a connected graph and so does their subset formed by the hard ones among them.

The lines that are external, together with the points out of which they emerge and with the largest (finite) box, form a graphical representation of the Wick monomial P selected by S.

The reader familiar with Feynman's graphs will recognize in such a representation of S something which can be called a "decorated Feynman graph", the decorations being the above collection of labels listed in (a)–(d) above. One also recognizes the connection between the above decorated graphs and trees and the basic notion of "forest" in (Zimmermann, 1969). To obtain bounds on $V^{(k)}$ consider the contribution to it by a choice of the summation indices S.

Denoting $V(\gamma;S)\,P_S$ such a contribution, where P_S denotes the Wick monomial selected by S and using (3.19) and (3.20) one finds after some meditation the ("good") estimate in terms of $\overline{\varepsilon} = \max(|\lambda|, |\alpha|, |\mu|, |\nu|)$:

$$|V(\gamma;S)|\,\gamma^{k\frac{d-2}{2}n^e_{0,v_0}}\gamma^{k(\frac{d-2}{2}+1)n^e_{1,v_0}} \le \overline{\varepsilon}^n. \tag{16.7}$$

$$\cdot \Big[\prod_{\ell_0} \gamma^{\frac{d-2}{2} h_{\ell_0}}\Big]\Big[\prod_{\ell_1} \gamma^{(\frac{d-2}{2}+1)h_{\ell_1}}\Big]\Big[\prod_{\lambda} B\, e^{-\kappa\gamma^{h_\lambda}|\lambda|}\Big]$$

where $d > 2$, for simplicity, λ represents an inner line of frequency h_λ associated with S and $|\lambda|$ is the distnce bewtween the end points of λ; regarding the contracted lines of S as composed by two joined half lines, and regarding the external lines also as half lines, we find that the first product in the *r.h.s.* is over the half lines bearing no ∂ label while the second product is over the half lines bearing a ∂ label. The first nontrivial vertex of γ is denoted by v_0 and $B, \kappa > 0$ are suitable constants.

The factor multiplying the *l.h.s.* , $|V(\gamma; S)|$, has been introduced for convenience (it will be clear shortly that it is a natural multiplier in the *l.h.s.* of the inequality (16.7)).

The bound (16.6) is really trivial "power counting", once the presence of the exponential factors is understood. It arises from bounding $C^{(h)}_\lambda$ contributed by the hard lines λ in S, with frequency index h.

Recalling (16.6) one sees, for instance, that there is $B > 0$ such that if $\lambda = (a, b), a = \xi, b = \eta$

$$|C^{(h)}_{ab}| \le (\gamma^{\frac{d-2}{2}h})^2 B e^{-\kappa\gamma^h\,|\xi-\eta|} \tag{16.8}$$

or, if $a = \xi, b = (\xi', \partial)$:

$$|C^{(h)}_{ab}| \le (\gamma^{(\frac{d-2}{2}+1)h})\gamma^{\frac{d-2}{2}h} B e^{-\kappa\gamma^h|\xi-\eta|} \tag{16.9}$$

or, if $a = (\xi, \partial), b = (\xi', \partial)$:

$$|C^{(h)}_{ab}| \le (\gamma^{(\frac{d-2}{2}+1)h})^2 B e^{-\kappa\gamma^h|\xi-\eta|} \tag{16.10}$$

while $C^{(\le h-1)}_{ab}$, contributed by the soft lines, can be bounded only by (16.8), (16.9) and (16.10) without the last exponential factor (or, rather, with that factor replaced by $e^{-\kappa|\xi-\eta|}$, useless), provided $d > 2$ (if $d = 2$ an extra factor h has to be added in bounding $C^{(\le h-1)}_{ab}$).

One could write the exponential factor in (16.7) as $\prod_v e^{-\kappa\gamma^{h_v} d^*(X_v)}$, using the notations introduced in Sec. 12 (particularly Eq. (12.6)) to treat the cosine field; however, this remark has been made only for the sake of comparison and will not be needed in what follows.

It remains to cast (16.7) into a more usable form. Select a vertex $v \in \gamma$ and let $m_{2,v}, m_{4,v}, m_{2',v}$ be the numbers of vertices in the cluster $\boldsymbol{\xi}_v$ associated with v and bearing an index $\alpha = 2, 4, 2'$, respectively: $m_{2,v} + m_{4,v} + m_{2',v} = n_v =$ (number of points in $\boldsymbol{\xi}_v$), and note that if v is a nontrivial vertex of the tree, $n_v \ge 2$ because v represents a truncation operation. For each v introduce also:

$n^{inner}_{0,v}$ = number of lines without ∂ label before the contractions, contained in the box corresponding to v but not in any smaller one,

$n^{inner}_{1,v}$ = number of lines with a label ∂ before the contractions, contained in the box corresponding to v but not in any smaller one,

the number of lines being counted before the contractions means that each inner line of a graph S counts twice in the evaluation of n^{inner}, and

$n^e_{0,v}$ = number of lines without ∂ label before the contractions, emerging from the box corresponding to v
$n^e_{1,v}$ = number of lines with ∂ label before the contractions, emerging from the box corresponding to v

A simple calculation allows us to rewrite (16.7) as

$$\begin{aligned}&\bar\varepsilon\, \gamma^{\frac{d-2}{2}\,k\, n^e_{0,v_0}} \gamma^{\frac{d}{2}\,k\, n^e_{1,v_0}} \Big(\prod_\lambda B e^{-\kappa\gamma^{h_\lambda}|\lambda|}\Big)\cdot\\ &\cdot \Big(\prod_{v>r} \gamma^{h_v\,\frac{d-2}{2}\, n^{inner}_{1,v}}\Big),\end{aligned} \tag{16.11}$$

where r is the root vertex of the tree γ.

Let γ have n endpoints labeled $\xi_1, \ldots, \xi_n$ and let the external lines of the graph S emerge from the first p points $\xi_1, \ldots, \xi_p$, as it can be assumed without loss of generality. Then one is interested, according to the general ideas developed in Sec. 12 in connection with the asymptotic freedom notion and the interpretation of the effective potential as potential for a continuous spin system, in bounding the quantity

$$\begin{aligned}M_S(\Delta_1, \ldots, \Delta_p) = &\int_{\Delta_1\times\ldots\times\Delta_p\times\Lambda^{n-p}} |V(\gamma; S)|\cdot\\ &\cdot \sup \big|P_S(\varphi^{(\le k)}, \partial\,\varphi^{(\le k)})\big| d\boldsymbol{\xi}_1 \cdots d\boldsymbol{\xi}_n\end{aligned} \tag{16.12}$$

where $\Delta_1, \ldots, \Delta_p$ are cubes of side size γ^{-p} in which the points appearing as labels to fields in P_S vary in (16.12): these cubes are extracted from a pavement Q_P of Λ. The supremum in (16.12) is over the fields $\varphi^{(\le k)} = \sum_{j=0}^k \varphi^{(j)}$ with $\varphi^{(j)}$verifying (3.20). If one denotes $\widetilde B = \sup|B_\Delta|$ (see (3.20)), one finds (setting $n^e_{0,v_0} = n^e_0$, $n^e_{1,v_0} = n^e_1$, $n^e = n^e_0 + n^e_1$)

$$\sup |P_S| \le \gamma^{\frac{d-2}{2}\,k\, n^e_0} \gamma^{\frac{d}{2}\,k\, n^e_1} \widetilde B^{n^e} \mathcal N \tag{16.13}$$

and the constant $\mathcal N$ depends only on the degree of the polynomial P_S (hence it depends neither on j nor on the degree n of γ: in fact $\mathcal N = O(n^\varepsilon!)$).

Inserting (16.13) into (16.12) one finds that (16.12) is estimated by $\widetilde B^{n^e}\mathcal N$ times the integral over $\Delta_1, \ldots, \Delta_p \times \Lambda^{n-p}$ of the *r.h.s.* of (16.7) (and this explains also why the factor in the *r.h.s.* of (16.7) is a natural one to introduce).

The only term in (16.7) which is not constant is the last factor: its integral over the set indicated in (16.13) has already been considered in Sec. 12 (see (12.15)) –see remark following (16.8)– and the result is expressed by

$$B_1^n\Big(\prod_{v>r}\gamma^{-d\,h_v\,(s_v-1)}\gamma^{-k\,d}\Big)\,e^{-\frac{\kappa}{2}\,d(\Delta_1,\ldots,\Delta_n)\,\gamma^k},\quad(16.14)$$

where $B_1>0$ is a suitable constant and s_v is the number of branches emerging from v in γ (see Appendix D for a roof of (16.4)).

Using (16.13),(16.11) and (16.14). we can bound the integral (16.12) by

$$\begin{aligned}&M_S(\Delta_1,\ldots,\Delta_p)\le\gamma^{-\frac{\kappa}{2}\gamma^k d(\Delta_1,\ldots,\Delta_p)}\,\overline{\varepsilon}\,\widetilde{B}^{n^e}\mathcal{N}\cdot\\&\cdot\gamma^{-k\,d}\gamma^{k\frac{d-2}{2}n_0^e}\gamma^{\frac{d}{2}\,k\,n_1^e+k\,\frac{d}{2}n_1^e}\cdot\\&\cdot\Big(\prod_{v>r}\gamma^{h_v\frac{d-2}{2}n_{0,v}^{inner}}\gamma^{h_v\frac{d}{2}n_{1,v}^{inner}}\gamma^{-h_v\,d\,(s_v-1)}\Big)\end{aligned}\quad(16.15)$$

The latter estimate can be elaborated by using the identity $\sum_{v'<v}(s_v-1)=(n_{v'}-1)$, see (12.17). Remembering that the end points of γ are not considered as vertices of γ and denoting simply $m_2,m_4,m_{2'},n_1^e,n_0^e,n^e$ the $m_{2,v_0},m_{4,v_0},\ldots$, respectively, if v_0 is the first nontrivial vertex of γ following the root, one finds

$$\begin{aligned}&M_S(\Delta_1,\ldots,\Delta_p)\le\overline{\varepsilon}^n\widetilde{B}^{n^e}\overline{B}_1^n\gamma^{-dk}\\&\Big[\prod_{v>r}\gamma^{-d(h_v-k)(s_v-1)}\gamma^{\frac{d-2}{2}(h_v-k)n_{0,v}^{inner}}\gamma^{\frac{d}{2}(h_v-k)n_{1,v}^{inner}}\Big]\cdot\\&\cdot\gamma^{-dk\sum_v(s_v-1)}\gamma^{\frac{d-2}{2}\,k\,(2m_1+4m_4-n_0^e)}\gamma^{\frac{d}{2}\,k\,(2m_{2'}-n_1^e)}\cdot\\&\cdot\gamma^{\frac{d-2}{2}\,n_0^e\,k}\gamma^{\frac{d}{2}\,n_1^e\,k}e^{-\frac{\kappa}{2}\gamma^k d(\Delta_1,\ldots,\Delta_p)}\mathcal{N}\equiv\\&\equiv\overline{\varepsilon}^n\widetilde{B}^{n^e}\overline{B}_1^n\mathcal{N}\,e^{-\frac{\kappa}{2}\gamma^k d(\Delta_1,\ldots,\Delta_p)}\gamma^{-k\,(2m_2+(4-d)m_4)}\cdot\\&\cdot\Big[\prod_{v>r}\gamma^{-d(h_v-k)(s_v-1)}\gamma^{\frac{d-2}{2}(h_v-k)n_{0,v}^{inner}}\gamma^{\frac{d}{2}(h_v-k)n_{1,v}^{inner}}\Big]\end{aligned}\quad(16.16)$$

Denoting v' the vertex preceding v in γ and denoting $\widetilde{n}_{j,v}^{inner}$, $j=0,1$, the number of lines before contractions (i.e. half lines), inside the box corresponding to v (which is not necessarily the first box containing them; i.e. $\widetilde{n}_{j,v}^{inner}\ge n_{j,v}^{inner}$ in general) and using

$$\begin{aligned}&\sum_{v>r}(h_v-k)(s_v-1)\equiv\sum_{v>r}(h_v-h_{v'})(n_v-1),\\&\sum_{v>r}(h_v-k)\,n_{j,v}^{inner}\equiv\sum_{v>r}(h_v-h_{v'})\,\widetilde{n}_{j,v}^{inner},\quad(16.17)\\&\widetilde{n}_{0,v}^{inner}\equiv 2m_{2,v}+4m_{4,v}-n_{0,v}^e,\\&\widetilde{n}_{1,v}^{inner}\equiv 2m_{2',v}-n_{1,v}^e,\quad j=0,1\end{aligned}$$

one realizes from (16.16) that

$$\begin{aligned}&M_S(\Delta_1,\ldots,\Delta_p)\le\mathcal{N}\,\overline{\varepsilon}^n\,\widetilde{B}^{n^e}\,\overline{B}_1^n\,e^{-\frac{\kappa}{2}\gamma^k d(\Delta_1,\ldots,\Delta_p)}\cdot\\&\cdot\gamma^{-k\,(2m_2+(4-d)m_4)}\prod_{v>r}\gamma^{-\rho_v\,(h_v-h_{v'})}\end{aligned}\quad(16.18)$$

with

$$\rho_v=-d+2m_{2,v}+(4-d)m_{4,v}+\frac{d-2}{2}n_{0,v}^e+\frac{d}{2}n_{1,v}^e.\quad(16.19)$$

Therefore recalling that the contribution to $V^{(k)}$ of the trees of given shape is obtained by summing over all the possible choices S and over all the possible frequency assignments to the vertices of the trees (i.e. over all the possible values of $h_v-h_{v'}>0$, $h_v<N$), one realizes that the estimate (16.18) and (16.19) for (16.12) implies ultraviolet finiteness if for all S and all tree shapes it is $\rho_v>0$.

However, clearly, there are plenty of cases with $\rho_v\le 0$ for some v, if $d\ge 2$.

The situation would be slightly better if one had started with a more restrictive interaction $\mathcal{I}$ (see (16.1)) –e.g. if $\mathcal{I}$ had been replaced by

$$\int_\Lambda\big(-\lambda:\varphi^{(\le N)^4}-\mu:\varphi^{(\le N)^2}--\gamma\big)\quad(16.20)$$

In this case it is easily realized that, in (16.18) and (16.19), one has just to take $n_{1,v}^e=0,m_{2',v}=0$. This implies $\begin{cases}d=2\ \Rightarrow\ \rho_v>0,\ \forall v\\ d=3\ \Rightarrow\ \rho_v>0\end{cases}$ unless $n_{0,v}^e=2$ and $m_{2,v}+m_{4,v}=2$, or $n_{0,v}^e=0$ and $m_{2,v}+m_{4,v}=0$: i.e. the theory (16.20) is ultraviolet finite in dimension $d=2$. However if $d=3$ it is not ultraviolet finite and one has to check if it is renormalizable.

Going back to (16.1) for $d=4$ we discover many cases with $\rho_v\le 0$; in general it, however, clear that $\rho_v>0$ if there are too many lines external to the box corresponding to v, i.e. if $n_{0,v}^e\ge 5$.

The above discussion completes the analysis of the origin of the divergences and of their strength. In the next sections the problem of renormalizing the theory (16.1) will be studied and solved for $d\le 4$.

A final but, as it will turn out, very important remark is that the above method allows us to produce estimates of (16.12) when the rule to compute $V(\gamma;S)$ is modified by replacing the $\lambda^{(\alpha)}$ contributions from the end points of γ with constants $r^{(\alpha)}(h_j)$ with h_j being the frequency of the vertex to which the j-th end point is attached by its tree branch.

Suppose that

$$r^{(\alpha)}(h)=\gamma^{(4-d)\,h\,\delta_{\alpha,4}+e\,h\,\delta_{\alpha,2}+d\,h\,\delta_{\alpha,0}}\quad(16.21)$$

and repeat the power counting argument leading to the bound (16.18). In this case the result will be, for $n>1$,

$$\begin{aligned}&M_S(\Delta_1,\ldots,\Delta_p)\le\mathcal{N}\widetilde{B}_1^n e^{-\frac{\kappa}{2}d(\Delta_1,\ldots,\Delta_p)\,\gamma^k}\cdot\\&\cdot\Big(\prod_{v>r}\gamma^{-\rho'_v\,(h_v-h_{v'})}\Big)\cdot\Big(\prod_{j=1}^n\overline{r}^{(\alpha_j)}(h_j)\Big)\end{aligned}\quad(16.22)$$

and

$$\rho'_v = -d + \frac{d-2}{2} n^e_{0,v} + \frac{d}{2} n^e_{1,v} \tag{16.23}$$

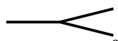
i.e. the lines coming from vertices of type $\alpha = 2$ acquire the "same dimension" as those coming from the vertices of type $\alpha = 4$.

In the bounds (16.22) and (16.23) the values α_j must be nonzero so that the factor $\gamma^{d\,h\,\delta_{2,0}}$ plays no role in deducing them. It has been inserted only for later reference.

xvii. Renormalization of φ^4-field to second order

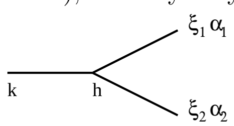
The application of the general renormalization theory (see Secs. 7 and 8) to cure the ultraviolet instability pointed pout in Dec. 16 follows the same scheme met in the case of the cosine field, in Sect. 12.

It is slightly more complex, because polynomials do not have nice multiplication properties, not as nice as those of complex exponentials multiplication rules which plaid a (hidden) role in simplifying the algebra in the discussion of the cosine interaction.

However it is still true that, to proceed, one has to understand in detail only the renormalization theory to second order, i.e. the definition of the subtraction operation $\mathcal{L}^{(\sigma_0)}_k$ with $\sigma_0 =$ , the other cases being easily understandable in terms of this special case.

A detailed understanding of the above simple case is absolutely essential and the inexperienced reader should check the minutest details of the following few straightforward but lengthy calculations, which are the heart of renormalization theory (contrary to what is sometimes asserted about the true difficulties being connected with the "overlapping divergences", a term that is not even defined here).

To proceed as in Sec. 7 one starts by defining the trees dressed to order 1: which are just the trees considered in Sec. 16. The one considers the trees of degree two (i.e. with two endpoints); actually they are



$\alpha_1, \alpha_2 = 2, 2', 4$ (46)

They have been estimated in Sec. 16, but it is easy to compute them explicitly from their expressions in Sec.6; after integration over the end points position labels ξ_1, ξ_2 they contribute to $V_1^{(k)}$

$$\begin{aligned}\frac{1}{2} V_1^{(k,\alpha_1,\alpha_2)} &= \frac{1}{2}\int_\Lambda d\xi_1 d\xi_2 \cdot \\ &\cdot \mathcal{E}_{k+1}\cdots\mathcal{E}_{h-1}\mathcal{E}^T_h\big(v_h^{(\alpha_1)}(\varphi^{(\le h)}_{\xi_1}, \varphi^{(\le h)}_{\xi_2})\big)\end{aligned} \tag{17.1}$$

A simple calculation which the reader should perform at least once in his life, in spite of its length (after all non so bad) gives

$$\begin{aligned}
&(1)\ V^{(k,2,2)} = \mu^2\binom{2}{1}^2 \int :\varphi_1\varphi_2: C^{(k)}_{12}\, d\xi_{12} + \\
&\qquad + \mu^2 2! \int \big(C^{(\le h)\,2}_{12} - C^{(\le k)\,2}_{12}\big)\\
&(2)\ V^{(k,2,2')} = \mu\alpha\binom{2}{1}^2 \int :\varphi_1\partial\varphi_2: \partial_2 C^{(k)}_{12}\, d\xi_{12} + \\
&\qquad \mu\alpha 2! \int \big((\partial_1 C^{(\le h)}_{12})^2 - (\partial_1 C^{(\le k)}_{12})^2\big)\\
&(3)\ V^{(k,2,4)} = \mu\lambda\binom{2}{1}^2\binom{4}{1} \int :\varphi_1\varphi_2^3: C^{(k)}_{12}\, d\xi_{12} + \\
&\qquad + \mu\lambda 2!\binom{4}{3} \int :\varphi_2^2: \big(C^{(\le h)\,2}_{12} - C^{(\le k)\,2}_{12}\big)\\
&(4)\ V^{(k,2',2')} = \alpha^2\binom{2}{1}^2\binom{4}{1} \int :\partial\varphi_1\,\partial\varphi_2^3: \partial^2_{12} C^{(k)}_{12} + \\
&\qquad + \alpha^2 2!\binom{4}{3} \int \big(\partial_{12} C^{(\le h)\,2}_{12} - \partial_{12} C^{(\le k)\,2}_{12}\big)\, d\xi_{12}\\
&(5)\ V^{(k,2',4)} = \alpha\lambda\binom{2}{1}^2\binom{4}{1} \int :\partial\varphi_1\varphi_2^3: C^{(h)}_{12}\, d\xi_{12} + \\
&\qquad + \alpha\lambda 2!\binom{4}{3} \int :\varphi_2^2: \big(\partial_1 C^{(\le h)\,2}_{12} - \partial_1 C^{(\le k)\,2}_{12}\big)\\
&(6)\ V^{(k,4,4)} = \lambda^2\binom{4}{1}^2 1! \int :\varphi_1^3\varphi_2^3: C^{(h)}_{12}\, d\xi_{12} + \\
&\qquad + \lambda^2\binom{4}{2}^2 2! \int |\varphi_1^2\varphi_2^2: (C^{(\le h)\,2} - C^{(\le k)\,2}) d\xi_{12} + \\
&\qquad + \lambda^2\binom{4}{3}^2 3! \int :\varphi_1\varphi_2: (C^{(\le h)\,3}_{12} - C^{(\le k)\,3}_{12}) d\xi_{12} + \\
&\qquad + \lambda^2\binom{4}{4}^2 4! \int \big(C^{(\le h)\,4}_{12} - C^{(\le k)\,4}_{12}\big)
\end{aligned} \tag{17.2}$$

where φ_1, φ_2 mean $\varphi^{(\le k)}_{\xi_1}, \varphi^{(\le k)}_{\xi_2}$, and $C^{(\cdot)}_{12}$ means $C^{(\cdot)}_{\xi_1\xi_2}$, $d\xi_{12} = d\xi_1 d\xi_2$, $\partial_1 \equiv \frac{\partial}{\partial\xi_1}, \partial_2 \equiv \frac{\partial}{\partial\xi_2}$, $(\partial_1 C^{(\cdot)})^2 \equiv \partial_1 C^{(\cdot)} \cdot \partial_1 C^{(\cdot)}$; the symbol $V^{(k,\alpha_1,\alpha_2)}$ denotes $V^{(k)}(\gamma)$ with γ given by Fig.46.

Some of the above integrals are not ultraviolet stable, once appropriately summed over h (i.e. for $h \in [k+1, N]$), as it is easy to check using the bounds of Sec.16 and showing that they admit "good bounds" (or by direct computation from (17.2)); see the following table where (i),(ii),(iii),(iv) mean "first addend", "second addend", "third addend","fourth addend" (when present in the rows of (17.2)), and S, U mean "stable" or "unstable"; $d \ge 2$ is the dimension of the theory

	(i)	(ii)	(iii)	(iv)
(1)	S	U if $d = 4$	-	-
(2)	S	U if $d \ge 2$	-	-
(3)	S	U if $d = 4$	-	-
(4)	U	U if $d \ge 2$	-	-
(5)	S	U if $d \ge 2$	-	-

(6)	S	U if $d=4$	U if $d\ge 2$	U if $d\ge 2$	$RP\,=\,P$	otherwise

Using (17.2) and proceeding according to the theory of Sec.7 we can find counterterms $V_{2,N}$ to V_1 so that the effective potentials $V_2^{(k)}$ of $V_1+V_{2,N}$ are ultraviolet finite to second order.

Following the ideas developed in Sec. 7, one can start by trying to define the operation $\mathcal{L}_k$ making (7.10), i.e. $(1-\mathcal{L}_k)$ applied to (17.1) and summed over $h\in]k+1,N]$ diverge for $\xi_1=\xi_2$.

Therefore one can think of defining $\mathcal{L}_k$ by specifying its action on functions F having the form of the *r.h.s.* of (17.2) with kernels in front of the Wick monomials replaced by general kernels $w(\xi_1,\xi_2)$:

$$F=\int w(\xi_1,\xi_2)\,P\,d\xi_1 d\xi_2 \tag{17.3}$$

with the restriction that the w kernels are translation invariant on A (recall that periodic boundary conditions are imposed on Λ) and rotation covariant with respect to the rotations by $\frac{\pi}{2}$ around the coordinate axes (which are the only meaningful rotations on the torus Λ). The covariance here refers to the fact that the Wick monomials in (17.2) may contain derivatives of the fields: each derivative bears an index denoting to which component it refers and hence will bear corresponding indices– i.e. it will be a tensor; this fact does not explictly show up in (17.2) or in the upcoming formulae because of the convention used here that suppresses the indices of the derivatives, for simplicity of notation.

Once the action of $\mathcal{L}_k$ is specified on the functions of the form of (17.3) it will be extended to their linear combinations by linearity, some more comments on $\mathcal{L}_k$ as an operator will be made later after discussing its action on F's like (17.3).

To produce the cancellations of the divergences which appear once the *r.h.s.* of (17.2) are summed over h, generating expressions which are linear combinations of expressions like (17.3) diverging for $\xi_1=\xi_2$, the action of $(1-\mathcal{L}_k)$ should result in replacing the monomial P in (17.3) by a new expression RP vanishing as $\xi_2-\xi_1\to 0$ to an order so high that the integrals are no longer divergent.

An examination of the integrals shows that the following choice of RP would produce ultraviolet finite integrals:

$$\begin{aligned}
&R1\;=0\\
&R\,:\varphi_1\varphi_2\;:=:\;\varphi_1\big(\varphi_2-\varphi_1-(\xi_2-\xi_1)\partial\varphi_1-\\
&\;-\frac{1}{2}(\xi_2-\xi_1)\times(\xi_2-\xi_1)\partial^2\varphi_1\big)\;:,\\
&R\,:\varphi_1\partial\varphi_2\;:=:\;\varphi_1\big(\partial\varphi_2-\partial\varphi_1-(\xi_2-\xi_1)\cdot\partial\,\partial\varphi_1\big)\;:-\\
&R\,:\partial\varphi_1\partial\varphi_2\;:=:\;\partial\varphi_1\big(\partial\varphi_2-\partial\varphi_1\big) \qquad\qquad (17.4):,\\
&R\,:\varphi_1^2\varphi_2^2\;:=:\;\varphi_1^2(\varphi_2-\varphi_1)^2\;:,\\
&R\,:\varphi_1\varphi_2^3\;:=:\;\varphi_2^2(\varphi_1-\varphi_2)^2\;:,
\end{aligned}$$

and using (3.20) (recall that the regularization being used here has $n=3$), one sees that the replacement of P by RP replaces P by a Wick polynomial which has a zero of order, respectively, ∞, third, second, first, first, zero.

Hence if there is an operation $\mathcal{L}_k$ such that $(1-\mathcal{L}_k)$ acting on the integrals in (17.2) just changes P into RP, then $\mathcal{L}_k$ has the property that (7.10) is, in the present case, ultraviolet finite because the above mentioned orders of zero of RP are sufficient, in the worst cases, to make the expressions (17.3) ultraviolet finite.

From (17.4) one deduces that the operation $\mathcal{L}_k$ "which identifies the divergent parts" of $V_1^{(k)}$ to second order has to act on the integrals (17.2) or more generally (17.3) as

$$\begin{aligned}
&\mathcal{L}_k\int w(\xi_1,\xi_2)\,d\xi_1 d\xi_2\;=\;\int w(\xi_1,\xi_2)\,d\xi_1 d\xi_2\\
&\mathcal{L}_k\int w(\xi_1,\xi_2)\,\varphi_{\xi_1}\varphi_{\xi_2}\;:=\int w(\xi_1,\xi_2)\,\varphi_{\xi_1}\big(\varphi_{\xi_1}+\\
&+(\xi_2-\xi_1)\partial_{\xi_1}\;+\frac{1}{2}(\xi_2-\xi_1)^2\times\partial^2\varphi_{\xi_1}\big)\\
&\mathcal{L}_k\int w(\xi_1,\xi_2)\,\varphi_{\xi_1}\partial\varphi_{\xi_2}\;:=\int w(\xi_1,\xi_2)\,\varphi_{\xi_1} \qquad (17.5)\\
&\big(\varphi_{\xi_1}+(\xi_2-\xi_1)\partial_{\xi_1}\cdot\partial\varphi_{\xi_1}\big)\\
&\mathcal{L}_k\int w(\xi_1,\xi_2)\,\partial\varphi_{\xi_1}\partial\varphi_{\xi_2}\;:=\int w(\xi_1,\xi_2)\,\partial\varphi_{\xi_1}\big(\partial\varphi_{\xi_1}\big)^2\\
&\mathcal{L}_k\int w(\xi_1,\xi_2)\,\partial\varphi_{\xi_1}^2\partial\varphi_{\xi_2}^2\;:=\int w(\xi_1,\xi_2)\,(\varphi_{\xi_1})^4,\\
&\mathcal{L}_k\int w(\xi_1,\xi_2)\,\partial\varphi_{\xi_1}\partial\varphi_{\xi_2}^3\;:=\int w(\xi_1,\xi_2)\,(\varphi_{\xi_1})^4,\\
&\mathcal{L}_k\equiv 0,\qquad \text{otherwise}
\end{aligned}$$

so that the action of $(1-\mathcal{L}_k)$ on the integrals like (17.3) is precisely obtained by replacing in them P by RP.

One has to check that $\mathcal{L}_k$ takes values in the space of the interactions; this is in fact the basic reason why the theory is renormalizable.

Possibly integrating by parts or using the rotation invariance properties of the coefficients $w(1,2)$, on can easily check that the action of $\mathcal{L}_k$ on the integrals in (17.5) is equivalent to the action of the following operator $\overline{\mathcal{L}}$ on the Wick monomials inside the integrals (here $\varphi\equiv\varphi^{(\le k)}$, $\theta,\theta'=1,2,\ldots,d$, $\partial_\theta=\frac{\partial}{\partial\theta^{(\theta)}}$, if $\xi^{(\theta)}$ is the θ-th component of the point ξ):

$$\begin{aligned}
&\overline{\mathcal{L}}\,1\;=\;1\,,\\
&\overline{\mathcal{L}}\,:\varphi_{\xi_1}\varphi_{\xi_2}\;:=:\;\varphi_{\xi_1}^2\;:-\frac{(\xi_2-\xi_1)^2}{2d}\;:(\partial\varphi_{\xi_1})^2\;:,\\
&=\,1\,,\\
&\overline{\mathcal{L}}\,1:\,\partial\varphi_{\xi_1}\partial\varphi_{\xi_2}\;:==:\;(\partial\varphi_{\xi_1})^2\;: \qquad (17.6)\\
&\overline{\mathcal{L}}\,:\varphi_{\xi_1}^2\varphi_{\xi_2}^2\;=::\varphi_{\xi_1}^4\;:,\\
&\overline{\mathcal{L}}\,:\varphi_{\xi_1}\varphi_{\xi_2}^3\;:=:\varphi_{\xi_1}^4\;:,\\
&1\,=\,1\,,
\end{aligned}$$

$$\overline{\mathcal{L}} = 0 \qquad \text{otherwise}$$

This proves that the range of $\mathcal{L}_k$ is in the space of the interactions if one takes $\mathcal{L}_k$ to defined by acting on expressions like (17.3) by replacing P inside them by $\overline{\mathcal{L}}P$ (see (17.6)).

The above e analysis shows, also, that trivially the action of $(1-\mathcal{L}_k)$ on expressions like (17.3) is precisely the substitution of P by RP.

It is convenient to stop to point out the following. The operation $\mathcal{L}_k$ defined above is not unambiguously defined as an operator in the sense of functional analysis: to let $\mathcal{L}_k$ act on functions like (17.1) one has, by definition, first to express them as a sum of functions like (17.3) and then to act term by term replacing P by $\overline{\mathcal{L}}P$ (see (17.6)).

However the expression of (17.1) as a linear combination of expressions like (17.3) is not unique.

Therefore, in order that the above definition of $\mathcal{L}_k$ makes sense one has also to prescribe how one writes (17.1), or more generally a function in the domain of $\mathcal{L}_k$, as a linear combination of terms like (17.3). Expression (17.2) is the prescription used here for the functions of interest. Also later on we shall have to use a well-defined prescription for the decomposition of the effective potentials as sums of terms on which the action of the higher order subtraction operations $\mathcal{L}^{(\sigma)}$ is defined. The prescription for the decomposition of the effective potential has therefore to be thought of as part of the definition of $\mathcal{L}_k$.

Taking into account the above comment we then check the relation (7.13) first by verifying that the prescription to decompose the interesting functions (i.e. the effective potentials) as a sum of terms in the domain of the $\mathcal{L}_k$ operations commutes with the expectations $\mathcal{E}_{p+1}\cdots\mathcal{E}_k$, and second by asking whether $\mathcal{L}_k$ also commutes (in the sense of (7.13) with them.

Both the above checks are very simple in our case; actually, the systematic use of Wick ordered interactions and of Wick monomials has the basic motivation of making this check an essentially trivial consequence of (4.2) (implies by Wick ordering) and the definition of $\mathcal{L}_k$.

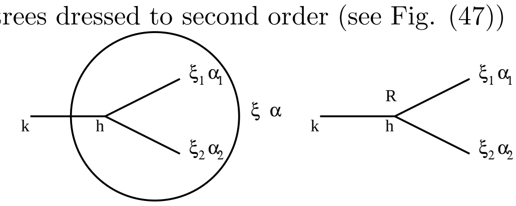
From (17.6) and (17.4) and applying the general theory of Sec. 7 (see (7.14) and (7.16)) one finds easily the following expression of the counterterms $V_{2,N}$; if $\varphi\equiv\varphi^{(\le N)}$ it is

$$\begin{aligned}
V_{2,N} = -\int d\xi_1 \sum_{h=0}^{N} \Big\{ &:\varphi^2_{\xi_1}: \int d\xi_2 \Big[\frac{1}{2}\mu^2 \binom{2}{1}^2 C^{(h)}_{12} + \\
&+\mu\lambda\, 2! \binom{4}{2} (C^{(\le h)^2} - C^{(<h)^2}) + \\
&+\alpha\lambda 2! \binom{4}{2} (\partial_1 C^{(\le h)^2} - \partial_1 C^{(<h)^2}) + \\
&+\lambda^2 \frac{3!}{2} \binom{4}{3}^2 (C^{(\le h)^3} - C^{(<h)^3}) + \\
&+\mu\alpha \frac{1}{2}\binom{2}{1} \partial_1 C^{(h)}_{12}\Big] + \\
&:(\partial_1\varphi_{\xi_1})^2: \int d\xi_2 \Big[-\mu\alpha\binom{2}{1}^2 \frac{\xi_2-\xi_1}{d}\partial_2 C^{(\le h)} - \\
&-\mu^2\frac{1}{2}\binom{2}{1}^2 \frac{(\xi_2-\xi_1)^2}{2d} C^(h)_{12} - \\
&\lambda^2 \frac{3!}{2}\binom{4}{3}^2 \frac{(\xi_2-\xi_1)^2}{2d} (C^{(\le h)^3}_{12} - C^{(<h)^3}_{12}) + \\
&+\frac{\alpha^2}{2}\partial_{12} C^{(h)}_{12}\Big] + :\varphi^4_{\xi_1}: \int d\xi_2 \Big[\mu\lambda \binom{2}{1}\binom{4}{1} C^{(h)}_{12} + \\
&+\frac{\lambda^2}{2} 2! \binom{4}{2} (C^{(\le h)^2} - C^{(<h)^2})\Big] + \\
&{}^1 \int d\xi_2 \Big[\frac{\mu^2}{2} 2! (C^{(\le h)^2} - C^{(<h)^2}) + \\
&+\mu\alpha\, 2!\, (\partial_1 C^{(\le h)^2} - \partial_1 C^{(<h)^2}) + \\
&+\frac{\alpha^2}{2} 2!\, (\partial_{12} C^{(\le h)^2} - \partial_{12} C^{(<h)^2}) + \\
&+\frac{\lambda^2}{2}\, 4! \binom{4}{4}^2 (C^{(\le h)^4} - C^{(<h)^4})\Big]\Big\}
\end{aligned} \tag{17.7}$$

It should be stressed that for some terms in (17.2) the rule (17.6) produces needless subtractions, as far as the ultraviolet stability is concerned; in fact rule (17.6) coincides with the "usual rule" in the literature only in the "usual" case $\alpha=\mu=0$, $d=4$; if $d<4$ then (17.6) is oversubtracting even in this case.

Nevertheless the "universal rule" (17.6) will be used for simplicity of exposition; it would probably be not difficult to make the theory of Secs. 7 and 8 more flexible so that more refined subtraction methods become possible permitting us to introduce counterterms only for "really divergent" parts of the effective interaction.

It is easy to compute in the above cases the meaning of the trees dressed to second order (see Fig. (47))



(47)

According to Sec. 7 (see (7.10)) the framed tree represents one of the terms in (17.7) with the summations over h ranging from 0 to k (rather than to N) and with φ now meaning $\varphi^{(\le k)}$, up to a factor 2. Precisely select the contribution to (17.7) from the term $V^{(k,\alpha_1,\alpha_2)}$ in (17.2) (or $V^{(k,\alpha_2,\alpha_1)}$, whichever is present in (17.2)) containing $v^{(\alpha)}_N$, $\alpha=4,2,2',0$, i.e. containing $:\varphi^4:$, $:\varphi^2:$, $:(\partial\,\varphi)^2:$, 1; then the frame in Fig. (47) means

$$r^{(\alpha)}(\sigma,k)\, v^{(\alpha)}_k(\varphi^{(\le k)}, \partial\,\varphi^{(\le k)}), \tag{17.8}$$

where the r coefficient is the coefficient of the term in (17.7) just selected but with the summation over h ranging from 0 to k; here σ is a symbol for the tree shape framed in Fig. (47). clearly $r^{\alpha)}(\sigma,k)$ is proportional to $\lambda^{(\alpha_1)}\lambda^{(\alpha_2)}$.

The unframed dressed tree in Fig. (47) represents, if we follow the rules of Secs 7,8, exactly (17.2) with the replacement induced by (17.4), $P \to RP$, see (7.16). Thus if we introduce the new fields

$$\begin{aligned}
&D_{\xi_1,\xi_2} \stackrel{def}{=} \varphi_{\xi_1} - \varphi_{\xi_2},\\
&D^1_{\xi_1,\xi_2} \stackrel{def}{=} \partial\varphi_{\xi_1} - \partial\varphi_{\xi_2},\\
&S^1_{\xi_1,\xi_2} \stackrel{def}{=} \partial\varphi_{\xi_1} - \partial\varphi_{\xi_2} - (\xi_1-\xi_2)\cdot\boldsymbol{\partial}\partial\varphi_{\xi_2},\\
&S_{\xi_1,\xi_2} \stackrel{def}{=} \varphi_{\xi_1} - \varphi_{\xi_2} - (\xi_1-\xi_2)\cdot\boldsymbol{\partial}\varphi_{\xi_2},\\
&T_{\xi_1,\xi_2} \stackrel{def}{=} \varphi_{\xi_1} - \varphi_{\xi_2} - (\xi_1-\xi_2)\cdot\boldsymbol{\partial}\varphi_{\xi_2} -\\
&\qquad -\frac{1}{2}(\xi_2-\xi_1)^2 \times \boldsymbol{\partial}^2\varphi_{\xi_2},
\end{aligned} \tag{17.9}$$

where $\boldsymbol{\delta}^2 \times \boldsymbol{\partial}^2$ means, if $\boldsymbol{\delta}$ is a vector in $\mathbb{R}^d$,

$$\sum_{i,j=1}^{d} \delta_i\delta_j \frac{\partial^2}{\partial\xi_j\partial\xi_j},$$

then the contributions from the unframed tree in Fig. (47) to the effective potential $V_2^{(k)}$ due to $V_1 + V_{2,N}$ at fixed h, α_1, α_2 are, if $\varphi \equiv \varphi^{(\le k)}$,

$$\frac{1}{2}\mu^2\binom{2}{1}^2 \;:\varphi_{\xi_1}T_{\xi_1\xi_1}:\; C^{(h)}_{\xi_2\xi_1}, \quad \alpha_1,\alpha_2 = 2, \tag{17.10}$$

$$\mu\alpha\binom{2}{1}^2 \;:\varphi_{\xi_1}S^1_{\xi_2\xi_1}:\; \partial_{\xi_1}C^{(h)}_{\xi_2\xi_1}, \quad \alpha_1 = 2.\alpha_2 = 2', \tag{17.11}$$

$$\mu\lambda\binom{2}{1}\binom{4}{1} \;:\varphi^3_{\xi_1}D_{\xi_2\xi_1}:\; C^{(h)}_{\xi_2\xi_1}, \quad \alpha_1,\alpha_2 = 4, \tag{17.12}$$

$$\frac{1}{2}\alpha^2\binom{2}{1}^2 \;:\partial\varphi_{\xi_1}D^1_{\xi_2\xi_1}:\; \partial_{\xi_1\xi_2}C^{(h)}_{\xi_2\xi_1}, \quad \alpha_1,\alpha_2 = 2' \tag{17.13}$$

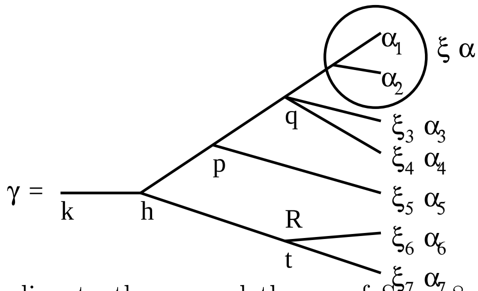
$$\alpha\lambda\binom{2}{1}\binom{4}{1} \;:\partial\varphi_{\xi_1}\varphi^3_{\xi_2}:\; \partial_{\xi_1}C^{(h)}_{\xi_2\xi_1}, \tag{17.14}$$

$\alpha_1 = 2', \alpha_2 = 4$(unchanged), and for $\alpha_1, \alpha_2 = 4$:

$$\begin{aligned}
&\frac{\lambda^2}{2}\Big[\binom{4}{1}^2 1! \;:\varphi^3_{\xi_1}\varphi^3_{\xi_2}:\; C^{(h)}_{\xi_2\xi_1} +\\
&+\binom{4}{2}^2 2! \;:\varphi^2_{\xi_1}(\varphi_{\xi_1}+\varphi_{\xi_2})D_{\xi_1\xi_2}:\; (C^{(\le h)\,2}_{\xi_2\xi_1} - C^{(<h)\,2}_{\xi_2\xi_1}) +\\
&+\binom{4}{3}^2 3! \;:\varphi_{\xi_1}T_{\xi_1\xi_2}:\; (C^{(\le h)\,3}_{\xi_2\xi_1} - C^{(<h)\,3}_{\xi_2\xi_1})\Big],
\end{aligned} \tag{17.15}$$

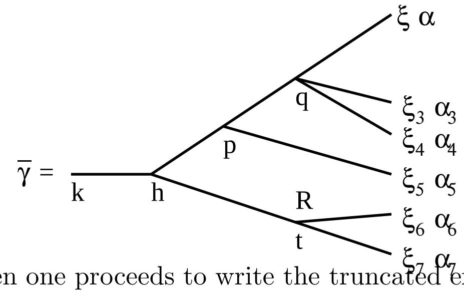
A simple way to describe the construction of (17.10)–(17.15), i.e. to interpret the R over the vertex of the tree in Fig. (47), is to think that the tree in Fig. (47) is computed from the values of the same tree without the R followed by the replacement of P by RP in the result.

It is also easy to compute the meaning of the most general tree dressed to order 2 (see Sec. 8 for the notation), as with the example in Fig. (48) below:



(48)

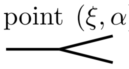
According to the general theory of Sec. 8 the first act will be to "trim" the tree γ of the frame and of its contents (if there are more frames one trims all of them), obtaining a simpler tree $\overline{\gamma}$; e.g. in the case of Fig. (48) the result would be



(49)

Then one proceeds to write the truncated expectation formula for the evaluation of the contribution to $V^{(k)}$ of the tree $\overline{\gamma}$, ignoring the presence of the R superscript (see comments in Sec. 7 after Fig. (13)). The vertex bearing the R contributes

$$\mathcal{E}^T_t\Big(v^{(\alpha_6)}(\varphi^{(\le t)}_{\xi_6}), v^{(\alpha_7)}(\varphi^{(\le t)}_{\xi_7})\Big) \tag{17.16}$$

in the above above example and a similar expression in general; then one just replaces in (17.16) the Wick monomials P by RP according to (17.4). Finally one replaces the factor $\lambda^{(\alpha}$ contributed to the effective potential by the end point (ξ, α) with the factor $r^{(\alpha)}(\sigma, q)$, σ being the shape , framed according to (17.8).

This completes the analysis of the second-order renormalization. It justifies calling (17.8) *form factors with structure* σ.

It will be clear that a detailed check of all the above formulae is the heart of renormalization theory and therefore the inexperienced reader should proceed only after having well understood the details of the above calculations.

As an exercise the reader can consider the theory of renormalization to second order of the following problems.

(1) Let $\mathcal{I}_N$ be

$$-\lambda\int_\Lambda :\varphi^{(\le N)\,4}_\xi:\, d\xi \tag{17.17}$$

and show that if $d = 2$ one can take $\mathcal{L}^{(\sigma)}_k \equiv 0$, i.e. no renormalization is necessary.

(2) Let $\mathcal{I}_N$ be

$$\int_\Lambda (-\lambda :\varphi_\xi^{(\le N)\,4}: -\mu :\varphi_\xi^{(\le N)\,2}: -\nu)\, d\xi \tag{17.18}$$

and work put the renormalization to second order in the case $d=3$, proving that one could use as a definition of $\mathcal{L}_k$, instead of (17.6), the following

$$\overline{\mathcal{L}}1 = 1,\ \overline{\mathcal{L}} :\varphi_{\xi_1}\varphi_{\xi_2}: =: \varphi_\xi^{(\le N)\,2}: \tag{17.19}$$

(3) Let $\mathcal{I}_N$ be

$$\int_\Lambda (-\mu :\varphi_\xi^{(\le N)\,2}: -\alpha :(\partial\varphi_\xi^{(\le N)})^2:)\, d\xi \tag{17.20}$$

and work out in detail renormalization theory showing that, unless $\alpha=0$, one still needs nontrivial renormalization. However the theory can be rigorously built if μ,α are small or non negative.

(4) Show that the theory with interaction (17.18) is not renormalizable if $d=4$, not even to second order, in the sense of Secs. 6–8 (not identical, although trivially related, to the one usual in the literature).

xviii. Renormalization and ultraviolet stability to any order for φ^4 fields

Section]sec(17) has shown that renormalization to second order suggests a representation of the effective potential in terms of Wick monomials more general that the ones used in Sec. 16, (16.4), and precisely as a sum of contributions like

$$\sum_P \int \frac{V(\gamma;P)}{n(\gamma)}\, P\, d\boldsymbol{\xi} \tag{18.1}$$

where P has the form (here $\varphi\equiv\varphi^{(\le k)}$ and the symbols in (17.9) are used)

$$P =: \Big(\prod_j \varphi_{\rho_j}^{n_j}\Big)\cdot\Big(\prod_j D_{\eta_j\eta_j'}^{m_j}\Big)\cdot\Big(\prod_j D^{1\,n_j'}_{\theta_j\theta_j'}\Big)\cdot \\ \cdot\Big(\prod_j \partial\varphi_{\xi_j}^{p_j}\Big)\cdot\Big(\prod_j S_{\xi_j\xi_j'}^{q_j}\Big)\cdot\Big(\prod_j S^{1\,r_j}_{\zeta_j\zeta_j'}\Big)\cdot\Big(\prod_j T^{t_j}_{\mu_j\mu_j'}\Big) : \tag{18.2}$$

with $n_j\le 4$, $\rho_j, m_j, n_j, q_j, r_j, t_j \le 2$.

The most naive way to proceed is to define recursively the localization operations $\mathcal{L}_k^{(\sigma)}$ associated with tree shapes of degree $p+1$ (i.e. with $p+1$ end points) partially dressed to order p simply by using again the localization prescription (17.6) and the corresponding renormalization prescriptions for the interpretation of the vertices with R-superscripts (17.4): the idea being that, as suggested by (16.18) and (16.19), the divergences are caused by the contributions to $V(\gamma;P)\,P$ from the vertices v of $|g$ describing a Wick monomial of degree ≤ 4.

However if P is given a general form (18.2) it is clear that there will be plenty of monomials of order ≤ 4 which do not appear in (17.4) and for which the operations R and $\mathcal{L}$ are not defined yet. The first task is to classify them.

One assumes inductively that the renormalized effective potential corresponding to an interaction renormalized to order p:

$$V_P = V_1 + V_{2,N} + V_{3,N} + \ldots + V_{p,N} \tag{18.3}$$

is still described in terms of Decorated Feynman graphs S as

$$\sum_{n=1}^{\infty}\int \sum_{\gamma;degree\,\gamma=n}\ \sum_S \frac{V(\gamma;S)}{n(\gamma)} \tag{18.4}$$

where now the graphs S will bear more decorations (compared to the cases treated in Sec. 16 to describe the "effects" of the renormalization.

One checks this inductive assumption in the case $p=2$ first, where it can be checked, because $V_{2,N}$ has already been studied in Sec. 17.

Let γ be any tree dressed to order 2, e.g. see fig. 48. Trim γ of the endframes and consider one decorated Feynman graph S corresponding to the evaluation of the effective potential due to the trimmed tree $\overline{\gamma}$ but ignoring the superscripts R.

Draw a box B_v around the cluster of position labels of $\overline{\gamma}$ corresponding to the vertex v of $\overline{\gamma}$: the box B_v will be drawn so that the lines of S with frequency index h_v are inner to B_v but not inner to v' if $v'>v$, as in Sec. 16.

Therefore, out of each box B_v emerge lines of S possibly carrying ∂ labels, as in Sec. 16 will be systematically used below; for instance, $n^e_{1,v}$ and $n^e_{0,v}$ will be. respectively, the number of lines emerging from B_v and carrying or not carrying a ∂ label; n^e_v will be defined to be the sum of the above two numbers.

So each box B_v represents a Wick monomial P_v, as in Sec. 16. Consider the vertices v bearing in $\overline{\gamma}$ an R: note that they must correspond to some innermost nontrivial clusters and precisely to those with two points in them.

Let one such v represent, in the given S, a Wick monomial P_v: one replaces it by $R\,P_v$, see (17.4).

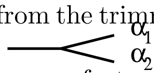
If $R\,P_v = P_v$ nothing has to be said; but if $R\,P_v \ne P_v$ one has to interpret that the vertex v contributes, via the graph S, $R\,P_v$ rather than P_v to the evaluation of the truncated expectations corresponding to $\overline{\gamma}$.

If $R\,P_v$ is a Wick monomial in the fields $\varphi, \partial\varphi, D, D^1, S, S^1, T$, see (17.9), then one denotes this operation of substitution of P_v by RP_v by simply adding an index 0 to the box B_v; but in some cases, actually only in one case among those so far considered, RP_v may not be a Wick monomial in the above fields. In fact the $R :\varphi_\xi^2\varphi_\eta^2:$ is, by (17.4), $:\varphi_\xi^2(\varphi_\eta^2-\varphi_\xi^2) :=: \varphi_\xi^3 D_{\eta\xi} : + : \varphi_\xi^2\varphi_\eta D_{\eta\xi}) :$, i.e. a sum of two monomials rather than a single monomial.

If RP_v is a Wick polynomial, sum of various monomials numbered from 0 to m, then one attaches to the box B_v a label $b_v = 0, 1, \ldots, m$ to indicate which term in $R\,P_v$ one selects in the evaluation of the truncated expectations as a contribution from v.

One takes into account this index β_v by changing accordingly the meaning of the lines of S emerging from the box B_v, e.g. the line representing φ_2 in $:\varphi_1\varphi_2:$ is replaced by T_{21}, that representing $\partial\varphi_2$ in $:\varphi_1\partial\varphi_2:$ is replaced by S^1_{21} that representing $\partial\varphi_2$ in $:\partial\varphi_1\partial\varphi_2:$ is replaced by D^1_{21}, the one representing one of the two φ_2's in $:\varphi_1^2\varphi_2^2:$ is replaces by D_{21} if the index β_v appended to the box B_v (which now takes values 0 or 1) is $\beta_v = 1$ while, if $\beta_v = 0$ then one of the two lines representing $|f_2^2$ is replaced by a line representing φ_1 and the other by one representing D_{21} (which one is replaced by which is irrelevant, e.g. one can decide on the basis of the identity indices appended to the lines emerging from a point, say lexicographically); in the $:\varphi_1\varphi_2^3:$ case the line representing φ_1 is replaced by by D_{12}.

Then the evaluation of $V(\overline{\gamma}; S)$ proceeds as before with the consequent change of meaning of the covariances associated with the contracted lines (when two lines are contracted, they give rise to the covariance between the two fields that they represent, of course).

Clearly at the end of the computation one still has to replace the $\lambda^{(\alpha)}$ contributed by the end points of $\overline{\gamma}$ (coming from the trimmed end frames bearing inside the shape $\sigma =$ 

, and attached to a vertex of frequency h) by new factors $r^{(\alpha)}(\sigma; h)$ as explained in Sec. 17, see (17.8) and the related discussion.

The result of the above procedure is a formula like (18.4) for the effective potential $V_2^{(k)}$ due to $V_2 + V_{2,N}$. Hence the inductive assumption is satisfied for $p = 2$.

Assume (18.4) for $p = 2, 3, \ldots, p_0$ and let γ be a tree dressed to order p_0 and of degree $p_0 + 1$; assume to have already defined operations $\mathcal{L}_k^{(\sigma)}$ for all the shapes of degree $\le p_0$.

Assume also that the result of the action of such operations leads to a rule of evaluation of the trees with no frames (and possibly some R indices) which consists in examining successively the boxes B_v corresponding to the vertices of a tree, starting from the innermost ones, and changing successively the monomials P_v, which each of them represents, into a new monomial in the fields $\varphi, \partial\varphi, D, D^1, S, S^1, T$ appearing in the polynomial RP_v defined as follows.

If P has one of the forms contemplated in (17.4), then RP is defined as in (17.4), i.e. if $\varphi_j = \varphi^{(\le k)}_{\xi_j}$, $\delta_{ij} = \xi_i - \xi_j$,

$$
\begin{aligned}
&R\,1 = 0\\
&R :\varphi_1\varphi_2: = :\varphi_1 T_{21}:\\
&R :\varphi_1\partial\varphi_2: = :\varphi_1 S^1_{21}:\\
&R :\partial\varphi_1\partial\varphi_2: = :\partial\varphi_1 D^1_{21}:\\
&R :\varphi_1^2\varphi_2^2: = :\varphi_1^3 D_{21} + :\varphi_1^2\varphi_2 D_{21}:\\
&R :\varphi_1\varphi_2^3: = :D_{12}\varphi_2^3:,
\end{aligned}
\tag{18.5}
$$

where (18.5) is just a way of rewriting (17.4) in the new notations (17.9) and the $D, S, T.D^1, S^1$ fields have indices j which mean ξ_1, ξ_2 and have also frequency indices which are the same as those of $|f$ and which are not explicitly written.

With the same notations the action of R on other monomials of degree ≤ 4 is defined by

$$
\begin{aligned}
&R :\varphi_2 D_{12}: = :\varphi_2 T_{12}:\\
&R :\varphi_2 S_{12}: = :\varphi_2 T_{12}:\\
&R :\varphi_1 S_{23}: = :D_{12} S_{23}: + R :\varphi_2 S_{21}:\\
&R :\varphi_1 S_{12}: = :D_{12} S_{12}: + :D_{21} T_{12}: + :\varphi_1 T_{12}:\\
&R :\varphi_1 D_{32}: = R :D_{12} D_{32}: + R :\varphi_2 D_{23}: =\\
&\quad = - :S_{12} S_{32}: + :S_{12} D_{32}: + :D_{12} S_{32}: +\\
&\quad + :D_{21} T_{32}: + :\varphi_1 T_{32}:,\\
&R :D_{12} D_{32}: = - :S_{12} S_{32}: + :S_{12} D_{32}: +\\
&\quad + :D_{12} S_{32}:\\
&R :D_{12} D_{34}: = - :S_{12}\,\delta_{34}\,D^1_{34}: + :D_{12}\,\delta_{34}\,D^1_{24}: +\\
&\quad - :S_{12} S_{34}: + :S_{12}\,D_{34}: + :D_{12} S_{34}:\\
&R :\partial\varphi_1\partial\varphi_2: = :\partial\varphi_1 D^1_{21}:\\
&R :\varphi_1\partial\varphi_2: = :\varphi_1 S^1_{21}:\\
&R :\partial\varphi_1 D_{21}: = :\partial\varphi_1 S_{21}:\\
&R :\varphi_1 D^1_{21}: = :\partial\varphi_1 S^1_{21}:\\
&R :\varphi_3 D^1_{21}: = :D_{31} D^1_{21}: + :D_{13} S^1_2: + :\varphi_3 S^1_{21}:\\
&R :\partial\varphi_1 D_{23}: = :D^1_{13} D_{23}: + :\partial_3 S_{23}:\\
&R :\varphi_1\varphi_2^2\varphi_3: = :D_{12}\varphi_2^2 D_{32}: + :\varphi_1\varphi_2^2 D_{32}:\\
&R :\varphi_1\varphi_2\varphi_3\varphi_4: = :\varphi_1 D_{21}\varphi_3\varphi_4: + :\varphi_1 D_{12} D_{31}\varphi_4: +\\
&\quad :\varphi_1\varphi_2 D_{31}\varphi_4: + :\varphi_1 D_{12} D_{13} D_{41}: +\\
&\quad :\varphi_1\varphi_2 D_{13} D_{41}: + :\varphi_1\varphi_2\varphi_3 D_{41}: +\\
&\quad :\varphi_1\varphi_2^2\varphi_3 D_{41}: .
\end{aligned}
\tag{18.6}
$$

The action of R on the monomials which differ from the ones listed above by a sign (e.g. $\varphi_2 D_{21}$) is that R acts by changing the

sign of the *r.h.s.* ; for the remaining monomials one puts $R\,P = P$.

The basic idea informing the definitions (18.5) and (18.5) is to subtract from each monomial P its "value at coinciding points" (defined below by the $\overline{\mathcal{L}}$ operation) to an order such that RP contains a zero of order

1 if degree of $P = 4$ and $n^e_{1,v} = 1$

$$\begin{array}{lll} 3 & \text{if degree of } P = 2 & \text{and } n^e_{1,v} = 1 \\ 2 & \text{if degree of } P = 2 & \text{and } n^e_{1,v} = 1 \\ 1 & \text{if degree of } P = 2 & \text{and } n^e_{1,v} = 1 \\ \infty & \text{if degree of } P = 0 & \text{and } n^e_{1,v} = 1 \end{array}$$

In other words if we call $\widetilde{\rho}_v$ the above order of zero then $\widetilde{\rho}_v$ is defined, see (16.23), as the smallest integer for which $\widetilde{\rho}_v + \rho'_v > 0$.

Furthermore, the definition of R is such that each of the monomials on the *r.h.s.* can be thought of as obtained by substituting for one of the factors in P an "improved" factor climbing the chains $\varphi \to D \to S \to T$ or $D \to S \to T$ or $D \to D^1 \to S^1$ or $\partial\varphi \to D^1 \to S^1$ or $D^1 \to S^1$ or $S \to T$.

In analogy with the second order case of Sec. 17 it is natural to try to define the operation $\mathcal{L}^{(\sigma)}_k$ on the contribution of the tree γ with shape $|s$ to the effective potential $V^{(k)}_{p_0}$, assuming that the tree has degree $p_0 + 1$ but that it is dressed to order p_0 only (this is the situation that has to be considered according to the general theory of Sec. 8). If this contribution is denoted

$$\sum_S \int V(\gamma; S)\, P_S\, d\boldsymbol{\xi}, \tag{18.7}$$

then, in analogy with Sec. 17, $\mathcal{L}^{(\sigma)}_k$ should act on (18.7) by just changing P_S into $\overline{\mathcal{L}}P_S$ and $\overline{\mathcal{L}}$ should be defined so that for any kernel w verifying the translation invariance and rotation covariance (for rotations of $\frac{\pi}{2}$ around the coordinate axes only, since Λ is taken with periodic boundary conditions) it should be $\int w\,(1-\overline{\mathcal{L}})P\, d\boldsymbol{\xi} = \int w\, RP\, d\boldsymbol{\xi}$.

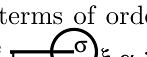
After some thought one realizes that this aim can be achieved by defining the action of $\overline{\mathcal{L}}$ to be that of replacing a non local Wick monomial P by its Taylor expansion truncated to an order $\widetilde{\rho}_v - 1$.

Since P is nonlocal it will have to be decided around which of the points appearing as indices of the fields in P the Taylor expansion will be made. For instance one could choose any of them and then symmetrize the result over the choices; however it is notationally and practically simpler to choose one among them giving the simplest form to the result; sometimes this may still leave some ambiguity; the ambiguity will be resolved by arbitrary choices guided by the labels j on the points ξ_j. Of course this implies that one has to be careful in imagining to draw the trees on the plane always in a standard way (a precaution ignored so far), i.e. picking up systematically one representative from each equivalence class and numbering the end points also in a standard way (e.g. from top to bottom).

In the expression below the indices of the fields in P are always supposed to be $\xi_{j_1}, \xi_{j_2}, \xi_{j_3}, \xi_{j_4}$ but the shortened notation $\varphi_j = \varphi^{(\le k)}_{\xi_j}$ will be used as well as $\delta_{ij} = (\xi_{j_i} - \xi_{j_j})$. Also if θ, θ' are component indices we set $\delta^2_{ij} \times \partial^2\varphi_j \overset{def}{=} \sum_{\theta,\theta'=1}^d \frac{\partial^2}{\partial\xi_\theta\,\partial_{\xi'_\theta}}$. Then with the above conventions

$$\begin{aligned}
&\overline{\mathcal{L}}\,1 = 1\\
&\overline{\mathcal{L}}\ :\varphi_1\varphi_2: \;=\; :\varphi_1(\varphi_1 + \delta_{21}\partial\varphi_1 + \tfrac{1}{2}\delta^2_{21}\times\partial^2\varphi_1):\\
&\overline{\mathcal{L}}\ :\partial\varphi_1\partial\varphi_2: \;=\; :\partial\varphi_1\partial\varphi_1:\\
&\overline{\mathcal{L}}\ :\varphi_1\partial\varphi_2: \;=\; :\varphi_1(\partial\varphi_1 + \delta_{21}\cdot\partial\,\partial\varphi_1):\\
&\overline{\mathcal{L}}\ :\varphi_1 D_{21}: \;=\; :\varphi_1(\delta_{21}\cdot\partial\,\varphi_1 + \tfrac{1}{2}\delta^2_{21}\cdot\partial\,\varphi_2):\\
&\overline{\mathcal{L}}\ :D_{13}D_{23}: \;=\; :\delta_{13}\cdot\partial\,\varphi_3\,\delta_{23}\cdot\partial\,\varphi_3:\\
&\overline{\mathcal{L}}\ :\varphi_1 D_{23}: \;=\; :\delta_{13}\cdot\partial\,\varphi_3\,\delta_{23}\cdot\partial\,\varphi_3: +\\
&\qquad + :\varphi_3(\delta_{23}\cdot\partial\,\varphi_3 + \tfrac{1}{2}\delta^2_{23}\times\partial^2\varphi_3):\\
&\overline{\mathcal{L}}\ :D_{12}D_{34}: \;=\; :\delta_{12}\cdot\partial\,\varphi_2\,\delta_{34}\cdot\partial\varphi_2:\\
&\overline{\mathcal{L}}\ :\varphi_1 S_{21}: \;=\; \tfrac{1}{2}:\varphi_1\delta_{21}\times\partial^2\varphi_1:,\\
&\overline{\mathcal{L}}\ :\varphi_1 S_{12}: \;=\; \tfrac{1}{2}:\varphi_2\delta_{12}\times\partial^2\varphi_2:,\\
&\overline{\mathcal{L}}\ :\varphi_1 S_{23}: \;=\; \tfrac{1}{2}:\varphi_3\delta_{23}\times\partial^2\varphi_3:,\\
&\overline{\mathcal{L}}\ :\varphi_1 D^1_{21}: \;=\; :\varphi_1\delta_{21}\cdot\partial\,\partial\varphi_1:,\\
&\overline{\mathcal{L}}\ :\varphi_1 D^1_{23}: \;=\; :\varphi_3\delta_{23}\cdot\partial\,\partial\varphi_3:,\\
&\overline{\mathcal{L}}\ :\partial\varphi_1 D^1_{21}: \;=\; :\partial\varphi_1\delta_{21}\cdot\partial\,\varphi_1:,\\
&\overline{\mathcal{L}}\ :\partial\varphi_1 D^1_{23}: \;=\; :\partial\varphi_3\delta_{23}\cdot\partial\,\varphi_3:,\\
&\overline{\mathcal{L}}\ :\varphi_1\varphi_2^3: \;=\; :\varphi_2^4:,\quad \overline{\mathcal{L}}\ :\varphi_1^2\varphi_2^2: \;=\; :\varphi_1^4:,\\
&\overline{\mathcal{L}}\ :\varphi_1\varphi_2^2\varphi_3: \;=\; :\varphi_1^4:,\\
&\overline{\mathcal{L}}\ :\varphi_1\varphi_2\varphi_3\varphi_4: \;=\; :\varphi_1^4:,
\end{aligned} \tag{18.8}$$

and $\overline{\mathcal{L}}P = 0$ if P does not differ by just a sign from one of the above monomials, $\overline{\mathcal{L}}P = -\overline{\mathcal{L}}(-P)$ if P differs by a sign from one of the above monomials.

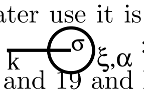
If the above is taken as definition of $\overline{\mathcal{L}}$ one can find $\mathcal{L}^{(\sigma)}_k$ and hence, by the general algorithm of Sec. 8, the counterterms of order $p_0 + 1$ as well as the meaning of the tree .

Recall that one is proceeding inductively and the definition of the counterterms (and the meaning of the dressed trees) is supposed known for trees of degree $\le p_0$. Of course one has first to check that the operation $\mathcal{L}^{(\sigma)}_k$ has range in the space of the interactions (see Secs. 7 and 8). This follows, as in the case of second order renormalization, by studying the integrals of expressione like (18.8) times kernels satisfying the translation invariance and rotation covariance mentioned above (and possibly integrating by parts to obtain expressions of the appropriate form).

It is perhaps worth saying why $\mathcal{L}^{(\sigma)}_k$ bears an index σ: in fact $\overline{\mathcal{L}}$ is defined independently of σ. However $\mathcal{L}^{(\sigma)}_k$ acts on the functions of the form (18.7) and a function F can be written in several ways in the form (18.7). As discussed in Sec. 17 the operation $\mathcal{L}^{(\sigma)}_k$ acts on the effective potential written in the form (18.7) as it arises from the prescriptions of the calculation to be followed in evaluating the contribution of the graph S to the effectivepotential once the tree γ is given (such prescriptions are the ones discussed in detail in Sec. 16): the prescription depends on the shape σ of γ; hence so does $\mathcal{L}_k^{(\sigma)}$. To be more precise in Sec. 16 the prescriptions for the evaluation of the effective potential in terms of decorated Feynman graphs were given in the absence of renormalization: but renormalization just allows more complex Wick monomials and therefore a possibility of giving to the graphs lines the meaning of more complex fields and can still use the same graphical rule to build the evaluation of the expectations via the Wick rules.

Therefore it will be decided to choose as definition of $\mathcal{L}_k^{(\sigma)}$ on the expressions (18.7) the action of the operations $\overline{\mathcal{L}}$ on the integrands. Then, by the above construction, the action of $(1 - \overline{\mathcal{L}})$ generates an interpretation of the R superscripts on the trees dressed to order $p_o + 1$ as meaning that the Wick monomial represented in a given graph S by a vertex v of order $p_0 + 1$ has to be replaced by RP defined by (18.5), (18.6).

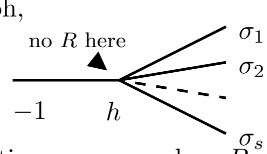
This means that the inductive assumption is indeed verified for $p = p_0 + 1$ and hece for all p. It also means that $\mathcal{L}_k^{(\sigma)}$ depends on σ only through the tree shape $\tau\sigma$ obtained by deleting the frames σ as well as their contents, a necessary property in order to apply the resummation theory of Sec. 9 to the present problem.

For later use it is convenient to recall the meaning of the tree  : it is obtained by the rules of 16; see Fig. 18 and 19 and Eq. (8.5).

One starts by erasing the frame around the shape s and its labels ξ, α. Then one attributes frequency indices to the vertices of σ which are outside the remaining frames, and one also attributes position indices to the unframed end points of $|s$ and to the endframes of σ: in this way one builds a partially dressed tree $\gamma \stackrel{def}{=} (\sigma^{\mathbf{h}}, \boldsymbol{\xi})$, because the first vertex of γ bears no superscript R (because before erasing the frame it was enclosed inside it and therefore had no R superscript).

Suppose that the indices $\mathbf{h}$ are such that the root frequency is -1: $h_r = -1$.

One proceeds by computing, with the rules explained above, the effective potential $V(\gamma; S)$, where S is a decorated Feynman graph,

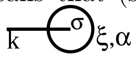


(50)

with enough decorations on every box B_v to allow recognizing which choice among the monomials of RP_v is made at that vertex: as explained above, this is done by adding an index β_v corresponding to a vertex v bearing a superscript R and β_v can take only a few values (from (18.5) and (18.6) one sees that $\beta_v = 0, 1, 2, 3, 4, 5, 6$ are enough in the mostcomplex cases).

Since the *r.h.s.* of (18.8) is made up of local expressions in the fields and the coefficients are kernels with translation invariance and rotation covariance (in the sense considered above), it follows that the integrals over the position labels of $V(\gamma; S)P_S$ summed over S can be cast in the form "of an interaction":

$$\begin{aligned}&\int_\Lambda (I^{(4)}(\sigma^h) : \varphi_\xi^{(-1)\,4} : + \\ &+ I^{(2)}(\sigma^h) : (\partial\varphi_\xi^{(-1)})^2 : + I^{(0)}(\sigma^h))\, d\boldsymbol{\xi},\end{aligned} \tag{18.9}$$

and this means that (see (8.5)) the form factor corresponding to  : is

$$r^{(\alpha)}(\sigma, k) = \sum_{h=0}^{k} \sum_{\mathbf{h}'} \frac{I^{(\alpha)}(\sigma^{\mathbf{h}})}{n(\sigma)} \tag{18.10}$$

where h denotes the frequency index of the first vertex of $\sigma^{\mathbf{h}}$ after the root and $\mathbf{h}'$ are the frequency indices on the higher vertices (and the root frequency is supposed to be -1).

Naturally the N dependence of (18.10) is in the fact that the summation indices over $\mathbf{h}'$ run with upper bounds equal to N; nevertheless, it will appear that $r^{(\alpha)}(\sigma, k)$ admits a limit as $N \to \infty$, at fixed k.

This completes the inductive description of the counterterms and of their effects on the tree representation of the effective potentials.

The final result is that after complete renormalization

$$V^{(k)} = \sum_{n=1}^{\infty} \int \sum_{\substack{\gamma\,:\,k(\gamma)=k \\ degree\ \gamma=n,\xi(\gamma)=\xi \\ \gamma\ dressed}} \sum_S \frac{V(\gamma; S)}{n(\gamma)} P_S \, d\boldsymbol{\xi}, \tag{18.11}$$

where the sum runs over the Feynman graphs S associated with the trimmed tree $\overline{\gamma}$ (i.e. γ deprived of the outer frames and of their contents), decorated by boxes (defining the clusters associated with the vertices v of $\overline{\gamma}$) bearing indices β_v explaining the selection to be made in evaluating RP_v (the index β_v can take at most seven values). Furthermore the graph S bears all the other decorations already possible in the nonrenormalized cases (i.e. frequency, character, identity and ∂ indices, see Sec. 16).

It remains for s to check that, with the above definitions of the subtraction operations, the new theory is ultraviolet finite.

Given a dressed tree $\overline{\gamma}$ with no frames (i.e. with every vertex of γ bearing an index R) one has to study, given $\varphi^{(\leq k)}$ verifying (3.20) (with $n = 3$), the expression (see the analog (16.12))

$$\begin{aligned}&M_S(\Delta_1, \ldots, \Delta_p) = \int_{\Delta_1 \times \ldots \times \Delta_p \times \Lambda^{n-p}} \cdot \\ &\cdot |V(\overline{\gamma}; S)| \sup |P_S| \, d\boldsymbol{\xi}, \qquad \Delta_j \in Q_k,\ p \leq n,\end{aligned} \tag{18.12}$$

where S is a given decorated Feynman graph: n is the degree of γ, k is the root frequency.

Clearly the integral (18.12) is evaluated by just the same type of analysis leading to the bounds (16.18) in the case of no renormalization. One has only to replace some covariances with new covariances due to the fact that some lines have the meaning of new fields (D, S, T, D^1, S^1).

However a few remarkable improvements are generated by such changes.

Call a line of S representing fields like (17.9) a "renormalized line". Below $\overline{\gamma}$ and S will be fixed.

Looking at the graph S one can see which is the vertex v "causing" the change of meaning of a renormalized line compared to the meaning that the line would have in the graph S_0 obtained from S by erasing all the decorations which allow one to interpret it as a renormalized graph. It must be a vertex v corresponding to a box B_v , which in S_0 would determine a monomial P_v on which R acts non-trivially ($RP_v \neq P_v$, see (17.4),(18.5) and (18.6)). The actual meaning of a renormalized line cannot be determined by v alone. In fact, as (18.6) shows, it may happen that its meaning is changed again in correspondence of a vertex $v' < v$ such that $R_{v'}$ contains two external lines.

But the change of meaning cannot take place more than four times, because the meaning of the line "keeps improving" (i.e. the corresponding order of zero in the RP polynomial keeps increasing): a φ line can become a D or S or T line, a D line can become an S or T line, and S line can become a T line, a $\partial\varphi$ line can become a D^1 or S^1 line, a D^1 line can become an S^1 line; and R is the identity when acting on monomials containing S^1 or T fields.

So, given $\overline{\gamma}, S$ and a renormalized line of S once can define the first vertex responsible for its change of meaning with respect to the meaning it would have in S_0; one can also define the vertices $v_1, v_2, \ldots$ following v where the line again changes meaning before acquiring its final meaning; from (18.6) and (18.5) one sees that this change of meaning cannot take place more than a fixed number of times (four). Finally one can define the vertex w where the line becomes internal to a box B_w for the first time: $w = r =$ root of γ if the line is external.

Call ρ_v the parameter associated with the vertex v (see (16.18),(16.19)) in the graph S_0. Then it is clear that the fact that the line has changed meaning introduces in the basic bound (16.7) an extra factor given, at least, by

$$B_2\,(\gamma^{h_w} d(\boldsymbol{\xi}_v))^{\delta_v} \equiv B_2\,\gamma^{-(h_v-h_w)\delta_v}\,(\gamma^{h_v} d(\boldsymbol{\xi}_v))^{\delta_v}, \tag{18.13}$$

where B_2 is suitable and δ_v is the variation of the order of zero, as $d(\boldsymbol{\xi}_v) \to 0$, introduced in P_v by the R operation via the change of meaning of the line under consideration.

Therefore every time a given line changes meaning at vertices $v_1 > v_2 > \ldots$ new factors like (18.13) arise in the bounds on $M_S(\Delta_1, \ldots, \Delta_p)$, and by construction the sum over the lines λ that change meaning and over the vertices v of the quantities δ_v is such that

$$\sum_\lambda \sum_v \delta_v\,(h_v - h_w) \geq \sum_v \widetilde{\rho}_v\,(h_v - h_{v'}), \tag{18.14}$$

if v' is the vertex immediately preceding v in γ, and $\widetilde{\rho}_v = -\rho_v + 1$ if $\rho_v \leq 0$ and $\widetilde{\rho}_v = 0$ otherwise. Eventually the bound on $M_S(\Delta_1, \ldots, \Delta_p)$ becomes

$$\begin{aligned} &\Big[\int_{\Delta_1\times\ldots\times\Delta_p} \Big(\prod_\lambda e^{-\kappa\gamma^{h_\lambda}|\lambda|}\Big)\Big(\prod_v \gamma^{-(h_v-h_{v'})\,\widetilde{\rho}_v}\Big)\cdot \\ &\cdot\Big(\prod_v (\gamma^{h_v} d(\boldsymbol{\xi}_v))^{\widetilde{\theta}}\Big)\,d\xi\Big]\cdot \\ &\cdot\,\overline{\varepsilon}^n \mathcal{N} B^{n_e} \widetilde{B}_2^{n_e} B_2^{4n} \gamma^{\frac{d-2}{2}\,k\,n^e_{0,v_0}}\gamma^{\frac{d}{2}\,k\,n^e_{1,v_0}}\cdot \\ &\cdot\prod_{v>r}\big(\gamma^{h_v\frac{d-2}{2}\,n^{inner}_{0,v}}\gamma^{h_v\frac{d}{2}n^{inner}_{0,v}}\big) \end{aligned} \tag{18.15}$$

where $\widetilde{B}_2$ is defined by a formula like (16.13) in which P_S has the new meaning (and the *r.h.s.* is changed accordingly in the natural way: note that the new *r.h.s.* will contain, in general factors like (18.13) when P_S contains renormalized fields and use is made of (3.20) to exhibit the order of zero in the D, S, T, D^1, S^1 fields; the constant $\widetilde{\theta}_v$ can, in principle, be read by comparing (18.13) and (18.15)).

The exponents $\widetilde{\theta}_v$ can be bounded by the maximum of $\widetilde{\rho}_v$ (i.e. three) times the number of times a line can change meaning (i.e. four at most) times the number of lines that do change meaning at the vertex v (by (18.5) and (18.6)): actually this happens only when P_v looks like $\varphi_1\varphi_2\varphi_3\varphi_4$, see (18.5). Call T th above bound ($T = 3^2\,4$).

Hence for all $\zeta > 0$ it is, if $d(\boldsymbol{\xi})$ is the graph distance between the points of $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$

$$\prod_v (\gamma^{h_v} d(\boldsymbol{\xi}_v))^{\widetilde{\theta}_v} \leq \big(\frac{T!}{\zeta}\big)^{4n} e^{\zeta T \sum_{v>r}\gamma^{h_v} d(\boldsymbol{\xi}_v)} \tag{18.16}$$

where the $4n$ arises from the fact that the lines changing meaning at v can become internal at different vertices w: at most four.

This is used to choose ζ so that $\zeta T < \frac{1}{4}\kappa\,(-\gamma^{-1})$, which can be used together with the inequality

$$\sum_\lambda \gamma^{h_\lambda}|\lambda| \geq (1-\gamma^{-1})\sum_{v>r}\gamma^{h_v} d(\boldsymbol{\xi}_v), \tag{18.17}$$

a consequence of $\gamma^h \geq (1-\gamma^{-1})(1+\gamma^{-1}+\gamma^{-2}+\ldots+\gamma^{-h})\gamma^h$ and of elementary geometry, to bound (18.15) by (see Appendix D for the bound (16.14) on the integral)

$$\begin{aligned} &\int_{\Delta_1\times\ldots\times\Delta_p} e^{-\kappa\sum_\lambda \gamma^{h_\lambda}|\lambda|}\prod_v (\gamma^{h_v} d(\boldsymbol{\xi}_v)) \leq \\ &\leq e^{-\frac{\kappa}{4}\gamma^k d(\Delta_1\times\ldots\times\Delta_p)}\big(\frac{T!}{\zeta}\big)^{4n}\cdot \\ &\cdot\int_{\Lambda^{n-1}} \gamma^{-kd} e^{-\frac{\kappa}{4}\sum_\lambda \gamma^{h_\lambda}|\lambda|} d\xi_2\ldots d\xi_n \end{aligned} \tag{18.18}$$

which, inserted in (18.15) and after appropriate power counting, becomes

$$M_s(\Delta_1 \times \ldots \times \Delta_p) \le \overline{\varepsilon}\mathcal{N}B^{n^e}\widetilde{B}_3^n e^{-\frac{\kappa}{4}\gamma^k d(\Delta_1\times\ldots\times\Delta_p)} \cdot$$
$$\cdot\, \gamma^{-k(2m_2+(4-d)m_4)} \prod_{v>r} \gamma^{-(\rho_v+\widetilde{\rho}_v)(h_v-h_{v'})} \tag{18.19}$$

for a suitable $\widetilde{B}_3$, if v' denotes the vertex immediately before v in γ and with ρ_v defined in (16.19), using the graph S_0 obtained from S by erasing all the labels referring to the renormalization, and

$$\rho_v + \widetilde{\rho}_v = -d + 2m_{2,v} + (4-d)m_{4,v} +$$
$$+ \frac{d-2}{2}n^e_{0,v} + \frac{d}{2}n^e_{1,v} + \delta_{n^e_{0,v},4}\delta_{n^e_{1,v},0} + 3\delta_{n^e_{0,v},2}\delta_{n^e_{1,v},0} +$$
$$+ 2\delta_{n^e_{0,v},1}\delta_{n^e_{1,v},1} + 1\delta_{n^e_{0,v},0}\delta_{n^e_{1,v},2} \ge \frac{1}{2} \stackrel{def}{=} \overline{\rho} \tag{18.20}$$

(here $n^e_v, n^e_{1,v}, n^e_{0,v}$ are counted as they appear in S_0).

Actually, for later use, one can remark that if $2m_{2,v} + (4-d)m_{4,v}$ is replaced by 0 in (18.20) one obtains a new expression $\rho'_v + \rho_v$ which, nevertheless, is still larger than $\overline{\rho} = \frac{1}{2}$, see (16.20)–(16.23).

Expressions (18.19) and (18.20) prove the ultraviolet finiteness for the trees which are dressed but contain no frames.

If γ bears frames enclosing shapes $\sigma_1, \ldots, \sigma_m$, $m \le n =$ degree of γ, attached to the trimmed tree $\overline{\gamma}$, obtained from $|g$ by trimming it at the vertices of frequency $h_1, \ldots, h_m$ (allow here the convention that the unframed end points are regarded as frames by a frame containing the trivial shape, as already done in the previous sections), then the bound (18.17) is obviously replaced by

$$M_s(\Delta_1 \times \ldots \times \Delta_p) \le \mathcal{N}B^{n^e}\widetilde{B}_3^n e^{-\frac{\kappa}{2}\gamma^k d(\Delta_1\times\ldots\times\Delta_p)} \cdot$$
$$\cdot\, \gamma^{-(2m_2+(4-d)m_4)k} \cdot \prod_{v>r} \gamma^{-(\rho_v+\widetilde{\rho}_v)(h_v-h_{v'})} \cdot$$
$$\cdot \prod_{j=1}^m |r^{(\alpha_j)}(\sigma_j; h_j)|, \tag{18.21}$$

where the factors $r^{(\alpha)}(\sigma; h)$ are the form factors associated with the shapes σ (see Secs. 8,17 and (17.8)) defined by (18.10), $r^{(\alpha)}(\sigma, h) \equiv \lambda^{(\alpha)}$ if the shape σ enclosed in the frame is trivial.

Consider $d = 4$ and suppose that one could prove that

$$|r^{(\alpha)}(\sigma; h)| \le \gamma^{2h\delta_{\alpha,2}}\gamma^{4h\delta_{\alpha,0}} h^s \overline{\varepsilon}^s C_s \tag{18.22}$$

where s is the degree of the shape σ and

$$\overline{\varepsilon} = \max(|\lambda|, |\mu|, |\alpha|, |\nu|) = \max_\alpha |\lambda^{(\alpha)}|.$$

Then, as already noted in Sec.16 and after (18.20) above, the factors $\gamma^{2h\delta_{\alpha,2}}$ would affect the bounds (18.19) and (18.20) by replacing $\widetilde{\rho}_v + \rho_v$ by $\rho'_v + \rho_v$ and $2m_{2,v}$ by 0, so that (18.19) becomes

$$M_s(\Delta_1 \times \ldots \times \Delta_p) \le \mathcal{N}B^{n^e}\widetilde{B}_3^n e^{-\frac{\kappa}{2}\gamma^k d(\Delta_1\times\ldots\times\Delta_p)} \cdot$$
$$\cdot \sum_{\mathbf{h}} \prod_{v>r} \gamma^{-\overline{\rho}\,(h_v-h_{v'})} \cdot \prod_{i=1}^m h_i^{s_i} C_{s_i}, \tag{18.23}$$

and the ultraviolet finiteness would follow also for the frame bearing dressed trees.

It is convenient to remark that the bound (18.23) can be considerably improved at no cost if one notes that, by the nature of the bounds leading to the $d(\Delta_1 \times \ldots \times \Delta_p)$ in the exponential, one could have obtained instead the quantity $d_S(\Delta_1 \times \ldots \times \Delta_p)$, where this is defined as the sum of the distances between the cubes Δ joined in S by a hard line. It is clear that this is a much better bound for very structured graphs.

It remains for us to prove (18.22); however in Sec. 19 a much stronger bound, compared to (18.22) (easy as it will appear) will be proved. Therefore the proof of (18.22) is postponed to Sec. 19.

The results of this section basically contain the "Hepp theorem" ((Hepp, 1966, 1969)): this theorem provided the first completely rigorous proof of ultraviolet stability (see also (Eckmann and Epstein, 1979; Speer, 1974; Zimmermann, 1969)).

xix. "$n!$ bounds" on the effective potential

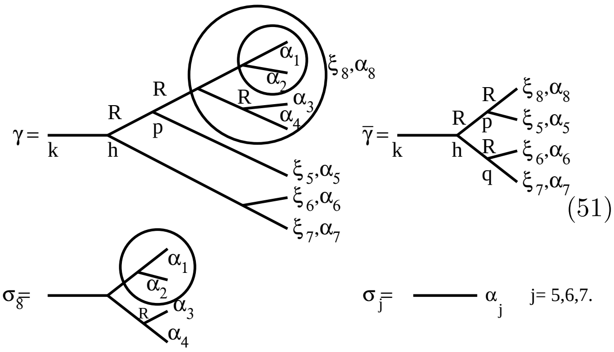
It is now possible to find concrete bounds on the coefficients of the effective potentials. In this section we take $d = 4$, for simplicity (the cases $d < 4$ are similar and slightly easier).

From the preceding analysis emerges the following organization of the contributions to $V^{(k)}$ of the trees of degree n.

A dressed tree γ will be described by its trimmed part $\overline{\gamma}$. obtained by cutting out of γ all frames and their contents, and by the actual contents of the external frames of γ: one per end point of γ which bears a frame; for uniformity of notation one imagines here that all the end points of the dressed trees are framed so that if k———— ξ,α; is an endbranch of γ which bears no frames one imagines to transform it into k——(σ) ξ,α .

The degree of γ will in general be larger than or equal to the degree m of $\overline{\gamma}$, which will be called the "renormalization degree" of γ.

So a dressed tree γ will be described by $\overline{\gamma}$ and m shapes $\sigma_1, \sigma_2, \ldots, \sigma_m$, which have to be enclosed in frames attached to the end points of $\overline{\gamma}$ to rebuild γ: if γ has degree n and σ_i degree n_i it must be $n = \sum_{i=1}^m n_i$. For instance the following picture shows a tree γ together with its trimmed part $\overline{\gamma}$ and the shapes $\sigma_1, \sigma_2, \ldots, \sigma_m$,



(51)

The number of shapes of degree s can be easily estimated by D_1^s for some D_1 (one can tale $D_1=2^4$).

Consider the contribution to $V^{(k)}$ from the trees of degree n:

$$V^{(k),n}=\int\sum_{\substack{k(\gamma)=k\\ degree\ \gamma\ =n\\ \boldsymbol{\xi}(\gamma)=\boldsymbol{\xi}}}\sum_{S}\frac{V(\gamma;S)}{n(\gamma)}P_S\,d\boldsymbol{\xi} \tag{19.1}$$

where P_S has the form (18.2) and S is a decorated Feynman graph as described in Sec. 18. The aim of this section is to show that if $\xi_1,\ldots,\xi_m$ are the endframe labels of $\overline{\gamma}$, then there are γ–independent constants B,κ,D,b such that if $B=\sup_\Delta B_\Delta$ in (3.20) and $\overline{\varepsilon}=\max_\alpha|\lambda^{(\alpha)}|$ it is

$$\begin{aligned}M(\Delta_1,\ldots,\Delta_p)=&\int_{\mathcal{D}(\Delta_1,\ldots,\Delta_p)}d\boldsymbol{\xi}\sum_{\substack{k(\gamma)=k\\ degree\ \gamma\ =n\\ \boldsymbol{\xi}(\gamma)=\boldsymbol{\xi}}}\sum_{\substack{S\\ P_S=P}}\sup|P|\,\cdot\\ \cdot\,\frac{|V(\gamma;S)|}{n(\gamma)}\le&\ \mathcal{N}B^n\overline{\varepsilon}^ne^{-\kappa\gamma^kd(\Delta_1,\ldots,\Delta_p)}\,n!\sum_{j=1}^{n-1}\frac{(bk)^j}{j!},\end{aligned} \tag{19.2}$$

where $\mathcal{D}(\Delta_1,\ldots,\Delta_p)\overset{def}{=}\Delta_1\times\ldots\Delta_p\times\Lambda\times\ldots\times\Lambda$ or a domain obtained by permuting such faactors; $\Delta_j\in Q_k$ and the supremum of P means supremum over the fields $\varphi^{(\le k)}=\sum_{j=0}^k\varphi^{(j)}$ with $\varphi^{(j)}$ satisfying (3.20); $\mathcal{N}$ depends on the degree of P only: $\mathcal{N}=O((n^e)!)$.

Equation (19.2) will be called the $n!$ bound: this bound was obtained in a slightly different form (i.e. as a bound on the Schwinger functions rather than on the effective potentials, and in "momentum space" rather than in "position space") and with a somewhat different method in the remarkable work (DeCalan and Rivasseau, 1982). The approach presented below follows essentially (Gallavotti and Nicolò, 1985a,b).

The first problem is to find explicit combinatorial estimates on the number of terms in (19.2).

Since S has the interpretation of a decorated Feynman graph with m vertices, m being the renormalized degree of γ (i.e. the degree of its trimmed part), and since the decorations consist of finitely many indices attached to each line and vertex of the graph (see Sec. 18 for an explicit description of such indices, each of which can take a number of values which is finite and graph independent, except for the frequency indices) it follows that one can bound the number of terms in $\sum_S$ in (19.2) at fixed γ by a constant of the form D_2^m times the number of Feynman graphs, which can be built by joining pairwise $4m_2+2m_2+2m_{2'}$ lines emerging from $m=m_4+m_2+m_{2'}$ vertices out of m_4 of which emerge four distinct lines, while out of the other $m_2+m_{2'}$ emerge only two lines, possibly leaving aa few lines unpaired. This number is clearly boundeb by $(2m_4+m_2+m_{2'})!\,4^{2m_4+m_2+m_{2'}}\le(2m)!\,2^{4m}$, and this is therefore an estimate of the number of terms in the $\sum_S$.

However the above number is too big, and it can be replaced by a better bound. This is so because the $(2m)!\,2^{4m}$ ways described above come from multiplying the $\le\sim m!2^{4m}$ connectd graphs built with m unlabeled points ("topologically distinct graphs") times the $m!$ ways of labeling such points by $\xi_1,\ldots,\xi_m$. But the rules of construction of a graph S associated with a tree γ are such that if a graph S is given and can arise in the sum (19.2) for a given $|g$, i.e. $\overline{\gamma}$, then the same graph with the vertices relabeled dos not necessarily arise.

Given a graph G with no labels, one can consider the number N of ways of labeling G compatibly with $\overline{\gamma}$ and with given numbers n_v^e of external lines (of any type) emerging from the subgraph of G associated with the vertices of $\overline{\gamma}$. Then N is bounded by $n(\sigma)\,C_\varepsilon^ne^{\varepsilon\sum_n n_v^e}$, for all $\varepsilon>0$ and suitable C_ε, if σ is the shape of $\overline{\gamma}$ and $n(\sigma)$ is the corresponding combinatorial factor. This bound replaces an incorrectone in a previous version, and I am indebted to G. Felder for pointing out the error and its correction (see Appendix F, vy G. Felder).he bound will be combined with the remark that the summation over $\overline{\gamma}$ can in fact be thought of as a sum over the shapes σ and the frequency labels $\mathbf{h}$ assigned to the vertices of σ. However various frequency assignments $\mathbf{h}$ to the vertices of $|s$ produce the same $\overline{\gamma}=(\sigma,\mathbf{h})$, because of our convention on the trees equivalence, and the correct relaton between the sum over $\overline{\gamma}$ and that over $(\sigma,\mathbf{h})$ is $\sum_{\overline{\gamma}}\frac{1}{n(\overline{\gamma})}=\sum_\sigma\sum_{\mathbf{h}}\frac{1}{n(\sigma)}$.

Let then $\gamma=(\sigma_0,\mathbf{h},\boldsymbol{\sigma},\boldsymbol{\xi})$ be the dressed tree obtained by choosing a trimmed shape σ_0, labeling its vertices with frequency indices $\mathbf{h}$, and then choosing m dressed tree shapes $\sigma_1,\ldots,\sigma_m$) of given degrees $n_1,\ldots,n_m$ such that $\sum_i n_i=n$, frames inside endframes attached to the end points of σ_0 and bearing position indices $\boldsymbol{\xi}=(\xi_1,\ldots,\xi_m)$.

Let S be a decorated Feynman graph, compatible with γ, such that P_S is a given P and such that the number of external lines n_v^e emerging from the subgraph of S corresponding to the vertex v of $|s_0$ are given. Then $M(\cdot)$ in (19.2) can be obviously bounded, by taking into account the above combinatorial considerations, as

$$\begin{aligned}M(\Delta_1,\ldots,\Delta_p)\le&\sup_{\substack{\sigma_0,\sigma_1,\ldots,\sigma_m\\ S}}m!\,D_3^n\sum_{\{m_v^e\}}\sum_{\{h_v\}}e^{\varepsilon\sum_v n_v^e}\cdot\\ \cdot&\int_{\mathcal{D}(\Delta_1,\ldots,\Delta_p)}d\boldsymbol{\xi}\,|V^{(k)}(\boldsymbol{\xi};S)\sup|P|\,\le\end{aligned} \tag{19.3}$$

$$\le e^{-\kappa\gamma^k d(\Delta_1,\ldots,\Delta_p)}\,\mathcal{N}\, m!\, D_4^m \gamma^{-e m_{2,v} k}\sum_{\mathbf{h}}\cdot$$
$$\cdot\Big(\prod_{v>r}\sum_{n_v^e} e^{\varepsilon n_v^e}\gamma^{-(h_v-h_{v'})(\widetilde{\rho}_v+\rho_v)}\Big)\Big(\prod_j |r^{(d_j)}(\sigma_j,h_j)|\Big),$$

if D_3, D_4 are suitable constants and the notatios of Sec. 18 are used; furthermore, the summations over n_v^e from 0 to ∞ can be controlled by

$$\sum_{n_v^e=0}^{\infty} e^{\varepsilon n_v^e}\gamma^{-(h_v-h_{v'})(\widetilde{\rho}_v+\rho_v)} \le const\,\gamma^{-(h_v-h_{v'})\overline{\rho}+2m_{2,v}} \tag{19.4}$$

because of (18.18) and $h_v - h_{v'} \ge 1$, if ε (arbitrary so far) is chose small enough. Here the notations of Sec. 18 are used: in particular v' denotes the vertex immediately preceding v in γ.

Therefore the bound (19.4) reduces the problem to that of the coefficients $r^{(\alpha)}(\sigma,h)$ which end up, in this way, playing the central role in the quantitatiove theory of renormalization.

The theory of the coefficients $r^{(\alpha)}(\sigma;h)$, to a degree of depth allowing the proof of the $n!$ bound, is in fact easy as soon as one makes the right guess as to what to prove; the guess has to be made by trial and error methods, and it is pointless to repeat the search here. The result is that one should try to prove that there exist constants $b>0, D_5>0$ such that (always for $d=4$)

$$|r^{(\alpha)}(\sigma,k)| \le \overline{\varepsilon}^n D_5^{n-1}(n-1)!\cdot \cdot\sum_{j=0}^{n-1}\frac{(b\,k)^j}{j!}\gamma^{2k\delta_{\alpha,2}+4k\delta_{\alpha,0}} \tag{19.5}$$

where n is the degree of σ and, as usual, $\overline{\varepsilon}=\max_\alpha|\lambda^{(\alpha)}|$.

Before proving (19.5) we shall find it reassuring to check that (19.5) is really what one needs. In fact, inserting (19.5) in (19.4) one estimates the *r.h.s.* of (19.4) by using the remarks following (18.20) and leading to (18.21); it follows that

$$M(\Delta_1,\ldots,\Delta_p) \le e^{-\kappa\gamma^k d(\Delta_1,\ldots,\Delta_p)} m!\mathcal{N}\, D_4^m\overline{\varepsilon}^n D_5^{n-m}\cdot \sum_{\mathbf{h}}\Big(\prod_{v>r}\gamma^{-(h_v-h_{v'})\overline{\rho}}\Big)\Big(\prod_j (n_j-1)!\sum_{p=0}^{n_j-1}\frac{(b\mu h_j)^p}{p!}\Big) \tag{19.6}$$

where n_j is the degre of σ_j:

$$\sum_{j=1}^m n_j = n \tag{19.7}$$

This gives immediately (19.2) via the inequality

$$\sum_{\mathbf{h}}\prod_{v>r}\gamma^{-\overline{\rho}(h_v-h_{v'})}\prod_{j=1}^m\Big((n_j-1)!\sum_{p=0}^{n_j-1}\frac{(bh_j)^p}{p!}\Big) \le \le D_6^m(n-m)!\sum_{p=0}^{n-m}\frac{(bh_j)^p}{p!} \tag{19.8}$$

valid for suitably chosen b, D_6. The latter remarkable inequality can be proved by induction on the number of vertices, and its (simple) proof is in Appendix E.

Coming back to the proof of (19.5) one shall again proceed by induction. Consider a shape σ enclosed in a frame f_0 and fix it.

Therefore the shape σ will have *no R* superscript on the first nontrivial vertex. Let σ_0 be the shape obtained by trimming σ of the outer frames and their contents; let $m\ge m$ be the degrees of σ and σ_0: of course no confusion should arise with the quantities with the same names used in the first part of this section. It is convenient to avoid proliferation of the symbols, but the reader should bear in mind that what follows is the proof of (19.5), quite independent of the first part of the section.

If f is any frame in σ and if m_f denotes the degree of the trimmed tree inside the frame f, it is

$$n-1=\sum_f(m_f-1) \tag{19.9}$$

where the sum runs over the frames of σ *and* on the frame f_0 enclosingσ (so that $m_{f_0}=m$), which one imagines to have erased in setting up the computation of the form factor $r^{(\alpha)}(\sigma;h)$ as prescribed in Sect.18 (see (18.10) and the discussion preceding it). Relation (19.9) is basically the same relation used several times (see, for instance, the comments before (16.16) or (12.17)).

As discussed in Sec.18 ((18.9) and (18.10)), it follows from the general theory of Secs.7 and 8 that $r^{(\alpha)}(\sigma;k)$ can be estimated in terms of the coefficients $V(\gamma;S)$ corresponding to the Feynman graphs S such that P_S has degree 4, 2 or 0 and $\gamma=(\sigma^{\mathbf{h}},\boldsymbol{\xi})$ is the tree obtained by attributing to σ frequency labels $\mathbf{h}$ and enframe position labels $\boldsymbol{\xi}$ so that the root of γ receives frequency index -1 and the first nontrivial vertex of γ receives frequency index $h\le k$. Note that γ is only partially dressed, because by construction the vertex v_0 bears no R superscript, having been obtained by deleting the frame f_0 originally containing it.

Assuming, inductively, that the r coefficients $r^{(\alpha)}$ verify bounds (19.5) when the degree of γ is less than n (trivially true for $n=1$), one sees that (19.4) and (18.21) together with the previous counting estimates imply (if $d=4$ and just applying the definitions)

$$|r^{(\alpha)}(\sigma;k)| \le D_7 m! D_4^m\sum_{h_{v_0}=0}^{h}\sum_{\mathbf{h}'}\overline{\varepsilon}^m D_5^{n-m}\cdot \cdot\big(\prod_{v>v_0}\gamma^{-\overline{\rho}_{v_0}h_{v_0}}\big)\gamma^{-\rho_{v_0}h_{v_0}}\prod_{j=1}^m\Big((n_j-1)!\sum_{p=0}^{n_j-1}\frac{(bh_j)^p}{p!}\Big) \tag{19.10}$$

where $\overline{\rho}\le\rho'_v+\rho_v$ is fixed and $\rho_{v_0}\ge -4+n^e_{0,v_0}+2n^e_{1,v_0}=-4\delta_{\alpha\,0}-2\delta_{\alpha\,2}$, see (16.9), because the first vertex v_0 has no superscript R; hence no improvement on ρ_{v_0} is provided by the renormalization ("no renormalization is operating on v_0"); in (19.10) h_j denotes the frequency of the vertex at which the j–th endline of σ_0 is attached to $\sigma_0^{\mathbf{h}}$.

Using the inequality (19.8) one can easily estimate the sum over $\mathbf{h}' = \{h_v\}_{v>v_0}$. Suppose that at v_0 bifurcate $\overline{m}$ branches, each of degree $\overline{n}_1, \ldots, \overline{n}_{\overline{m}}$ so that $\sum \overline{n}_j = n$; then by (19.8) being applied to each branch

$$|r^{(\alpha)}(\sigma;k)| \le D_7 m! D_4^m \sum_{h=0}^{k} \gamma^{4h\delta_{\alpha 0}+2h\delta_{\alpha 2}} \overline{\varepsilon}^m D_5^{n-m} \cdot$$
$$\cdot \prod_{s=1}^{\overline{m}} \big((\overline{n}_s - \overline{m}_s)! \sum_{p=0}^{\overline{n}_s-\overline{m}_s} \frac{(bh)^p}{p!} D_6^{\overline{m}_s}\big) \tag{19.11}$$

where $\overline{m}_m$ is the number of end points of the s-th branch, after trimming it of its endframes: $\sum_s \overline{m}_s = m$.

Then one can use the following bound valid for all non-negative integers $a_1, \ldots, a_q$:

$$\prod_{s=1}^{q} \Big(a_s! \sum_{j=0}^{a_j} \frac{(bh)^j}{j!}\Big) \equiv$$
$$\equiv \sum_{r=0}^{\sum a_s} \frac{(bh)^r}{r!} \Big(\sum_{\substack{j_1,\ldots,j_q=0 \\ j+1+\ldots+j_q=r}}^{a_1,\ldots,a_q} \frac{r!}{j_1!\cdots j_q!} a_1! \cdots a_q! \Big) \le$$
$$\le (\sum_s a_s)! \sum_{r=0}^{\sum a_s} \frac{(bh)^r}{r!} \tag{19.12}$$

following from the fact that the large parentheses in the intermediate step arebounded by the square bracket on the *r.h.s.* ; a proof of this elementary combinatorial inequality can be found by induction.

The bound (19.12) can be used in (19.11) to infer

$$|r^{(\alpha)}(\sigma;k)| \le D_7 m! D_4^m \overline{\varepsilon}^m D_5^{n-m} D_6^m \cdot$$
$$\cdot \gamma^{4k\delta_{\alpha 0}+2k\delta_{\alpha 1}} (n-m)! \sum_{h=0}^{k} \sum_{r=0}^{n-m} \frac{(bh)^r}{r!} \tag{19.13}$$

and using

$$\sum_{h=0}^{k} h^r \le k^r + \int_0^k h^r dh = k^r + \frac{k^{r+}}{r+1} \tag{19.14}$$

implying

$$\sum_{h=0}^{k} \sum_{r=0}^{n-m} \frac{(bh)^r}{r!} \le \frac{b+1}{b} \sum_{r=0}^{n-m+1} \frac{(bh)^r}{r!}, \tag{19.15}$$

one deduces the bound

$$|r^{(\alpha)}(\sigma;k)| \le D_7 \frac{b+1}{b} D_5^m (\overline{\varepsilon} D_6 D_4 D_5^{-1})^m \gamma^{4k\delta_{\alpha 0}+2k\delta_{\alpha 1}} \cdot$$
$$\cdot (n-1)!\, m \sum_{r=0}^{n-m+1} \frac{(bh)^r}{r!},\ m > 1 \tag{19.16}$$

where $D_7 > 0$ is a suitable constant.

Thus if D_5 is chosen so large that

$$D_5 D_7 \frac{b+1}{b} (D_6 D_4 D_5^{-1})^m m < 1,\ \forall m > 1 \tag{19.17}$$

the (19.5) follows by induction: in fact the bound (19.5), as already remarked, holds for $m = 1$ (trivial shape of s), and the above chain of inequalities proves that the bound holds for trees of degree n, if it holds for trees of lower degree. The constant b is not arbitrary because it must be such that (19.8) holds. The constant D_5 can be take $\overline{\varepsilon}$-independent.

By repeating the same argument and taking into account that $n - m + 1$ can be considerably smaller than $n - 1$, one could improve (19.6) as

$$|r^{(\alpha)}(\sigma;k)| \le \overline{\varepsilon}\, (\overline{\varepsilon}\, \overline{D})^{n-1} (n-1)! \cdot$$
$$\cdot \sum_{j=0}^{f} \frac{(bk)^j}{j!} \gamma^{4k\delta_{\alpha 0}+2k\delta_{\alpha 1}}, \tag{19.18}$$

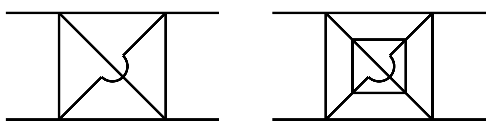
where $f - 1$ is the number of frames in σ: this bound shows that the number of frames in $|s$ measures the rate of growth of$r^{(\alpha)}(\sigma;k)$ with k, or at least bounds it.

xx. An application: planar graphs and convergence problems. A heuristic approach

Consider the power series for the effective potentials and, given a dressed tree γ, consider the contribution $\int V(\gamma;S)\, P_S d\boldsymbol{\xi}$, associated with γ, to the effective potential coming from a decorated Feynman graph S, as explained in the previous sections.

Most of the graphs S have a complicated topological structure and i will be impossible to draw them on a plane (without causing line intersections which are not, actually, graph vertices or without enclosing one of the external lines inside a region surrounded by internal lines).

For instance the graphs in Fig. (52) are nonplanar (if the bumpy crossings are not graph vertices)



(52)

The planar φ^4 theory is the set of power series for the effective potentials (as well as for Schwinger functions) obtained by restricting the summation

$$\int \sum_{\gamma} \sum_{G} \frac{V(\gamma;G)}{n(\gamma)}\, P_G\, d\boldsymbol{\xi} \tag{20.1}$$

to the planar graphs G only; of course such a restriction also applies in the graphs arising in the evaluation of the counterterms and of the "form factors" $r^{(\alpha)}(\sigma;h)$ (otherwise one would lose ultraviolet stability).

For what concerns the physical as well as the mathematical meaning of such a planar theory perhaps the best interpretation id that of "leading order" in a N^{-1} expansion in a vector $(\boldsymbol{\varphi}^2)^2$ theory, where $\boldsymbol{\varphi}$ is a $N\times N$ matrix with $(\boldsymbol{\varphi}^2)^2 \stackrel{def}{=} \mathrm{Tr}\,(\boldsymbol{\varphi}^*\boldsymbol{\varphi})^2$ (see ('t Hooft, 1982b, 1983, 1984; Rivasseau, 1985)).

Therefore in this paper the pplanar field theory for φ^4 will be considered only as a set of formal power series and as a prototype of a situation in which the resummation ideas of Sec.]sec(9) can be applied.

The main property of the planar graphs is that the unlabeled planar graphs, "topological planar graphs", are not too many and their number can be bounded by N_0^n where N_0 is some constant and $n = m_4 + m_2 + m_{2'}$ is the number of vertices. One can take $N_0 = 3^6$ (see (Koplik *et al.*, 1977)).

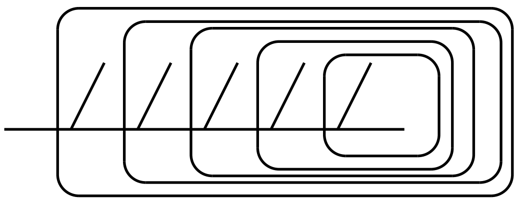
Without our entering one more into the details, it should be quite clear, or at least reasonable, that the whole theory of the preceding sections for the shape form factors $r^{(\alpha)}(\sigma;k)$ remains essentially unchanged, except that facotrs like $n!\,n(\gamma)$ estimating the number of graphs relevant for a tree γ with n endframes are now replaced by factors $N_0^n n(\gamma)$.

The basic bound (19.5), proved in the same way, becomes

$$r^{(\alpha)}_{planar}(\sigma;k) \le\le \varepsilon\,(\overline{\varepsilon}\overline{D})^{n-1} \sum_{j=0}^{f} \frac{(bk)^j}{j!} \tag{20.2}$$

where $(f-1)$ is the number of frames inside σ: in other words, instead of $(n-1)!$ one finds $f!$ (note that $f \le n-1$); compare this with the improved bound (19.18) to understand a little howthis is possible.

The improvement over (19.5) and (19.18) is clearly very string when $f \ll n$. However f can be as large as $n-1$, and therefore the sums (20.1) still present convergence problems of a major nature being a power series in the renormalized couplings $\boldsymbol{\lambda} = (-\lambda, -\alpha, -\mu, -\nu)$ with factorially growing coefficients coming from the trees γ with f of order of the number of vertices of γ, e.g. see Fig.(53)



(53)

To understand better the problem of convergence one can consider the resummation procedures outlined in Sec.9. Precisely consider the pruning oparation τ (see Sec.9) cutting out od a tree all the frames. The resulting resummation equation (9.9) for the fully summed coefficient $r^{(\alpha)}(k)$ (called in Sec.9 $\lambda^{(\alpha)}(k)$) becomes

$$\begin{aligned} r^{(\alpha)}(k) = \lambda^{(\alpha)} + \sum_{r=2}^{\infty}\sum_{h=0}^{k} \sum_{\substack{h_1,\ldots,h_r\ge h\\ \alpha_1,\ldots,\alpha_r}} \cdot \\ \cdot\, \overline{B}^{(\alpha)}_{\alpha_1,\ldots,\alpha_r} \prod_{i=1}^{r} r^{(\alpha_i)}(h_i) \end{aligned} \tag{20.3}$$

The simplest, rigorously correct, interpretation of (20.3) is that it can be used to generate recirsively a power series expansion for the functions $r^{(\alpha)}(k)$ in the renormalized coupling constants.

It is convenient to recall that in the previous sections this expansion was studied in some detail and led to the representation

$$r^{(\alpha)}(k) = \lambda^{(\alpha)} + \sum_{\sigma} r^{(\alpha)}(\sigma;k) \tag{20.4}$$

where σ are all the possible shapes of trees (see Fig.(24)).

Clearlym (20.4) is a power series in $\boldsymbol{\lambda}$ and $r^{(\alpha)}(\sigma;k)$ is part of the polynomial of degree equal to thedegree of σ in the expansion of $r^{(\alpha)}(k)$.

From the general theory of Sec.9 it follows that (20.4) must verify, if thought of as a formal power series, the relation (20.3) and therefore (20.4) can be generated by solving recursively (20.3) as an expansion for $\mathbf{r}(k)$ with $\boldsymbol{\lambda}$ as inoput. It is not surpprising that once the coefficients $\overline{B}$ in (20.3) are known one can reduce the problem of computing $\sum_{degree\,\sigma=m} r^{(\alpha)}(k) \equiv r^{(\alpha)}_m(k)$ to a simple "algebraic" problem; i.e. that of iterating m times (20.3)., retaining only the m–th order monomial in $\boldsymbol{\lambda}$.

From the definitions it is clear that the computation of the coefficients $\overline{B}$ is a necessary prerequisite for the computation of $r^{(\alpha)}(\sigma;h)$, since computing the $\overline{B}$ factors amounts precisely to computingthe dressed trees with no frames. In fact, recall that the computation of $r^{(\alpha)}(\sigma;k)$ for general σ is reduced inductively to the no-frame case; on this fact are based the $n!$ estimates of Sec.19. But it is quite evident that (20.3) provides a very economic and systematic way of reorganizing the calculations of the factors $r^{(\alpha)}(\sigma;k)$. Equation (20.3) is similar to the Callan-Symanzik equations (Callan, 1970; Symanzik, 1966).

From the work of Secs. 16–19 the coefficients $\overline{B}$ can be easily computed for small r and estimated for large r, uniformly in the ultraviolet cut–off N (in fact they are N-independent, as the reader should eventually realize, but they depend on the regularization chosen, as it will be pointed out later).

The coefficients $\overline{B}$ can be bounded following the same procedures used in Secs.18 and 19; one just has to take into account that only planar graphs will ever be considered. The work is a repetition of what was done there, and it will not be reproduced here. The coefficients $\overline{B}$ arise from the computation of trimmed trees, i.e. from trees with no frames so that $f = 1$, by keeping only the planar graphs, so that the factorials $m!\,n(\gamma)$ are replaced by $N_0^m n(\gamma)$. And no factorials arise inthe estimates of B. It is, see also (Gallavotti and Nicolò, 1985a,b), for some C_1

$$\sum_{\substack{h_i\ge h\\ h \text{ fixed}}} |B^{(\alpha)}_{\alpha_1,\ldots,\alpha_r}(h;h_1,\ldots,h_r)\cdot \cdot\gamma^{\sum_i(\delta_{\alpha_2 2}+4\delta_{\alpha_i 0})h_i}\,\gamma^{-(\delta_{\alpha_2 2}+4\delta_{\alpha_i 0})h} \le C_1^{r-1} \tag{20.5}$$

For $r = 2$ one can perform some explicit easy calculations starting from (17.7): the coefficients $B^{(2)}_{\alpha_1\alpha_2}(h;h,h)$ are given by the h-independent constants $\beta^{(\alpha)}_{\alpha_1\alpha_2}$ with an error bounded by $O(h^p\gamma^{-2h})$ for some p:

$$\begin{aligned}
\beta^{(2)}_{22} =& -\gamma^{2h}\frac12\binom{2}{1}^2\int C^{(\le h)}_{12}\,d\boldsymbol{\xi}_2,\\
\beta^{(2)}_{42} =& -\frac{2!}{2}\binom{4}{2}^2\int (C^{(\le h)\,2}_{12}-C^{(<h)\,2}_{12})\,d\boldsymbol{\xi}_2,\\
\beta^{(2)}_{42'} =& -\gamma^{-2h}\frac{2!}{2}\binom{4}{2}\int ((\partial C^{(\le h)}_{12})^2-(\partial C^{(<h)}_{12})^2)\,d\boldsymbol{\xi}_2,\\
\beta^{(2)}_{44} =& -\gamma^{-2h}\frac{3!}{2}\binom{4}{3}^2\int (C^{(\le h)\,3}_{12}-C^{(<h)\,3}_{12})\,d\boldsymbol{\xi}_2,\\
\beta^{(2)}_{22'} =& -\gamma^{2h}\frac12\binom{2}{1}^2\int \frac{(\boldsymbol{\xi}_2-\boldsymbol{\xi}_1)}{d}\cdot\partial_2 C^{(h)}_{12}\,d\boldsymbol{\xi}_2,\\
\beta^{(2')}_{22} =& \gamma^{4h}\frac12\binom{2}{1}^2\int \frac{(\boldsymbol{\xi}_2-\boldsymbol{\xi}_1)^2}{2d} C^{(h)}_{12}\,d\boldsymbol{\xi}_2,\\
\beta^{(2')}_{44} =& \gamma^{4h}\frac{3!}{2}\binom{4}{3}^2\int \frac{(\boldsymbol{\xi}_2-\boldsymbol{\xi}_1)^2}{2d}(C^{(\le h)\,3}_{12}-C^{(<h)\,3}_{12})\,d\boldsymbol{\xi}_2,\\
\beta^{(4)}_{24} =& -\gamma^{2h}\frac12\binom{2}{1}\binom{4}{1}\int C^{(\le h)}_{12}\,d\boldsymbol{\xi}_2,\\
\beta^{(4)}_{44} =& -\frac{2!}{2}\binom{4}{2}^2\int (C^{(\le h)\,2}_{12}-C^{(<h)\,2}_{12})\,d\boldsymbol{\xi}_2,\\
\beta^{(0)}_{22} =& -\frac12 2!\int (C^{(\le h)\,2}_{12}-C^{(<h)\,2}_{12})\,d\boldsymbol{\xi}_2,\\
\beta^{(0)}_{22'} =& -\gamma^{-2h}\frac12 2!\int ((\partial C^{(\le h)}_{12})^2-(\partial C^{(<h)}_{12})^2)\,d\boldsymbol{\xi}_2,\\
\beta^{(0)}_{2'2'} =& -\gamma^{-4h}\frac12 2!\int ((\partial^2 C^{(\le h)}_{12})^2-(\partial^2 C^{(<h)}_{12})^2)\,d\boldsymbol{\xi}_2,\\
\beta^{(0)}_{44} =& -\gamma^{-4h}\frac12\binom{4}{4}^2 4!\int (C^{(\le h)\,4}_{12}-C^{(<h)\,4}_{12})\,d\boldsymbol{\xi}_2,
\end{aligned}\tag{20.6}$$

all other B's with $r = 2$ vanish or reduce to the above by $B^{(\alpha)}_{\alpha_1\alpha_2} = B^{(\alpha)}_{\alpha_2\alpha_1}$. It is convenient to introduce new form factors, more naturally depending on k; they are "adimensional form factors" defined by

$$r^{(\alpha)}(k) = \lambda^{(\alpha)}(k)\gamma^{(2\delta_{\alpha 2}+4\delta_{\alpha 4})\,k} \tag{20.7}$$

and one can rewrite (20.3) in terms of new functions $\beta^{(\alpha)}_{\alpha_1,\ldots,\alpha_r}(h;h_1,\ldots,h_r)$

$$\lambda^{(\alpha)}(k) = \lambda^{(\alpha)}\gamma^{-(2\delta_{\alpha 2}+4\delta_{\alpha 4})\,k} + \sum_{r=2}^\infty\sum_{h=0}^k \sum_{\substack{h_1,\ldots,h_r\ge h\\ \alpha_1,\ldots,\alpha_r}} \cdot \cdot\,\beta^{(\alpha)}_{\alpha_1,\ldots,\alpha_r}(h;h_1,\ldots,h_r)\gamma^{(h-k)(2\delta_{\alpha 2}+4\delta_{\alpha 4})}\prod_{i=1}^r \lambda^{(\alpha_i)}(h_i) \tag{20.8}$$

and it can be checked that

$$\begin{aligned}\lim_{h\to\infty}\beta^{(\alpha)}_{\alpha_1,\ldots,\alpha_r}(h;h_1,\ldots,h_r) &\stackrel{def}{=}\\ \stackrel{def}{=} \overline{\beta}^{(\alpha)}_{\alpha_1,\ldots,\alpha_r}&(h_1-h,\ldots,h_r-h)\end{aligned}\tag{20.9}$$

exist if $h_i - h$ are kept constant and the basic bounds of Sec.18 imply, via (20.5)

$$\sum_{\substack{h_i\ge h\\ h \text{ fixed}}} |\overline{\beta}^{(\alpha)}_{\alpha_1,\ldots,\alpha_r}(h_1-h,\ldots,h_r-h) \le C_1^{r-1} \tag{20.10}$$

therefore if we define

$$\begin{aligned}&\lambda(k) \stackrel{def}{=} -\lambda^{(4)}(k),\ \mu(k) \stackrel{def}{=} -\lambda^{(2)}(k),\\ &\alpha(k) = -\lambda^{(2')}(k),\ \nu(k) = -\lambda^{(0)}(k)\end{aligned}$$

the (20.8) can be written explicitly:

$$\begin{aligned}
\lambda(k) =& \lambda + \sum_{h=0}^k(\beta^{(4)}_{44}\lambda(h)^2 + 3\beta^{(2)}_{24}\lambda(h)\mu(h)) + \ldots\\
\alpha(k) =& \alpha - \sum_{h=0}^k(\beta^{(2')}_{44}\lambda(h)^2 - 2\beta^{(2')}_{24}\mu(h)\alpha(h)+\\
&+\beta^{(2')}_{22}\mu(h)^2) + \ldots\\
\mu(k) =& \mu\gamma^{-2k} + \sum_{h=0}^k \gamma^{2(h-k)}(\beta^{(2)}_{22}\mu(h)^2 + 2\beta^{(2)}_{42}\lambda(h)\mu(h)+\\
&+2\beta^{(2)}_{42'}\lambda(h)\alpha(h) + \beta^{(2)}_{44}\lambda(h)^2) + \ldots\\
\nu(k) =& \nu\gamma^{-4k} + \sum_{h=0}^k \gamma^{4(h-k)}(\beta^{(0)}_{22}\mu(h)^2 + 2\beta^{(0)}_{22'}\mu(h)\alpha(h)+\\
&+\beta^{(0)}_{2'2'}\alpha(h)^2) + \beta^{(0)}_{44}\lambda(h)^2 + \ldots
\end{aligned}\tag{20.11}$$

and the functions $\beta^{(\alpha)}_{\alpha_1\alpha_2} \equiv \beta^{(\alpha)}_{\alpha_1\alpha_2}(h;h,h)$ will have a well defined positive limit as $h \to \infty$, as follows from (20.6); the dots denote the "higher order terms", $r > 2$.

The limits $\overline{\beta}^{(\alpha)}_{\alpha_1\alpha_2}(0,0)$ of $\beta^{(\alpha)}_{\alpha_1\alpha_2}$ are reached exponentially fast ($O(\gamma^{-2h})$) and are not all independent (e.g. $\overline{\beta}^{(4)}_{44} = 8\overline{\beta}^{(0)}_{22} = 3\overline{\beta}^{(2)}_{42}, \ldots$).

Obviously, because of the meaning of the truncated expectations of the trees vertices, it follows that no $\nu(k)$ appears in the first three of (20.11). this means that the fourth equation in (20.11) decouples from the first three and determines $\nu(k)$ completely as soon as $\lambda^{(\alpha)}(h)$ are known for $\alpha = 4, 2, 2'$ (because, also, no $\nu(k)$ appears in the *r.h.s.* of the fourth term in (20.11)). For this reason the fourth equation is not too important in setting up the theory of renormalization.

The power series in (20.11) (in the variables $\boldsymbol{\lambda}(k)$) can be used, as already mentioned, to generate expressions of $\boldsymbol{\lambda}(k)$ as power series in $\boldsymbol{\lambda}$.

As proved in Secs.18 and 19, this power series has coefficients which are uniformly bounded in the ultraviolet cut-off and this also follows directly (but not independently of the theory of Secs. 18 and 19) from the bounds (20.10).

However it is clear that the coefficients one gets must coincide with the ones estimated in Sec.19, and which grow with the order n as $O(n!)$, even in the planar case being considered here (because of the contributions that these coefficients receive from the trees with many frames, see (20.10)). One can convince himself that such estimates are not pessimistic unless some cancellations take place.

In fact the bounds are reasonable and "optimal" on each individual graph, as one can easily identify graphs (planar) and trees giving contributions to the n–th order coefficients of $\boldsymbol{\lambda}(k)$ which are of the order of $n!$; this was pointed out in (Lautrup, 1977).

However cancellations between several big terms can take place and in various possible senses. A way of exhibiting such cancellations is to find a sequence $\{\boldsymbol{\lambda}(k)\}_{k=0}^{\infty} \equiv \underline{\underline{\lambda}}$ verifying (20.11). This sequence could then be taken as a definition of the sum of he power series in the $\boldsymbol{\varphi}\lambda$'s which define perturbatively $\boldsymbol{\lambda}(k)$ as a (probably divergent) power series in $\boldsymbol{\lambda}$.

To make sense of the *r.h.s.* of (20.11) it seems natural to impose on the sequence $\underline{\underline{\lambda}}$ a decay condition at $k = \infty$, in apparent contradiction with the bounds (20.2) which are strongly growing with k. So one introduces

$$|\underline{\underline{\lambda}}|_q \stackrel{def}{=} \sup_{k\ge 0}(1+k)^q|\boldsymbol{\varphi}\lambda(k)| \tag{20.12}$$

The bounds (20.10) allow one to define an operator $\mathcal{B}$ on the $\underline{\underline{\lambda}}$'s with $|\underline{\underline{\lambda}}|_q < \infty$ for some $q \ge 0$; in fact the bounds (20.10) imply (recall that they hold in the planar case only) that the operator $\mathcal{B}$,

$$\begin{aligned}(\mathcal{B}\underline{\underline{\lambda}})^{(\alpha)}(h) \stackrel{def}{=} \sum_{r=2}^{\infty} \sum_{\substack{h_1,\ldots,h_r\\ \alpha_1,\ldots,\alpha_r}} \cdot \\ \cdot\, \beta^{(\alpha)}_{\alpha_1,\ldots,\alpha_r}(h; h_1, \ldots, h_r) \prod_{i=1}^{r} \lambda^{(\alpha_i)}(h_i)\end{aligned} \tag{20.13}$$

has the property

$$|\mathcal{B}\underline{\underline{\lambda}}|_q \le C_1^{-1}(C_1|\underline{\underline{\lambda}}|_q)^2, \quad q = 0, 1, \ldots \tag{20.14}$$

and therefore $\mathcal{B}$ is well defined on the space (20.12) if for some $\eta > 0, B > 0$ and a suitably chosen $K_{\eta,B}$ and δ

$$|\underline{\underline{\lambda}}|_\eta < B, \quad \text{and} \quad \sup_{k\le K_{\eta,B}} |\boldsymbol{\lambda}(k)| < \delta \tag{20.15}$$

as follows from (20.10). Equation (20.11) becomes

$$\begin{aligned}\lambda^{(\alpha)}(k) =& \lambda^{(\alpha)}\gamma^{-(2\delta_{\alpha,2}+4\delta_{\alpha,0})k} + \\ &+ \sum_{h=0}^{k} \gamma^{(h-k)(2\delta_{\alpha,2}+4\delta_{\alpha,0})}(\mathcal{B}\underline{\underline{\lambda}})^{(\alpha)}(h)\end{aligned} \tag{20.16}$$

i.e. if $\lambda^{(\alpha)}(-1) \stackrel{def}{=} \lambda^{(\alpha)}\gamma^{(2\delta_{\alpha,2}+4\delta_{\alpha,0})}$:

$$\begin{aligned}\lambda^{(\alpha)}(k+1) =& \lambda^{(\alpha)}\gamma^{-(2\delta_{\alpha,2}+4\delta_{\alpha,0})}\lambda^{(\alpha)}(k) + \\ &+ (\mathcal{B}\underline{\underline{\lambda}})^{(\alpha)}(k+1), \quad k \ge -1\end{aligned} \tag{20.17}$$

and one looks for a solution $\underline{\underline{\lambda}}$ such that, say, $|\underline{\underline{\lambda}}|_1 < 1$ and $\sup_{k\le K_{1,1}} |\boldsymbol{\lambda}(k)| < \delta$.

In studying (20.17) one is thus interested in solutions $\boldsymbol{\lambda}(k) \xrightarrow[k\to\infty]{} 0$; therefore it is natural to replace $\mathcal{B}$ by its second order part $\mathcal{B}_2$ for the purpose of getting first an approximate solution.

$$(\mathcal{B}_2\underline{\underline{\lambda}})^{(\alpha)}(h) = \sum_{\alpha_1,\alpha_2} \beta^{(\alpha)}_{\alpha_1\alpha_2}(h)\lambda^{(\alpha_1)}(h)\lambda^{(\alpha_2)}(h) \tag{20.18}$$

(see (20.11)). In turn, since $\beta^{(\alpha)}_{\alpha_1\alpha_2}(k) \xrightarrow[k\to\infty]{} \overline{\beta}^{(\alpha)}_{\alpha_1\alpha_2}$, it is convenient to study first the relation, $k \ge 0$,

$$\lambda^{(\alpha)}(k+1) = \lambda^{(\alpha)}(k)\gamma^{-(2\delta_{\alpha 2}+4\delta_{\alpha 0})} + (\overline{\mathcal{B}}_2\underline{\underline{\lambda}})^{(\alpha)}(k) \tag{20.19}$$

with $\overline{\mathcal{B}}_2$ defined as (20.18) with $\beta^{(\alpha)}_{\alpha_1\alpha_2}(h)$ replaced by their limits $\overline{\beta}^{(\alpha)}_{\alpha_1\alpha_2}$ as $h \to \infty$. Explicitly the last equation is

$$\begin{aligned}\lambda(k+1) =& \lambda(k) + \overline{\beta}^{(4)}_{44}\lambda(k)^2 + 2\overline{\beta}^{(4)}_{24}\lambda(k)\mu(k), \quad (20.20)\\ \alpha(k+1) =& \alpha(k) - \overline{\beta}^{(2')}_{22}\mu(k)^2 - \overline{\beta}^{(2')}_{44}\lambda(k)^2 +\end{aligned}$$

$$\begin{aligned} &+2\overline{\beta}^{(2')}_{22'}\,\mu(k)\alpha(k),\\ \mu(k+1) =&\gamma^{-2}\mu(k)+\overline{\beta}^{(2)}_{22}\,\mu(k)^2+\\ &+2\overline{\beta}^{(2)}_{42}\,\lambda(k)\mu(k)+\overline{\beta}^{(2)}_{44}\,\lambda(k)^2,\\ \nu(k+1) =&\gamma^{-4}\nu(k)+\overline{\beta}^{(0)}_{22}\,\mu(k)^2+2\overline{\beta}^{(0)}_{22'}\mu(k)\alpha(k)+\\ &+\overline{\beta}^{(0)}_{2'2'}\alpha(k)^2+\overline{\beta}^{(0)}_{44}\,\lambda(k)^2. \end{aligned}$$

This relation cab be regarded as an iteration of a map T on $\mathbb{R}^4$, or $\mathbb{R}^3$ if one disregards the last (decoupled) equation. One can therefore apply the techniques developed in the general theory of maps to analyze (20.20).

One looks for data $\overline{\lambda},\overline{\alpha},\overline{\mu},\overline{\nu}$ for $\boldsymbol{\lambda}(0)$ such that $\boldsymbol{\lambda}(k)\xrightarrow[k\to\infty]{}0$. Their existence can be proved by using the general theory of the central manifold, (see (Lanford, 1973) and (Gallavotti, 1983b), Chap.5, Secs.6 and 8 and the related problems). There exists a surface Σ, in general nonunique,

$$\begin{aligned} \mu =&\mu(\alpha,\lambda)=A\alpha^2+L\lambda^2+I\alpha\lambda+\ldots\\ \nu =&\nu(\alpha,\lambda)=A'\alpha^2+L'\lambda^2+I'\alpha\lambda+\ldots \end{aligned} \qquad (20.21)$$

where the dots represent terms of higher order, which is invariant under the map T defined by (20.20) and such that the T–images of any point $\overline{\boldsymbol{\lambda}}$ close enough to the origin evolves under repeated iterations of T by approaching exponentially fast the surface Σ as long as they stay close enough to the origin.

A simple exercise ("substitute (20.21) in (20.20) to find $A, A'\ldots$") yields

$$\begin{aligned} I =&\frac{2\overline{\beta}^{(2)}_{42}}{1-\gamma^{-2}},\ A=\frac{-\overline{\beta}^{(2)}_{2'2'}}{1-\gamma^{-2}}\\ L =&\frac{-\overline{\beta}^{(2)}_{44}}{1-\gamma^{-2}},\ I'=0,\\ A' =&\frac{-\overline{\beta}^{(0)}_{2'2'}}{1-\gamma^{-4}},\ L'=\frac{-\overline{\beta}^{(0)}_{44}}{1-\gamma^{-4}} \end{aligned} \qquad (20.22)$$

and the map (20.20) becomes on Σ

$$\begin{aligned} \lambda(k+1) =&\lambda(k)+\overline{\beta}^{(4)}_{44}\,\lambda(k)^2+\ldots\\ \alpha(k+1) =&\alpha(k)-\overline{\beta}^{(2')}_{44}\,\lambda(k)^2+\ldots \end{aligned} \qquad (20.23)$$

where the dots represent terms of higher order. Neglecting the higher order corrections once more, and setting $\beta\stackrel{def}{=}\overline{\beta}^{(4)}_{44}>0$ $\beta'\stackrel{def}{=}\overline{\beta}^{(2')}_{44}>0$, one considers the relations

$$\begin{aligned} \lambda(k+1) =&\lambda(k)+\beta\,\lambda(k)\lambda(k+1),\\ \alpha(k+1) =&\alpha(k)-\beta'\,\lambda(k)\lambda(k+1), \end{aligned} \qquad (20.24)$$

which admit solutions with data $\overline{\lambda}$ and $\overline{\alpha}=-\beta;\beta^{-1}\overline{\lambda}$, with $\overline{\lambda}<0$:

$$\lambda(k)=\frac{\overline{\lambda}}{1-\beta\,k\,\overline{\lambda}},\qquad \alpha(k)=-\beta;\beta^{-1}\lambda(k). \qquad (20.25)$$

From general considerations of stability theory it follows that (20.23) also admits a solution behaving as $k\to\infty$ as (20.25) with initial data $\overline{\lambda}<0$ and $\overline{\alpha}=-\beta'\beta^{-1}\overline{\lambda}+O(\overline{\lambda}^2)$ and such that $\boldsymbol{\lambda}(k)\xrightarrow[k\to\infty]{}\mathbf{0}$ at fixed k.

This means, via (20.21), that (20.20) admits a solution with data $\overline{\lambda}<0$, $\overline{\alpha}=-\beta'\beta^{-1}\overline{\lambda}+O(\overline{\lambda}^2)$, $\overline{\mu}=O(\overline{\lambda}^2)$, $\overline{\nu}=O(\overline{\lambda}^2)$ which is such that $\lambda(k),\alpha(k)=O(k^{-1})$ and $\mu(k),\nu(k)=O(k^{-2})$ as $k\to\infty$ and such that $\boldsymbol{\lambda}(k)\to 0$ at fixed k when $\overline{\lambda}\to 0$.

Hence one finds a solution to (20.20) depending on one parameter $\overline{\lambda}$ such that $||\underline{\lambda}||_1\le O((\beta^{-1}))$ for $-\overline{\lambda}$ small and such that $\boldsymbol{\lambda}(k)$ is as small as one wishes for any fixed number of k's, say $k\le K_1$.

Hence such $\underline{\underline{\lambda}}$ is in the domain of the "beta function" $\mathcal{B}$ defined in (20.13) and by some more efforts of abstract perturbation theory it could be proved that there is a solution to (20.17) depending on one parameter $\overline{\lambda}<0$, with $\lambda(k),\alpha(k),\mu(k),\nu(k)$ given approximately by (20.25) and (20.21).

Such a solution will not be such that $|\underline{\underline{\lambda}}|_1$ is small (rather the above discussion suggests $|\underline{\underline{\lambda}}|_1=O(\beta^{-1})$), although $\boldsymbol{\lambda}(k)$ at fixed k will be small for small $\boldsymbol{\lambda}(0)$ or for small values of the parameter $-\overline{\lambda}$ on which the solution depends. This "nonuniform smallness" is related to the fact that $\underline{\underline{\lambda}}$ cannot be found perturbatively, although it has by construction the correct asymptotic series in $\overline{\boldsymbol{\lambda}}$.

Note also that (20.25) shows that (at least the approximating) $\underline{\underline{\lambda}}$ has singularities at points accumulating at $\overline{\lambda}=0$, as a function of $\overline{\lambda}$.

The renormalized couplings are defined by the (convergent) series, if $-\overline{\lambda}$ is small,

$$\lambda^{(\alpha)}=-\sum_{k=0}^{\infty}(\mathcal{B}\underline{\underline{\lambda}})^{(\alpha)}(k) \qquad (20.26)$$

obtained by setting $\lambda^{(\alpha)}(+\infty)=0$ in (20.16). Alternatively one can use

$$\begin{aligned} \lambda^{(\alpha)}(0) =&\lambda^{(\alpha)}+(\mathcal{B}\underline{\underline{\lambda}})^{(\alpha)}(0)\Rightarrow\\ \Rightarrow\lambda^{(\alpha)}=&\lambda^{(\alpha)}(0)-(\mathcal{B}\underline{\underline{\lambda}})^{(\alpha)}(0) \end{aligned} \qquad (20.27)$$

The family of solutions to (20.17) constructed above is a one parameter family; however one could alter the coefficients in front of the few covariances or their mass terms so that one has built a many parameter family of field theories "like φ^4_4 planar"; however, it does not seem possible to choose $\overline{\alpha}=0$ nor, by (20.27), $\overline{\mu}=0$, because $A\ne 0$ in (20.22) if one wishes that $\alpha(k),\mu(k)\to 0$ as $k\to\infty$.

The meaning of the statement that one has built planar φ^4_4 theory is explained below and is summarized in the statement "the resummed tree expansions for the effective planar potentials converge for small negative coupling".

The solution to Eq.(20.17) discussed above is roughly like (20.25) and (20.23), i.e. such that

$$|\underline{\underline{\lambda}}|_0 = \sup_k |\boldsymbol{\lambda}(k)| = \widetilde{\varepsilon} \xrightarrow[\overline{\lambda}\to 0]{} 0. \tag{20.28}$$

Therefore the effective potential of the planar theory corresponding to the above definition of $\boldsymbol{\lambda}(k)$ will be described by dressed trees with no frames but with "heavy end points" contributing to the effective potential the form factor $r^{(\alpha)}(h) = \lambda^{(\alpha)}(h)\gamma^{(2\delta_{\alpha 2}+4\delta_{\alpha 0})h}$ when they are attached to vertices of the tree bearing a frequency label h.

Furthermore, since one is considering only the planar theory, one evaluates the contributions of a tree γ to the effective potential by using the "few" $N_0^n M \equiv N_0^n M n(\gamma) C_{\varepsilon}^n e^{\varepsilon \sum_v n_v^e}$ planar graphs compatible with γ (see Sec.19).

This means that, by the theory of Secs.18 and 19, the bound (19.2) is replaced, if D_0 is a suitable constant, by

$$\mathcal{N} B^{n^e} \widetilde{\varepsilon}^n D_0^n e^{-\kappa\gamma^k d(\Delta_1,\ldots,\Delta_n)} N_0^n \tag{20.29}$$

with no $n!$, because $n!$ arose for two reasons: one was that the number of Feynman graphs associated with a treeγ were bounded by $M\, n!$, where n is the number of end points in the trimmed tree γ, and the other was the $n!$ in the form factors $r^{(\alpha)}(\sigma; h)$, see (19.5), due to the endframes of σ.

However in the planar theory the graphs are far fewer, and the form factors, still badly dependent on the degree of the shapes (as pointed out at the beginning of this section), are "resummed" to yield new form factors:

$$r^{(\alpha)}(h) = \lambda^{(\alpha)}(h)\gamma^{(4\delta_{\alpha 0}+2\delta_{\alpha 2})h}, \tag{20.30}$$

with $\lambda^{(\alpha)}(h) \xrightarrow[h\to\infty]{} 0$ (this quantity was not only not small in perturbation theory, but even divergent with h as $h \to \infty$). And at the same time the resummation leading to the form factors (20.30) eliminates the necessity of considering contributions from trees with frames to $V^{(k)}$, hence (20.29) is really a simple consequence of the estimates of Secs.19 (Eq. (19.18)) and 18 (Eq. (18.21)).

Since for $\widetilde{\varepsilon}$ small the (20.29) can be summed over n, one gets the effect, in the above considered planar theory, that the resummed series for the effective potentials is really convergent for small $-\overline{\lambda} > 0$ (i.e. small negative $\boldsymbol{\lambda}(0)$, i.e. small negative renormalized coupling). Therefore, in the planar theory the effective potentials can be defined beyond perturbation theory.

The series defining the effective potential is a power series in the resummed form factors (20.30): the form factors being non analytic near $\overline{\lambda} = 0$ (in the sense roughly expressed by approximation (20.25)) it is clear that one cannot expect that the effective potentials be analytic in the renormalized coupling constant near zero.

The resummation procedure induced by the beta function allows one to express the effective coupling constants or "form factors", (20.30), and provides a well defined resummation prescription. It seems highly plausible there is one which is the Borel sum of its perturbative nonresummed series; this was proved in the case $\alpha = \mu = \nu = 0$ (not covered here because I have chosen for simplicity the initial $\overline{\alpha}, \overline{\mu}$ so that $\boldsymbol{\lambda}(k) \to 0$ as $k \to \infty$, see ('t Hooft, 1983, 1984) and (Rivasseau, 1985)).

Another interesting possibility is that the series may converge even for some $\overline{\lambda} > 0$: the formula (20.25) allows the possibility that for $\overline{\lambda} > 0$ the effective potentials are defined for "most" values of $\overline{\lambda}$. The resemblance with the situation arising in classical mechanics in the Hamiltonian stability problems in connection with the appearance of small denominators seems interesting: maybe here one needs some imagination.

xxi. Constructing φ^4 fields in $d = 2, 3$

The theory of renormalization in dimension $d = 2, 3$ can be done in a much simpler way, compared to the $d = 4$ case. Of course there is no problem in repeating word by word the four-dimensional theory in dimension 2 or 3 (and in fact in Secs.16–20 one had never really used that $d = 4$ but only that $d < 5$).

The real simplification arises when one remarks that if $d = 2, 3$ one can study much simper theories which lead, or may lead, to nontrivial fields (i.e. fields with nonquadratic effective potentials $V^{(k)}$) of φ^4 type.

What is more important is that the simpler theories (which would not make sense if $d - 4$) can be treated rigorously for "small couplings" and really shown to exist beyond the level of formal perturbation theory.

The theories which make sense if $d = 2$ and that are simpler than the ones considered so far are those generated by the interaction $\mathcal{I}_N$

$$V_1 = -\lambda \int_\Lambda :\, \varphi_x^{(\le N)\,4} \,:\, d^2x \tag{21.1}$$

while if $d = 3$ a theory simpler than the one arising from (16.1) is provided by the interaction $\mathcal{I}_N$

$$V_1 = \int_\Lambda (-\lambda :\, \varphi_x^{(\le N)\,4} \,: -\mu :\, \varphi_x^{(\le N)\,2} \,: -\nu)\, d^2x \tag{21.2}$$

The main reason (21.1),(21.2) are much simper than (16.1) is that no resummations have to be devised to organize the corresponding renormalized perturbative series, because only finitely many trees lead to divergences.

The renormalizability in the above case with $d = 2$ follows immediately from the formulae and estimates of Sec.16 setting $n_{1,v}^e = m_{2,v} = m_{2',v} = 0$ so that (16.19) becomes, for all $m_{4,v} > 1$

$$\rho_v = -2 + 2m > 0 \tag{21.3}$$

and this not only shows renormalizability of the "pure φ^4 field" but also shows that *no* renormalization is ever necessary (this same remarkable conclusion would hold, when $d = 2$, for the most general Wick-ordered polynomial interaction).

If $d = 3$ and (21.2) is considered, one can still use the general bounds of Sec.16 setting $n^e_{1,v} = m_{2],v} = 0$ so that

$$\rho_v = -3 + 3m_{2,v} + \frac{1}{2} n^e_{0,v} > 0 \tag{21.4}$$

unless (recall that $m_{2,v} + m_{4,v} \geq 2$ in the nontrivial cases)

$$\begin{array}{llll} m_{4,v} = 1 & m_{2,v} = 1 & n^e_v = 0 & \text{impossible} \\ m_{4,v} = 2 & m_{2,v} = 0 & n^e_v = 0, 2 & \text{possible} \\ m_{4,v} = 3 & m_{2,v} = 0 & n^e_v = 0 & \text{possible} \end{array} \tag{21.5}$$

So only trees of degree 2 or 3 need the definition of the $\mathcal{L}^{(\sigma)}$ localization operations, and the only nontrivial case is $m_{4,v} = 2, n^e_v = 2$ ("mass diagrams") yielding $\rho_v = 0$. and therefore it can be cured by a simple subtraction

$$V_{2,N} = \int r^{(2)}_N \; : \varphi^{(\leq N)\,2}_x : \; d^3x \tag{21.6}$$

Formally $\mathcal{L}^{(\sigma)}$ is defined in terms of the action $\overline{\mathcal{L}}$ on Wick monomials, as in Secs.17 and 18:

$$\begin{aligned} \overline{\mathcal{L}} 1 &= 1 \qquad && \text{if degree } \gamma \leq 3 \\ \overline{\mathcal{L}} : \varphi_x \varphi_y : &= : \varphi^2_x : \qquad && \text{if degree } \gamma = 2 \end{aligned} \tag{21.7}$$

which leads to a simple expression for $V_{2,N}, V_{3,N}$, i.e. for the Counterterms (note that $V_{3,N}$ is a constant and that the nonconstant part of $V_{2,N}$ must have the form (21.6)):

$$\begin{aligned} V_{2,N} = &-\frac{\lambda^2}{2} \binom{4}{1}^2 3! \int C^{(\leq N)\,3}_{\xi_1 \xi_2} : \varphi^{(\leq N)\,2}_{\xi} : d\xi_1 d\xi_2 - \\ &- \frac{\lambda^2}{2} 4! \int C^{(\leq N)\,4}_{\xi_1 \xi_2} \, d\xi_1 d\xi_2, \\ V_{3,N} = &-\frac{\lambda^3}{3!} \int d\xi_1 d\xi_2 \cdot \\ &\mathcal{E}^T_{(\leq N)} \Big(: \varphi^{(\leq N)\,4}_{\xi_1} : \cdot : \varphi^{(\leq N)\,4}_{\xi_2} : \cdot : \varphi^{(\leq N)\,4}_{\xi_3} : \Big) \end{aligned} \tag{21.8}$$

The theory of Secs.16–18 now becomes much simpler and one can prove that the effective potential has the form

$$\int \sum_\gamma \sum_S \frac{V(\gamma; S)}{n(\gamma)} \, P_S \, d\boldsymbol{\xi} \tag{21.9}$$

where S represents the decorated Feynman graphs and P_S has the form (if $\varphi \equiv \varphi^{(\leq k)}$, $D \equiv D^{(\leq k)}$):

$$: \prod_i \varphi^{n_i}_{\eta_i} \prod_j D^{m_j}_{\xi_j \xi_{j'}} :, \tag{21.10}$$

where, as in Sec.17, $D_{\xi\eta} = \varphi_\xi - \varphi_\eta$.

The same techniques of Secs.16–19 (easier now, in practice) yield the bound

$$\begin{aligned} M(\Delta_1, \ldots, \Delta_p) \leq \mathcal{N} \, n! \, \overline{\varepsilon} \, (\overline{\varepsilon} D)^{n-1} \, k^2 \cdot \\ \cdot \, e^{-\kappa \gamma^k d(\Delta_1, \ldots, \Delta_p)} \gamma^{-nk} B^{n^e} \end{aligned} \tag{21.11}$$

with the same notations as in Secs.16–19, i.e. $n =$ degree of γ, $\overline{\varepsilon} = \max(|\lambda|, |\mu|, |n|)$, $k =$ (root frequency of the tree),

$$\begin{aligned} M(\Delta_1, \ldots, \Delta_p) \equiv \int_{\Delta_1 \times \ldots \times \Delta_p \times \Lambda \times \ldots \times \Lambda} d\boldsymbol{\xi} \, \cdot \\ \cdot \sum_{\substack{\gamma \\ degree\,\gamma = n}} \; \sum_{\substack{G \\ P_G = P}} \frac{|V(\gamma; G)|}{n(\gamma)} \sup |P| \end{aligned} \tag{21.12}$$

where the supremum, of $|P_G| = |P|$ is over the fields $\varphi^{(\leq k)}$ verifying $\varphi^{(\leq k)} = \varphi^{(0)} + \varphi^{(1)} + \ldots + \varphi^{(k)}$ and (3.15), (3.16) and $B = \sup B_\Delta$. Finally $\mathcal{N}$ is the "adimensional bound" on p: $\sup |P| \leq B^{n^e} \big(\prod_j (\gamma^k |\zeta_j - \zeta_{j'}|)^{\frac14} \gamma^{\frac12 n^e k} \big)$ and $\mathcal{N}$ depends om n^e only because there are only a finite number of Wick monomials P of degree n^e of the type (21.10), apart from the values of the position labels.

The presence of the factor γ^{-nk} in (21.11) proves that the theory is asymptotically free. In the case $d = 2$ one replaces, basically, γ^{-nk} by γ^{-2nk}.

The above bounds were found in special cases and by using the techniques of the previous sections in (Benfatto *et al.*, 1978, 1980a,b); for the Schwinger functions expansions analogous bounds hold and were well known; see for instance (Glimm and Jaffe, 1968, 1970a, 1981).

Since in the approach presented here there is little difference between $: \varphi^4 :_2$ and $: \varphi^4 :_3$, I shall focus on the $d = 3$ case, (21.2), in this section.

The actual construction of the theory can be easily performed by taking advantage of the asymptotic freedom just pointed out (see the factor γ^{-nk} in (20.11)) and following, basically word by word, the procedure adopted in the cosine interaction case (which is, in fact, equivalently hard). For simplicity of exposition it is convenient to choose the values of the renormalized coupling constants μ and ν equal to 0; of course this does not mean that the three dimensional parameter space in (21.2) is replaced by the one dimensional space in (21.1) but only that the theory has only one renormalized coupling, namely λ, but still the counterterms can generate nonzero constant and mass terms (which will be of higher order in λ).

The strength of the asymptotic freedom shows that if the integrals over the "small fields" are computed via the cumulant expansion, i.e. via expressions like (13.22), (see also (5.13)), the expansion must be carried out at least to third order, since only the remainders of order, in λ, larger or equal to four give rise to an error of controllable size; such remainders at order $t+1$ are now estimated by a bound analogous to (13.25) $|\Lambda|\sum_{p=0}^{\infty}\big(\lambda\gamma^{-kp}(1+p)^a\log(e+p+\lambda^{-1})\big)^{t+1}\gamma^3 p$ convergent for $t\geq 3$ (if $d=2$ one could control errors of order ≥ 2 so that the cumulant expansion could be carried out stopping only at order $t=1$, in practice a good simplification).

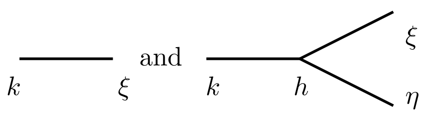
The other hard problem is that of the "large deviations" or "large fields". The D factors ($D_{\xi\eta}=\varphi_\xi-\varphi_\eta$) are dangerous much as the $(1-\cos(\varphi_\xi-\varphi_\eta))$ factors were in the cosine field case: they are treated exactly in the same way, because they appear with the right sign (i.e. the corresponding defective potentials tend to $-\infty$ when the field φ becomes so rough that $D_{\xi\eta}$ is too large compared to its covariance).

In this case there are also other dangerous terms in the third order effective potential, namely *all* the others. In fact, the field φ can be very large and make P itself very large; this was not a problem in the case of the cosine interaction, because there the fields appeared only inside trigonometric functions and therefore in a "bounded form".

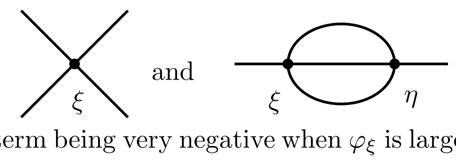
The large fields have to be treated by positivity arguments. The positivity properties needed in the theory are that the effective interaction contains the sum W of the following two terms:

$$\begin{aligned}&-\lambda\int_\Lambda :\varphi_\xi^{(\leq k)\,4}\,d\xi \qquad\qquad (21.13)\\ &-\frac{\lambda^2}{2}\binom{4}{1}^2 3!\int_{\Lambda^2}(C_{\xi\eta}^{(\leq N)\,3}-C_{\xi\eta}^{(\leq k)\,3}):D_{\xi\eta}^{(\leq k)\,2}:\,d\xi d\eta\end{aligned}$$

corresponding to contributions from the trees



due to the Feynman graphs



The first term being very negative when φ_ξ is large and the second being very negative when $D_{\xi\eta}$ is large compared to $(\gamma^k|\xi-\eta|)^{\frac14}$; here one uses $\lambda>0,\lambda^2>0$ (which must be a further restriction, although no restriction on the size of λ is necessary).

The details are essentially identical to those explained in the cosine case and they will not be repeated here; and the reader is referred to the literature, see (Benfatto *et al.*, 1978, 1980a,b). It is, however, important to stress once more that nothing is really different from the case of the cosine field treated in detail in Secs.13 and 14, as the reader can check by a glance at the above references.

The result of the analysis is the existence of a constant $E>0$ such that for all $f\in C^\infty$ with support in a set Λ_f it is

$$\int e^{V_{3,N}(\varphi^{(\leq N)})+\varphi^{(\leq N)}(f)}P(d\varphi^{(\leq N)})\leq e^{|\Lambda|\,E+||f||_\infty^2|\Lambda_f|} \tag{21.14}$$

which proves, up to technicalities, the existence of the limit of the "interaction measure" at least on subsequences as $N\to\infty$: i.e. it proves the "nonperturbative ultraviolet stability".

With some extra work and using the same ideas plus abstract arguments one could prove the actual existence of the ultraviolet limit (with no subsequences involved); this is not written explicitly in the literature but, at least for λ small, the result is known also by other methods.

As in the case of the cosine interaction the other limit, $\Lambda\to\infty$, the "infrared limit" has to be treated under extra assumptions (like λ small), because contrary to the ultraviolet limit, *in the cases considered so far*, it may be affected by nonuniqueness phenomena: "(infrared) phase transitions" corresponding to the ordinary phase transitions of statistical mechanics. Such transitions have to be expected here, too, as the main idea of the multiscale approach is that field theory can be reduced to the theory of a spin system on a lattice of scale 1. And such systems are known to exhibit phase transitions in the infrared limit (also called thermodynamic limit) $\Lambda\to\infty$, (Federbush and Battle, 1982; Feldman, 1974; Feldman and Ostervalder, 1976; Glimm *et al.*, 1975, 1976; Magnen and Seneor, 1976).

Finally let me mention that in some cases with $d=2,3$ the theory can be performed completely, i.e. up to the extent of really constructing a field theory verifying the Wightman axioms, hence with the proper interpretation of a physical quantum theory, describing in some of its states, interacting relativistic quantum particles, (Glimm *et al.*, 1973, 1975, 1976; Koch, 1980; Ma, 1976), however this kind of questions go beyond the scopes of the present review.

xxii. Comments on resummations. Triviality and nontriviality. Some apologies.

The reason one cannot perform the resummations, described in the preceding section, in a rigorous way is simply that the coefficients β of the "beta function" (20.13), formally defining the resummed "adimensional form factors" $\lambda^{(\alpha)}(k)$, $\alpha=4,2,2'.0$

$$\begin{aligned}(\mathcal{B}\underline{\lambda})^{(\alpha)}(k)&\stackrel{def}{=}\sum_{r=2}^{\infty}\sum_{\substack{h_1,\ldots,h_r\\ \alpha_1,\ldots,\alpha_r}}\cdot\\ &\cdot\beta^{(\alpha)}_{\alpha_1,\ldots,\alpha_r}(h;h_1,\ldots,h_r)\prod_{i=1}^{r}\gamma^{-\overline{\rho}(h-h_i)}\lambda^{(\alpha_i)}(h_i)\end{aligned} \tag{22.1}$$

are badly behaved in r as $r \to \infty$: i.e. they are bounded by $r!C^r$ (unless one restricts oneself to the planar theory where (20.10) holds, see Sec21). This is in conflict with the fact that the idea of using the equation (of "Callan-Symanzik)

$$\lambda^{(\alpha)}(k+1) = \lambda^{(\alpha)}(k) + (\mathcal{B}\underline{\underline{\lambda}})^{(\alpha)}(k+1) \tag{22.2}$$

to define the adimensional form factors in a nonperturbative way requires the existence of a sequence $\underline{\underline{\lambda}} = \{\lambda^{(\alpha)}(k)\}_{\alpha,k}$ of form factors for which $\mathcal{B}\underline{\underline{\lambda}}$ makes sense and verifies (22.2).

Because of the bad bounds on the β coefficients and because, as emerges from considering only the second order part of (22.2), a solution to (22.2) cannot tend to zero too fast as $k \to \infty$ (see (20.25)), the only way in which $\mathcal{B}\underline{\underline{\lambda}}$ could make sense for interesting sequences $\underline{\underline{\lambda}}$ is that there are cancellations in the β's (which are sums of many terms of uncontrolled signs) and, possibly, the existence of such cancellations might depend upon the sequences $\boldsymbol{\lambda}(k)$ chosen in (22.1) and not just on the β coefficients.

In this section I elaborate on what could happen if (22.2) admitted a solution verifying $\boldsymbol{\lambda}(k) \xrightarrow[k\to\infty]{} 0$ and providing the necessary cancellations needed to make sense of the *r.h.s.* of (22.1) and, consequently, of (22.2).

In this situation one should reasonably expect that the solution of (22.2) behaves as $h \to \infty$ exactly as the solution to an equation like (22.2) but with $\mathcal{B}$ replaced by its second order part (i.e. by the terms with $r = 2$ in (22.1)), see (Coleman and Weinberg, 1973).

Such an equation was the basis for the theory of the adimensional form factors in the "planar theory" of Sec.20 and, as discussed there, one expects that it has a solution in which $-\lambda^{(4)}(h)$ behaves as, see (20.23),

$$\simeq_{k\to\infty} \frac{\overline{\lambda}}{1-\beta\, h\,\overline{\lambda}} \tag{22.3}$$

and similarly should behave $\alpha(h)$, while $\mu(h), \nu(h)$ ought to go to zero as the square of (22.3). Then the following remarks can be made.

(1) In itself a solution to (22.2) behaving like (22.3) does not yet yield a solution to the problem of showing that the effective potentials $V^{(k)}$ are well defined as sums of resummed perturbation series (see ('t Hooft, 1983, 1984)).

In fact, the resummation operation just permits one to describe the effective potentials in terms of dressed trees "with no frames" and with end points (ξ, α) providing an adimensional form factor $\lambda^{(\alpha)}(h)$ rather than $\lambda^{(\alpha)}$, if h is the frequency index of the tree vertex to which they are joined by a branch of the tree.

Although this is a big improvement, as far as the k dependence of $V^{(k)}$ is concerned (it suffices to recall that the non resummed adimensional form factors were diverging with h as powers of an order depending on their degree of complexity and with no *a priori* bounds, see (19.5) and (19.18), while the resummed adimensional form factors even go to zero with the frequency h as $h \to \infty$) one is still confronted with the problem of summing the contributions to $V^{(k)}$ of the above "simple", i.e. frameless, trees.

One finds, in doing so, a power series in the resummed adimensional form factors (coming from the trees of order n) whose n-th terms can still be bounded only by $n!$. If we use the bounds of Sec.19, the effective potential is now given by an expression like (19.1) with a sum running only over the trees with no frames and such that the contributions from the trees of degree n can be bounded as in (19.2) with the last sum (divergent, *a priori*) replaced by k^{-n}, a rather minor gain as far as the n dependence is concerned.

The structures of the beta-function coefficients and those of the $V(\gamma; S)$ are obviously related, and "basically the same", so that if one is willing to accept the existence of cancellations allowing giving a meaning to $\mathcal{B}\underline{\underline{\lambda}}$ one should also accept that the very same mechanism might produce cancellations in the expression of the effective potential in terms of the resummed form factors $\underline{\underline{\lambda}}$.

However this cancellation mechanism is totally unclear (as this time the beta function cannot help, as it did in the planar case of Sec.21, to exhibit such cancellations) and it can only be hoped to exist.

(2) It might be that the parameter γ plays an important role in the theory: for instance, in (22.3) the singularities in $\overline{\lambda}$ are located at γ–dependent positions (in fact one could check that $\frac{\beta}{\log\gamma} \xrightarrow[\gamma\to 1]{} \beta_0 > 0$, by explicit calculation).

This leads to the possibility that the theory could be defined for many but not for all $\overline{\lambda}$'s near zero, e.g. for the values of the renormalized coupling constant which avoid a suitable set of small measure (union of small neighborhoods of the points $(\beta\, h)^{-1}$ in the case (22.3)) where the form factors could be singular functions of $\overline{\lambda}$. Such a situation is not uncommon in perturbation theory in classical mechanics and it might appear also in field theory.

(3) The possibility of the existence of cancellations mentioned in remark (1) above is hinted at also by the "triviality proofs" where, via some very special assumptions on the regularization and the form of the counterterms, one shows that the adimensional form factors $\lambda^{(\alpha)}(k; N)$ defined in the presence of an ultraviolet cut-off at length γ^{-N} vanish as $N \to \infty$: $\lambda^{(\alpha)}(k; N) \xrightarrow[N\to\infty]{} 0$.

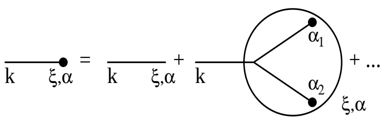
The fact that $\lambda^{(\alpha)}(k; \infty) = 0$ is a property that can be proved nonperturbatively under very special assumptions, (Aizenman, 1982; Frölich, 1982), hints at the existence of nontrivial cancellations mechanisms in the summations involved in the construction of the effective potentials and of the beta function. Paradoxically the "triviality arguments" might be interpreted as nontriviality arguments.

If we go back to a slightly more concrete frame of mind, some comments on the cut-off dependence of the above discussion, brought up in the last remark, as well as on the classical triviality arguments of Landau, (Landau, 1955; Landau and Pomeranchuck, 1955), (Thirring, 1958) p. 198, (Boboliubov and Shirkov, 1959), p. 528, seem appropriate here. In fact, they hinge upon the just brought up question of the cut-off and of the regularization dependence of the whole theory.

The form factor resummations can be studied with no formal change in the presence of an ultraviolet cut off γ^N. In the previous sections the N dependence of the form factors was seldom made explicit because one was interested in properties which were uniform in N.

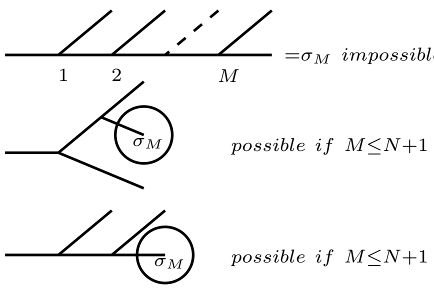
Contrary to what is sometimes stated, fixing N does not make the theory well defined; in fact one can easily see that there is a simple relation between the form factors of the theory with ultraviolet cut off N, denoted $\lambda^{(\alpha)}(k;N)$, and the bare coupling constants. Precisely, the bare couplings are $\lambda^{(\alpha)}(N;N)\gamma^{(2\delta_{\alpha,2}+4\delta_{\alpha,0})\,N}$. The reason the bare coupling constants are undefined even in a theory with cut-off is simply that $\boldsymbol{\lambda}_N \equiv \boldsymbol{\lambda}(N;N)$ are still power series in the renormalized couplings with only $n!$ bounds on their coefficients, i.e. they are formal power series, probably divergent.

One can use the resummation ideas of Secs.9 and 19 to try to say something about the bare couplings $\boldsymbol{\lambda}_N$; in fact, $\boldsymbol{\lambda}(h;N)$ is formally defined by the same recursion relation as $\boldsymbol{\lambda}(h) \equiv \boldsymbol{\lambda}(h;\infty)$:



(54)

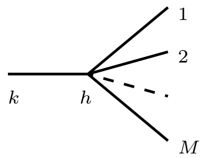
(see also Fig.31) the difference being that $k \le N$ and that everywhere only some tree shapes can appear. Thus if one fixes a frame and deletes all the inner frames and their contents, the tree shape left inside the selected frame has to be a shape which can arise in computing the effective potentials in the presence of a cut off γ^N; for instance



(55)

In fig.(55) the first tree is impossible if $M > N+1$, because one cannot attach allowed frequency labels h_i to the vertices of σ_M with $h_i < h_{i+1}$ and root at -1 (as should have been the case has σ_M been a tree which could have arisen in the presence of a cut off γ^N).

Note that $\lambda(N;N)$ is a (probably) divergent series, because there are infinitely many trees compatible, even with a finite cut off N, e.g.



and $M \ge h$. The equation in Fig.(54) is very similar to the equation discussed in Secs.19 and 20, and in fact it coincides with them if one restricts the k and h indices in (22.1) and (22.2) to be $\le N$.

It is therefore clear that in hte theory of (22.2) performed in the approximation in which the second order "dominates", i.e. in which (22.2) becomes equivalent to (20.19). and hence eventually to (20.20)and (20.24), on e can manage to find a solution to (20.2) with

$$\lambda^{(4)}(h;N) \simeq \frac{\overline{\lambda}}{1-\beta\,\overline{\lambda}\,h} \qquad \text{for large } h \tag{22.4}$$

which would lead to (setting $\lambda^{(4)}(N;N) = \lambda_N =$ bare coupling) the following surprising relation

$$\overline{\Lambda} = \frac{\lambda_N}{1+\beta\,N\,\lambda_N}, \tag{22.5}$$

where $\overline{\lambda} \equiv \lambda(0;N)$ is a "renormalized coupling" expressed in terms of the bare coupling λ_N.

Triviality follows from (22.5) which implies

$$\overline{\lambda} \xrightarrow[N\to\infty]{} 0 \;\Rightarrow \lambda(h;\infty) \equiv 0 \tag{22.6}$$

no matter how λ_N behaves provided $\lambda_N \ge 0$.

On the other hand $\lambda_N < 0$ is obviously not allowed as this would make the theory in presence of a cut off undefined. Of course the above argument is based on the identification of $\boldsymbol{\lambda}(N;N)$ with $\boldsymbol{\lambda}(N;\infty)$ which, to say the least, is not proved (even in an approximate sense).

To understand better the structure of (22.5) one can remark that the bare couplings $\boldsymbol{\lambda}(N;N)$ are a formal power series in the renormalized couplings (for simplicity take $\mu = \nu = \alpha = 0$ so that there is only one renormalized coupling),. The coefficients diverge with N as $N \to \infty$ like powers of N: precisely as N^{n-1} to order n.

The latter statement can be proved by going back to (19.18) which tells us that the bare couplings $\boldsymbol{\lambda}(N;N)$ can receive the "most divergent contribution" from the trees γ containing the largest number of frames. Such a number is, if n is the degree of γ, $f-1<n$. Furthermore the trees which contain the maximal number of frames, $f = n$, really give a contribution to the form factors like $\lambda\,(\lambda D)^{n-1}(bN)^{n-1}$ to leading order in N.

This can easily be seen by observing that $f = n$ implies that each vertex of σ is framed and gives rise to a bifurcation in just two branches (otherwise $f < n$). In other words the resummation of the most divergent contributions is obtained simply by considering what in Sec.9 was called the resummation of the most divergent graphs. In the language of Sec.20 and of this section this means replacing $\mathcal{B}$ by $\mathcal{B}_2$ in the beta function (so that one also finds the interpretation of the approximation in which $\mathcal{B}$ is replaced by $\mathcal{B}_2$: it just means that counting only trees simplest in structure and completely framed, i.e. with no renormalization vertex (i.e. no unframed vertex) allowed).

Since, as was explained in Sec.20, one knows that the well behaved solutions to

$$\boldsymbol{\lambda}(k+1) = \boldsymbol{\lambda}(k) + (\mathcal{B}\underline{\underline{\lambda}})(k+1) \tag{22.7}$$

behave like (22.4), one sees another interpretation of Landau's result: it leads to triviality if one neglects everything except the most divergent contributions to the adimensional form factors.

At the same time it also allows one to compute rigorously the most divergent contributions to the coefficients of the expansion of the bare couplings in terms of the renormalized ones. For example λ_N has, to order n in the renormalized coupling, a most divergent contribution exactly equal to

$$\lambda^n (\beta N)^{n-1} \tag{22.8}$$

while α_N has the contribution

$$-\beta' \beta^{-1} \lambda^n (\beta N)^{n-1} \tag{22.9}$$

with the notations of Sec.20, see (20.4).

A more detailed analysis allows one easily to select the Feynman graphs which, in the evaluation of the most divergent trees contributions really give the leading behavior in N: in the language of classical perturbation theory they are the so called "parquet graphs" and one could find (22.8) and (22.9). This involves quite hard work (see the appendix by Rivasseau in the paper (Gallavotti and Rivasseau, 1983)). This point will not be discussed further here because it involves too many new definitions necessary to establish contact between the formalism developed here and the classical language of the Feynman graphs.

I collect now a few concluding comments to stress some of the ideas and problems already foreshadowed in all the sections of this work.

(a) The assumption that the form factors $\lambda(h;N)$ verify essentially the same equation as the $\lambda(k;\infty)$ seems hard to accept (at least if one wishes to claim from this that $\lambda(k;N)$ and $\lambda(k;\infty)$ have the "same" properties) if one accepts that perturbation theory gives correctly the asymptotic expansion for the beta function when the renormalized couplings $\lambda, \alpha, \mu, \nu$ are suitably chosen (say as functions of λ).

In fact in order that this could be true some important cancellation effects must be present (to compensate the factorially growing coefficients) and the recursion relation for $\boldsymbol{\lambda}(k;N)$ being "slightly different" from those of $\boldsymbol{\lambda}(k;\infty)$ may just miss the cancellations.

(b) It is clear that, once an ultraviolet cut off is specified together with the bare Lagrangian, the coefficients $\boldsymbol{\lambda}(h;N)$ are well defined and can be expressed in terms of the bare coupling constants $\boldsymbol{\lambda}(N;N)$ both as formal power series or as true functions (of the bare couplings) or as formal power series in the renormalized form factors $\boldsymbol{\lambda}(0;N)$ or in the renormalized coupling constants.

On the other hand the functions $\boldsymbol{\lambda}(h;\infty)$ are perturbatively well defined as formal power series in the renormalized constants and, thought as formal power series, are *completely independent* of the regularization used.

The approach in which one prescribes the bare constants $\boldsymbol{\lambda}(N;N)$ and tries to study the renormalized constants $\boldsymbol{\lambda}(0;N)$ looks conceptually clearer; however it suffers from the drawback of necessarily replying on special assumptions on the cut off and on the regularization and on the bare Lagrangian.

For instance the well-known "lattice approximation" in which $\partial\varphi$ is the nearest neighbor difference and the Lagrangian is is taken to be

$$\gamma^{-4N} \sum_{\xi} (-\lambda_N : \varphi_\xi^4 : -\mu_N : \varphi_\xi^2 : -\nu_N) d\xi \tag{22.10}$$

with the free field distribution defined by

$$const\, e^{-\gamma^{-4N} \sum_\xi ((\partial\varphi_\xi)^2 + \varphi_\xi^2)} \prod_\xi d\varphi_\xi \tag{22.11}$$

has the drawback of making "indistinguishable" the "main" $(\partial\varphi)^2$ term from the similar "counterterm": whether this point is relevant or not is not known but it is certainly one of the main properties necessary in the existing triviality proofs of the lattice regularization of φ_4^4 (in the sense of (22.10) and (22.11)).

A sign that something might be wrong with the lattice regularization, with respect to the old problem of finding a meaning for the perturbation theory formal series, is that the most divergent contributions to the expansion of the bare couplings λ_N, α_N in a series of the renormalized couplings $\lambda, \alpha, \mu, \nu$ are (when $\mu = \alpha = \nu = 0$ for simplicity) all positive for λ_N and all negative for α_N, (Gallavotti and Rivasseau, 1983, 1985), hinting at the possibility that in the bare theory the counterterms in $(\partial\varphi)^2$ might be antiferromagnetic and therefore a detailed description of their form might be essential (e.g. whether $\alpha_N(\partial\varphi)^2$ contains the square of the nearest neighbor difference or a many neighbor version of it).

This also hints at the possibility that the convergence of the fields φ_ξ on the lattice to the continuum fields might be more complicated than the naive pointwise convergence of the Schwinger functions, even at distinct points.

(c) Expression (22.4) hints at the possibility that $\boldsymbol{\lambda}(k;N)$ could be defined for some values of $\overline{\lambda}$ which accumulate to 0 together with other values of $\overline{\lambda}$ for which $\boldsymbol{\lambda}(k;N)$ cannot be defined. Such regular and singular values of $\overline{\lambda}$

may depend on γ: i.e. the parameter γ itself may play a nontrivial role in defining the theory. The existence of another relevant parameter is somewhat necessary if one believes that the antiferromagnetic effects discussed above may have some importance: such a parameter should describe on which scale such effects are smoothed out (an event that should happen since the final Schwinger functions, as defined order by order by perturbation theory, are smooth except at coinciding points).

(d) Of course one cannot even exclude the possibility that $\overline{\lambda}$ could be negative (which might eliminate the singularities in $\overline{\lambda}$ for $\overline{\lambda}$ small, as shown in the approximate formulae (22.4)).

In fact from the observation that $\boldsymbol{\lambda}(N;N) \neq \boldsymbol{\lambda}(N;\infty)$ there seems to be little (or no) relation between the signs of $\boldsymbol{\lambda}(N;N) =$ (bare coupling) and those of the effective form factors $\boldsymbol{\lambda}(0;\infty)$ (which for small renormalized coupling should have the same sign as the renormalized coupling itself, called above $\overline{\lambda}$): at least unless special assumptions on the bare Lagrangian are made, see (Coleman and Weinberg, 1973) who prove that $\lambda(0;N) < 0$ implies $\lambda(N;N) < 0$ in a class of nonperturbative lattice regularized φ^4 models with a ferromagnetic kinetic term; see also (Aizenman, 1982; Frölich, 1982) for a rigorous version of the same result.

(e) If $d = 2$ or 3 one could still perform the (mostly unnecessary if $d = 3$ and totally unnecessary if $d = 2$) subtractions that one would perform in the case $d = 4$, as described in Secs.17 and 18. Contrary to what is sometimes stated, the problem is far from being easy in spite of the strength of the asymptotic freedom.

The bare couplings are still given by *a priori* nonconvergent series and the same happens for the form factors. The only gain is that the dimensionless form factors are bounded or grow with a power of the frequency index at any fixed order of perturbation theory and the power is a number independent on n However the dependence of the perturbation series coefficients is, at order n, bounded by $n!$.

Understanding whether, in spite of this, one can make sense, beyond perturbation theory, of $:\varphi^4:$ fields in dimension $d = 2,3$ with the subtractions of $:\varphi^4:_4$ would help in understanding the role of asymptotic freedom in constructive field theory. By "subtractions of $:\varphi^4:_4$" one means here essentially the usual zero-momentum subtractions "to first order for the four-external-lines diagrams and to second order for the two-external-lines diagrams". This problem, surprisingly, does not seem to have been considered in the literature.

I apologize for this section, which has a somewhat different character from the rest of the work, mostly dealing with open or ill-defined problems. The main reason for including it is to stress a fact that I think is a rather important one, namely that the problem of the construction of a nontrivial $:\varphi^4:_4$ field theory, or a proof of its triviality is still wide one and hard.

Acknowledgments

I am grateful to Elliot Lieb for his encouragement and for his proposal to write this paper. I am indebted to Francesco Nicoló for many clarifying discussions and suggestions. I am also indebted to Francesco Nicoló, Jürgen Renn and Adrian Steinmann for reading some sections of this work. It is a pleasure to thank Giovanni Felder for pointing out an error in an earlier version of this work (see Appendix F). This work has been partially supported by the Ministero della Pubblica Istruzione and the Scuola Normale di Pisa in Italy, and by Rutgers University and NSF grant No. MCS-8108814(AO3) at the Institute for Advanced Study in the U.S.A.

Appendix A1. Free fields covariance: hints

Let $H_{quantum} = -\frac{1}{2}A\Delta + V$ where Δ is the Laplace operator on the space $L_2(\mathbb{R}^{(L/a)^d}) \stackrel{def}{=} \mathcal{H}$, $A = \frac{\hbar^2}{\mu\, a\, D}$, see (1.14). The operator $H_{quantum} \equiv H$ has a simple lowest eigenvalue, because $V \to \infty$ at infinity, see (1.14): therefore if φ_x denotes also the multiplication operator on $\mathcal{H}$ by φ_x it is

$$\begin{aligned} C_{\xi\eta} &= (e_0, \varphi_{\overline{x}} T_t \varphi_{\overline{y}} e_0) \equiv (e_0, T_\tau \varphi_{\overline{x}} T_t \varphi_{\overline{y}} T_{\tau-t} e_0) = \\ &= \lim_{\tau\to\infty} \operatorname{Tr}(T_\tau \varphi_{\overline{x}} T_t \varphi_{\overline{y}} T_{\tau-t}) = \lim_{\tau\to\infty} \int \cdot \\ &\cdot T_\tau(\boldsymbol{\varphi}_{-\tau}, \boldsymbol{\varphi}_0)\, (\boldsymbol{\varphi}_0)_{\overline{x}}\, T_t(\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_t)\, (\boldsymbol{\varphi}_t)_y \cdot \\ &\cdot T_{\tau-t}(\boldsymbol{\varphi}_\tau, \boldsymbol{\varphi}_{-\tau})\, d\boldsymbol{\varphi}_{-\tau} d\boldsymbol{\varphi}_0 d\boldsymbol{\varphi}_t \end{aligned} \tag{A1.1}$$

where $\boldsymbol{\varphi} = (\varphi_x)_{x\in\Lambda\cap\mathbb{Z}^{D_a}}$ and $\xi = (\overline{x}, 0)$, $\eta = (\overline{y}, t)$. Using Trotter's formula, see comments before (2.7), one finds (if $b > 0$ and $2\tau/b = N$ is an integer)

$$\begin{aligned} C_{\xi\eta} &= \lim_{\tau\to\infty} \lim_{b\to 0} e^{\frac{2E_0\tau}{\hbar}} \int (e^{\frac{bA}{2}\Delta} e^{bV})^{\frac{\tau}{b}} (\boldsymbol{\varphi}_{-\tau}, \boldsymbol{\varphi}_0) \cdot \\ &\cdot (\boldsymbol{\varphi}_0)_{\overline{x}} (e^{\frac{bA}{2}\Delta} e^{bV})^{\frac{\tau}{b}} (\boldsymbol{\varphi}_0, \boldsymbol{\varphi}_t) \cdot \\ &\cdot (\boldsymbol{\varphi}_t)_{\overline{y}} (e^{\frac{bA}{2}\Delta} e^{bV})^{\frac{(t-\tau)}{b}} (\boldsymbol{\varphi}_t, \boldsymbol{\varphi}_{-\tau}) d\boldsymbol{\varphi}_{-\tau} d\boldsymbol{\varphi}_0 d\boldsymbol{\varphi}_t = \lim_{\tau,b} \\ &\frac{\int \prod_{j=0}^N \Big(e^{\frac{bA}{2}\Delta} (\boldsymbol{\varphi}_{t_j}, \boldsymbol{\varphi}_{t_{j+1}}) e^{bV(\boldsymbol{\varphi}_{t_j})} \boldsymbol{\varphi}_{(\overline{x},0)} \boldsymbol{\varphi}_{(\overline{y},t)}\, d\boldsymbol{\varphi}_{t_j} \Big)}{\int \prod_{j=0}^N \Big(e^{\frac{bA}{2}\Delta} (\boldsymbol{\varphi}_{t_j}, \boldsymbol{\varphi}_{t_{j+1}}) e^{bV(\boldsymbol{\varphi}_{t_j})}\, d\boldsymbol{\varphi}_{t_j} \Big)} \end{aligned} \tag{A1.2}$$

where $t_j = -\tau + b\,j$ and one assumes that τ/b is also an integer and the fields in the kernels have been denoted, for reasons which will be soon clear, with a "time index" t_j rather than by j itself. Also one writes $(\boldsymbol{\varphi}_\theta)_x = \varphi_{(x,\theta)}$ and $\boldsymbol{\varphi} = (\varphi_x)_{x\in\mathbb{Z}^d a\cup\Lambda}$. The denominator within the last limit is essentially $e^{-2E_0\tau\hbar^{-1}}$ being, after the limit $b\to 0$, equal to the trace of $e^{-\frac{2\tau}{\hbar}H_{quantum}}$. Using the explicit form of the heat equation kernel

$$\prod_{j=0}^{N} e^{\frac{bA}{2}\Delta}(\boldsymbol{\varphi}_{t_j},\boldsymbol{\varphi}_{t_{j+1}}) = \\ = e^{-\frac{1}{2}\frac{1}{Ab}\sum_{j=0}^{N}\sum_{x\in\Lambda}(\varphi_{x,t_j}-\varphi_{x,t_{j+1}})^2} \tag{A1.3}$$

and including the factor $\prod_{j=0}^{N} e^{bV(\boldsymbol{\varphi}_{t_j})}$ one gets

$$e^{-\frac{1}{2}\frac{ba^D\mu}{\hbar}\sum_{x\in\Lambda}\sum_{j=0}^{N}\sum_{i=1}^{D}(\varphi_{x+ae_i,t_j}-\varphi_{x+ae_i,t_{j+1}})^2}\cdot \\ \cdot e^{-\frac{1}{2}\frac{ba^D\mu}{\hbar}\left(\frac{m_0c^2}{\hbar^2}\right)^2\sum_{x\in\Lambda}\sum_{j=0}^{N}(\varphi_{x,t_j})^2} \tag{A1.4}$$

If ξ denotes a point on the $d = D+1$ dimensional lattice with spacing a in the first D directions and b in the last one, and if e_j, $j = 1,\ldots,D,0$ are unit vectors in the lattice directions, one finds that the integral in (A1.2) has the form

$$\lim_{\tau,b} const \int e^{-\frac{1}{2}(Q\varphi,\varphi)}\varphi_\xi\varphi\eta\prod_\omega f\varphi_\omega \tag{A1.5}$$

where the constant is a normalization constant and $Q = (Q_{\xi\eta})_{\xi,\eta\in\widetilde\Lambda}$, where $\widetilde\Lambda$ is $(\Lambda\cap\mathbb{Z}^d a)\times((-\tau,\tau)\cap\mathbb{Z}b)$ with "periodic boundary conditions" and $\xi = (x,0)$, $\eta = (y,t)$ is given by

$$(Q\varphi,\varphi) \stackrel{def}{=} \frac{\mu}{\hbar}ba^D\sum_{\xi\in\widetilde\Lambda}\Big(\big(\frac{(\varphi_{\xi+e_d b}-\varphi_\xi)^2}{b^2}+ \\ + c^2\sum_{j=1}^{D}\frac{(\varphi_{\xi+e_j a}-\varphi_\xi)^2}{a^2}\big)+\frac{(m_0c^2)^2}{\hbar^2}\varphi_\xi^2\Big) \tag{A1.6}$$

But the integral (A1.5) is simply $Q^{-1}_{\xi\eta}$ and Q^{-1} can be easily found by explicit diagonalization: because of the periodic boundary conditions the eigenvectors of Q are complex exponentials. In the limit $\Lambda\to\infty,\tau\to\infty$ the eigenvalues fill the Brillouin zone and Q^{-2} becomes, if $p = (\mathbf{p},p_0)\in\mathbb{R}^{D+1}$,

$$Q^{-1}_{\xi\eta} = \frac{\hbar}{(2\pi)^d\mu}\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} d\mathbf{p}\int_{-\frac{\pi}{b}}^{\frac{\pi}{b}} dp_0\cdot \\ \cdot\frac{e^{ip(\xi-\eta)}}{\frac{(m_0c^2)^2}{\hbar^2}+2\big(\frac{1-\cos bp_0}{b^2}+c^2\sum_{j=1}^{D}\frac{1-\cos ap_j}{a^2}\big)} \tag{A1.7}$$

and (2.8) follows from (A1.7) by letting $b\to 0$.

Appendix A2. Hint for (2.1)

For the proof of (2.10) one proceeds as in Appendix A. Everything is the same up to (A1.4) where, in the present case, an extra term appears:

$$e^{-\frac{ba^D\mu}{2\hbar}\sum_{\boldsymbol{\xi}} I(\varphi_{\boldsymbol{\xi}})} \tag{A2.1}$$

Setting $T = \tau$ one sees that the proof of (2.10) is the proof of admissibility of the interchange of two limits. This problem should be studied by the reader as a test of understanding of the theory of Brownian motion. On a heuristic level the reader can accept (2.10) and proceed to see what is done with it. The identity of the P in (2.10) and (2.11) is a byproduct of the above discussion.

Appendix A3. Wick monomials and their integrals

Let $x_1,\ldots,x_p$ be Gaussian random variables with covariance matrix

$$C_{ij} = \mathcal{E}(x_ix_j) \tag{A3.1}$$

One defines, for any of the above variables x,

$$:x^p: \stackrel{def}{=} (2\mathcal{E}(x^2))^{\frac{p}{2}}H_p(\frac{x}{(2\mathcal{E}(x^2))^{\frac{1}{2}}}) \tag{A3.2}$$

where H_p is the p-th Hermite polynomial defined by the generating function

$$\sum_{p=1}^{\infty}\frac{\alpha^p}{p!}H_p(\xi) = e^{-\frac{1}{4}\alpha^2+\alpha\xi} \tag{A3.3}$$

More generally one defines inductively

$$:x_1^{n_1+1}x_2^{n_2}\ldots x_p^{n_p} := x_1\ :x_1^{n_1}x_2^{n_2}\ldots x_p^{n_p}:- \\ -\sum_{j=0}^{p}C_{1j}\,n_j\ :x_1^{n_1}\ldots x_j^{n_j-1}\ldots x_p^{n_p}:, \tag{A3.4}$$

interpreting the last term as 0 if $n_j = 0$ and setting

$$:x_1^0\ldots x_p^0 := 1,\ :x_1^0\ldots x_k\ldots x_p^0 := x_k. \tag{A3.5}$$

Expressions (A3.4) and (A3.5) are a natural extension of the recursion relations for Hermite polynomials expressed by (if $C = \mathcal{E}(x^2)$

$$x^{n+1} := x\ :x^n: -m\,C\ :x^{n-1}:. \tag{A3.6}$$

Note that if $x_1 = x_2 = x$ it is $x_1^{n_1}x_2^{n_2} :=: x^{n_1+n_2}:$. The expression (A3.4) implies by induction

$$:(\sum_{i=1}^{p}\omega_ix_i)^q := \sum_{q_1+\ldots q_p=q}\frac{q!}{q_1!\ldots q_p!}\cdot \\ \omega_1^{q_1}\ldots\omega_p^{q_p}\ :x_1^{q_1}\ldots x_p^{q_p}: \tag{A3.7}$$

which id the "Leibnitz rule" for Wick monomials.

The basic property of the Wick-ordered monomials is the "Wick rule" for the expectations of products of Wick monomials. Let $D_1, \ldots, D_s$ be s subsets of $(1, 2, \ldots, n)$ and let

$$:x_{D_j}: =: \prod_{\alpha \in D_j} x_\alpha :, \tag{A3.8}$$

then the integral $\mathcal{E}(\prod_{j=1}^s : x_{d_j} :)$ is computed as follows.

Draw, say on a plane, s clusters of $|D_1|, \ldots, |D_s|$ points each and arbitrarily label the points in the cluster D_j.

Draw one line out of each of the vertices $\alpha \in D_j$ and think of it as representing the variable x_α.

Let $\mathcal{I}$ be the set of the graphs obtained by joining pairwise alla such lines in all possible ways so that no lines constituting a pair emerging from the same cluster are ever joined together. Denote (α, β) the elements of $\pi \in \mathcal{I}$ obtained by "joining" (or "contracting") a line emerging from the vertex α with a line emerging from the vertex β. Then denoting (α, β) the lines in π joining α and β, we have

$$\mathcal{E}(\prod_{j=1}^s : x_{D_j} :) = \sum_{\pi \in \mathcal{I}} \prod_{(\alpha,\beta) \in \pi} C_{\alpha\beta} \tag{A3.9}$$

Equation (A3.9) id the "Wick rule" and it is easily proved by induction from (A3.4) and its special case when D_j contains one point for each j.

The latter case is treated directly from the relation

$$\begin{aligned} &\mathcal{E}(\sum_{q=0}^\infty \frac{1}{q!}(\sum_i \omega_i x_i)^q) = \\ &= \sum_{n_1+\ldots+n_q} \frac{\omega_1^{n_1} \cdots \omega_q^{n_q}}{n_1! \cdots n_q!} \mathcal{E}(x_1^{n_1} \cdots x_q^{n_q}) = \\ &= \mathcal{E}(e^{\sum_i \omega_i x_i}) = e^{\frac{1}{2}\sum_{i,j} \omega_i \omega_j C_{ij}} \end{aligned} \tag{A3.10}$$

where the last integration is the general integration formula for the exponential of a Gaussian variable$x$: namely $\mathcal{E}(e^x) = e^{\frac{1}{2}\mathcal{E}(x^2)}$.

The formula that one is seeking follows from (A3.10) by developing the last exponential in powers and by identifying the coefficients of equal powers in the second and fourth terms of (A3.10) and then interpreting the result graphically.

But the most remarkable property of the Wick monomials is related to the possibility of simple formulae for the truncated expectations. In fact

$$\mathcal{E}^T(: x_{D_1} :, \ldots, : x_{D_p} :; s_1, \ldots, s_p) \tag{A3.11}$$

can be computed via the following rule: draw s_1 clusters of $|D_1|$ points each, s_2 clusters of $|D_2|$ points each, *etc*and label the points in in them by the elements of $D_1, D_2, \ldots, D_p$ respectively plus another index identifying which cluster is being considered among the s_j clusters of $|D_j|$ points.

Then consider all the possible graphs π obtained by joining pairs of such points, avoiding drawing lines joining points belonging to the same cluster and with the property that each graph π would be connected if all the points inside each cluster were considered identical or, the same thing, connected (i.e. π should be connected "modulo the clusters").

Then, if λ is a line in π joining the pair of points $(\alpha, \beta) = \lambda$, it is

$$\mathcal{E}^T(: x_{D_1} :, \ldots, : x_{D_p} :; s_1, \ldots, s_p) = \sum_\pi \prod_{\lambda \in \pi} C_{\alpha\beta} \tag{A3.12}$$

In particular it is remarkable that $\mathcal{E}^T(\cdot) \geq 0$ if $C_{\alpha\beta} \geq 0$ (which, however, is a property not necessarily rue because C is constrained only to be a positive definite matrix).

The (A3.12) can be generalized to the case where $x_i = y_i + z_i$ with y_i and z_i, $i = 1, \ldots, p$, being two sets of independent Gaussian random variables with covariances C_{ij}^0 and C_{ij}^1, and one considers

$$\mathcal{E}_1^T(: x_{D_1} :, \ldots, : x_{D_p} :; s_1, \ldots, s_p) \tag{A3.13}$$

where $\mathcal{E}_1$ means expectation (i.e. integration) with respect to the z variables at fixed y.

Let $\mathcal{I}$ denote now the set of the graphs obtained by joining pairs of points of different clusters as before but now allowing that some points stay disconnected from the others provided the set of lines joining the points still makes the set of clusters connected (if each of them is regarded as connected), let $: x_\pi :$ denote $| \prod_\alpha^* x_\alpha :$ where the product is over the points which in $\pi \in \mathcal{I}$ are left unconnected with other points. Then

$$\begin{aligned} &\mathcal{E}_1^T(: x_{D_1} :, \ldots, : x_{D_p} :; s_1, \ldots, s_p) = \\ &= \sum_{\pi \in \mathcal{I}} : y_\pi : \sum_{\substack{\tau \in \pi \\ \tau \in \mathcal{I}}} (\prod_{\lambda \in \tau} C_{\alpha\beta}^1)(\prod_{\lambda \notin \pi/\tau} C_{\alpha\beta}^0) \end{aligned} \tag{A3.14}$$

where the second sum runs over the subgraphs of π which are still elements of $\mathcal{I}$ (i.e. which still form a graph connected modulo the identification of the points in each cluster).

One first checks that (A3.14) is an immediate consequence of (A3.12) by writing $x_i = y_i + z_i$ and developing the sums using the Leibnitz rule: actually it is convenient to note from the beginning that (A3.14) is true in general if it is true for $s_1 = s_2 = \ldots = s_p = 1$. This is seen by using the identity valid for the truncated expectations:

$$\begin{aligned} &\mathcal{E}^T(x_1, \ldots, x_1, \ldots, x_p, \ldots, x_p; 1, 1, \ldots, 1) = \\ &\mathcal{E}^T(: x_{D_1} :, \ldots, : x_{D_p} :; s_1, \ldots, s_p) \end{aligned} \tag{A3.15}$$

if x_j is repeated s_j times in the *l.h.s.* of (A3.15). Then one checks that (A3.14) follows from (A3.12) by developing the summations mentioned above in the case $s_1 = \ldots = s_p = 1$.

Finally one checks (A3.12) as a consequence of another remarkable formula (in the case $s_1 = \ldots = s_p = 1$, which is not restrictive as noted above):

$$\begin{aligned}&\mathcal{E}^T(: e^{\omega_1 x_1} :, \ldots, e^{i\omega_p x_p} :; 1, 1, \ldots, 1) = \\ &= \sum_{\pi\in\mathcal{I}^*} \sum_{\substack{\lambda\in\tau \\ \lambda=(\alpha,\beta)}} (e^{\omega_\alpha\omega_\beta C_{\alpha\beta}} - 1)\end{aligned} \tag{A3.16}$$

where $\mathcal{I}^*$ is the set of graphs with lines joining p points and forming a connected set in which there are never two lines joining the same pair of vertices. Below the notation $\lambda = (\alpha, \beta)$ is used to identify a line with its end points; and one also defines

$$: e^{\omega x} := \sum_{p=0}^{\infty} \frac{\omega^p}{p!} : x^k := e^{-\frac{1}{2}\omega^2 C} e^{\omega x} \tag{A3.17}$$

where the latter equality follows from (A3.2) and (A3.3).

More generally, if $x_i = y_i + z_i$ and y, z are independent with covariances C^0, C^1, respectively,

$$\begin{aligned}&\mathcal{E}_1^T(: e^{\omega_1 x_1} :, \ldots, e^{i\omega_p x_p} :; 1, 1, \ldots, 1) = \\ &(\prod_{j=1}^{p} : e^{\omega_i y_i} :) \sum_{\pi\in\mathcal{I}^*} \prod_{\lambda\in\pi} (e^{\omega_\alpha\omega_\beta C^1_{\alpha\beta}} - 1)\end{aligned} \tag{A3.18}$$

as a consequence of (A3.16) and, see (A3.17), of

$$: e^{\alpha x} :=: e^{\alpha y} :: e^{\alpha z} : \tag{A3.19}$$

Equation (A3.12) follows from (A3.16) by expanding both sides in powers of ω and identifying equal powers of ω. Therefor the only formula that one must prove is (A3.16). One possible proof of (A3.16) can be given as follows. Consider

$$Z = \int e^{\sum_{j=1}^{p} \lambda_j : e^{i\omega_j x_j} :} P(d\mathbf{x}) \tag{A3.20}$$

where $P(d\mathbf{x})$ is the Gaussian distribution of $\mathbf{x}$ and $\lambda_j > 0$ $\omega_j \in \mathbb{R}$ and $i = \sqrt{-1}$. Then expanding in powers of λ one finds:

$$\begin{aligned}Z &= \sum_{n=0}^{\infty} \int \frac{(\sum_{j=1}^{p} \lambda_j : e^{i\omega_j x_j} :)^n}{n!} P(d\mathbf{x}) = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot \\ &\cdot \sum_{n_1+\ldots+n_p=n} \frac{n!}{n_1! \cdots n_p!} \lambda_1^{n_1} \ldots \lambda_p^{n_p} \cdot \\ &\mathcal{E}((: e^{i\omega_1 x_1} :)^{n_1} \ldots (: e^{i\omega_p x_p} :)^{n_p}) = \\ &= \sum_{n_1,\ldots,n_p} \frac{\lambda_1^{n_1} \ldots \lambda_p^{n_p}}{n_1! \cdots n_p!} \int \mathcal{E}(e^{i\sum_j \omega_j x_j}) \cdot \\ &\cdot e^{\frac{1}{2}\sum_j n_j \omega_j^2 C_{jj}} P d(\mathbf{x}) = \\ &= \sum_{n_j} (\prod_{j=1}^{p} \frac{\lambda_j^{n_j}}{n_j!} e^{-\frac{\omega_j^2}{2}(n_j^2 - n_j)}) \, e^{-\sum_{i<j} n_i n_j \omega_i \omega_j C_{ij}}.\end{aligned} \tag{A3.21}$$

Therefore one has to study

$$\begin{aligned}&\frac{\partial^p}{\partial\lambda_1 \ldots \partial\lambda_p} \log Z = \\ &= \mathcal{E}^T(: e^{i\omega_1 x_1} : \ldots : e^{i\omega_p x_p} :, 1, \ldots, 1)\end{aligned} \tag{A3.22}$$

and one realizes that, for this purpose, one can replace Z by Z' defined as by Z' defined as

$$\begin{aligned}&\sum_{n_j=0,1} \prod_j \text{-}j^{n_j} e^{-\sum_{i<j} \omega_i \omega_j n_i n_j C_{ij}} = \\ &= \sum_{X\subset\{1,\ldots,p\}} (\prod_{\xi\in X} \lambda_\xi) \, e^{-\sum_{(\xi,\eta)\in X} C_{\xi\eta}\omega_\xi\omega_\eta},\end{aligned} \tag{A3.23}$$

where the last sum is over the pairs (ξ, η) in $X = (x_1, \ldots, x_p)$; this fact follows from the last expression of (A3.21) (because $n_j^2 - n_j = 0$ if $n_j = 0, 1$).

One realizes that Z' defined by (A3.23) is the grand canonical partition function for a system of particles with variable activity λ_ξ sitting on a finite set $(1, 2, \ldots, p)$ and interacting with a pair potential $C_\xi\eta\, \omega_\xi\omega_\eta$.

The theory of the Mayer expansion teaches that the logarithm of Z' can be expanded in a series of the activities and the coefficients of this series are well known and can be obtained via a graphical algorithm: the coefficient of $\lambda_1 \cdots \lambda_p$ (which in any event is easy to compute independently of the theory of the Mayer expansion) is precisely

$$\sum_{\pi\in\mathcal{I}^*} \prod_{\lambda\in\pi} (e^{-\omega_\alpha\omega_\beta C_{\alpha\beta}} - 1) \tag{A3.24}$$

which proves (A3.16) replacing ω_α by $i\omega_\alpha$ (the imaginary unit has been introduces in (A3.20) to avoid convergence problems in the definition of Z as an integral).

Appendix A4. Proof of (16.14)

One has to show that

$$\begin{aligned}I(\gamma_0) &= \int_{\Lambda^n} \prod_\lambda e^{-\frac{\kappa}{4}\gamma^{h_\lambda}|\lambda|} d\xi_1 \ldots d\xi_n \le \\ &\le |\Lambda| B_1^{n-1} \prod_{v>r} \gamma^{-d\, h_v\, (s_v - 1)}, \; n > 1\end{aligned} \tag{A4.1}$$

which is clearly equivalent to (16.14). Here one imagines to have fixed a tree (with no decorations or frames but just frequency indices $\mathbf{n}$ and position labels $\xi_1, \ldots, \xi_n$ at the end points). The vertices v of he tree organize the end points into a hierarchy of clusters $\boldsymbol{\xi}_v$.

The lines λ are drawn so that the ones which join pairs $\xi, \eta \in \boldsymbol{\xi}_v$ which are not both located inside any smaller clusters (i.e. imagining that the points inside the smaller clusters are connected): for such lines λ it is $h_\lambda = h_v =$ frequency index of the vertex.

Define $I(\gamma_0) = |\Lambda|$ if $n = 1$,]ie if the tree is trivial. Assume that the tree γ_0 has root frequency k and has a first nontrivial vertex v_0 where it bifurcates into s subtrees $\gamma_1, \ldots, \gamma_s$, $s > 1$.

Clearly in proving (A4.1) it is not restrictive to suppose that the lines connecting the clusters $\boldsymbol{\xi}_{v_1}, \ldots, \boldsymbol{\xi}_{v_s}$ associated with the vertices immediately following v_0 in γ_0 do the connection in a simply connected way; otherwise one just deletes the extra factors in (A4.1).

Once this is supposed it is clear that one can perform the integrals in (A4.1) by keeping first all the points in $\boldsymbol{\xi}_{v_2} \ldots, \boldsymbol{\xi}_{v_s}$ fixed and the positions of the points in the first cluster fixed relative to the point ξ which is linked by a lineλ to the other clusters; here one is supposing that $\boldsymbol{\xi}_{v_1}$ is one of the (at least two) clusters connected to only one other cluster (which is no loss of generality).

The result of the integration, followed by the integration over the remaining coordinates, yields the inequality

$$
\begin{aligned}
I(\gamma_0) = &\Big(\int e^{-\frac{\kappa}{4}\gamma^h |p|} dp \Big) I(\gamma') \frac{I(\gamma_1)}{|\Lambda|} \le \\
&\le B_1 \frac{I(\gamma_1) I(\gamma')}{|\Lambda|} \gamma^{-d\,h}
\end{aligned}
\tag{A4.2}
$$

where γ' is the tree obtained from γ_0 by deleting the branch γ_1 and $B_1 \gamma^{-d\,h}$ is a Λ-independent bound on the integral in (A4.2); in deducing (A4.2) the translation invariance of the problem has been used; furthermore the above inequality holds even if one of the trees in the *r.h.s.* is trivial provided one defines, as above, $I = |\Lambda|$ for the trivial tree. Hence by iteration

$$
I(\gamma_0) \le B_1^{s-1} \frac{I(\gamma_1) \ldots I(\gamma_s)}{|\Lambda|^{s-1}}
\tag{A4.3}
$$

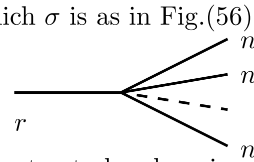
Obviously (A4.3) implies (A4.1) for γ_0 if (A4.1) is supposed valid for $\gamma_1, \ldots, \gamma_s$; hence the theorem follows by induction being true, by definition, for $n = 1$: note that here the relation used several times $\sum_{v' > v} (s_{v'} - 1) = n_v - 1$, is useful, see (12.17).

Appendix A5. Proof of (19.8)

Given a tree shape σ without any frames and with $m \ge 2$ final lines each carrying an index n_j, so that $\sum_j n_j = n$, $n_j >$, consider the sum

$$
\sum_{\mathbf{h}} \Big(\prod_{v > r} \gamma^{-\overline{\rho}\,(h_v - h_v)} \Big) \Big(\prod_{j=1}^{m} \big((n_j - 1)! \sum_{p=0}^{n_j - 1} \frac{(b\,h_j)^p}{p!} \big) \Big)
\tag{A5.1}
$$

where m is the degree of σ and $\mathbf{h}$ is a frequency assignment to the vertices of σ with root frequency k. Consider first the case in which σ is as in Fig.(56):



(56)

In this case one has to study, changing for convenience of notation n_j to $n_j + 1$,

$$
\begin{aligned}
&\sum_{h > k} \gamma^{-\overline{\rho}(h-k)} \prod_{j=1}^{m} \big(n_j! \sum_{p=0}^{n_j} \frac{(b\,h)^p}{p!} \big) = \\
&= \sum_{t=1}^{\infty} \gamma^{-\overline{\rho}t} \prod_{j=1}^{m} \big(n_j! \sum_{p=0}^{n_j} \frac{(b\,(t+k))^p}{p!} \big) = \\
&= \sum_{t=1}^{\infty} \gamma^{-\overline{\rho}t} \big(\prod_{j=1}^{m} n_j! \big) \sum_{j_1, \ldots, j_m} b^{j_1 + \ldots + j_m} \frac{(\sum j_i)!}{\prod j_i!} \cdot \\
&\cdot \sum_{r=0}^{j_1 + \ldots + j_m} \frac{t^{j_1 + \ldots + j_m - r} k^r}{r! (j_1 + \ldots + j_m - r)!} \le \\
&\big(\prod_{j=1}^{m} n_j! \big) \sum_{j_1, \ldots, j_m}^{n_1, \ldots, n_m} b^{j_1 + \ldots + j_m} \frac{(\sum j_i)!}{\prod j_i!} \cdot \\
&\cdot \sum_{t=1}^{\infty} \sum_{r=0}^{j_1 + \ldots + j_m} \frac{\gamma^{-\overline{\rho}t}\, t^{j_1 + \ldots + j_m - r} k^r}{r! (j_1 + \ldots + j_m - r)!}
\end{aligned}
\tag{A5.2}
$$

and for all $\theta > 0$ the *r.h.s.* of (A5.2) is bounded by

$$
\begin{aligned}
&\le \big(\prod_{j=1}^{m} n_j! \big) \sum_{j_1, \ldots, j_m}^{n_1, \ldots, n_m} \big(\frac{b}{\theta} \big)^{j_1 + \ldots + j_m} \frac{(\sum j_i)!}{\prod j_i!} \cdot \\
&\cdot \sum_{t=1}^{\infty} \gamma^{-\overline{\rho}\,t} e^{\theta t} \sum_{r=0}^{j_1 + \ldots + j_m} \frac{(\theta k)^r}{|} r!
\end{aligned}
\tag{A5.3}
$$

so that if $\gamma^{-\overline{\rho}} e^{\theta} < 1$ and $b < \theta$,

$$
\begin{aligned}
&Eq.(A5.3) \le \gamma^{-\overline{\rho}} e^{\theta} \frac{(\sum_i n_i)!}{1 - \gamma^{-\overline{\rho}} e^{\theta}} \sum_{r=0}^{n_1 + \ldots + n_m} \frac{(\theta k)^r}{r!} \cdot \\
&\cdot \frac{\prod_i n_i!}{(\sum_i n_i)!} \sum_{q \ge r} \big(\frac{b}{\theta} \big)^q \Big(\sum_{\substack{j1, \ldots, j_m \\ \sum j_i = q}}^{n_1, \ldots, n_m} \frac{(\sum j_i)!}{\prod j_i|} \Big) \\
&\le \frac{(\sum n_i)! \gamma^{-\overline{\rho}} e^{\theta}}{1 - \gamma^{-\overline{\rho}} e^{\theta}} \sum_{r=0}^{n_1 + \ldots + n_m} \frac{(\theta k)^r}{r!} \big(\frac{b}{\theta} \big)^r \frac{1}{1 - b\theta^{-1}} \le \\
&\le \frac{\gamma^{-\overline{\rho}} e^{\theta}}{1 - \gamma^{-\overline{\rho}} e^{\theta}} \frac{(\sum n_i)!}{1 - b\theta^{-1}} \sum_{r=0}^{n_1 + \ldots + n_m} \frac{(b\,k)^r}{r!}
\end{aligned}
\tag{A5.4}
$$

where we have used the inequality (to be proved by induction)

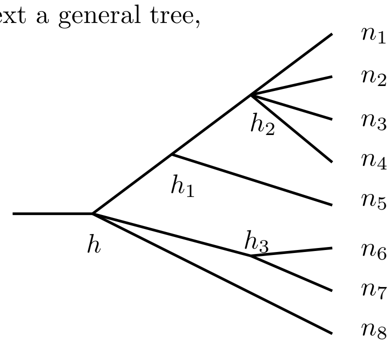
$$\frac{n_1!\cdots n_m!}{(n_1+\ldots+n_m)!}\sum_{\substack{j_1,\ldots,j_m=0\\ j_1+\cdots+j_m=q}}^{n_1,\ldots,n_m}\frac{q!}{j_1!\cdots j_m!}\le 1 \qquad (A5.5)$$

$\forall n_i, q$. Finally let

$$D_6 \stackrel{def}{=} \frac{\gamma^{-\overline{\rho}}e^{\theta}}{1-\gamma^{-\overline{\rho}}e^{\theta}}\frac{1}{1-b\theta^{-1}} \qquad (A5.6)$$

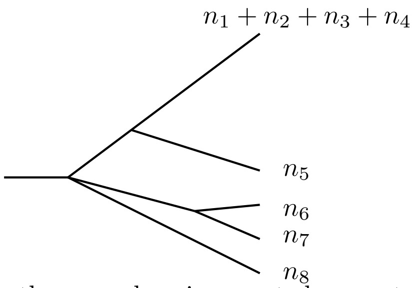
and (9.8) follows with $b=\frac{\theta}{2}$ suitably chosen (e.g. $b=\frac{\overline{\rho}}{4}\log\gamma$) and with D_6 replacing D_6^m, which is correct in the special case just considered.

Consider next a general tree,



(57)

Using recursively the bound found in the case of Fig.(56), one reduces the problem of estimating (19.8) to the problem of a similar estimate for a simpler tree. In fact summing over h_v, where v is one f the highest vertices of the tree, and if $m_v \ge 2$ is the number of branches emerging out of v, we find the estimate (19.8) to be reduced to that relative to the tree σ' without the vertex v and with the line $v'v$ joining v to the preceding vertex v' being a final line bearing an index $\sum_i n_i$, where the sum is over the end points indices (of σ) of the end points linked to v by a final branch of σ. For instance in the case of Fig.(57) one gets, if v is taken to be the vertex with frequency h_2, the tree in Fig.(58):



(58)

Every time the procedure is repeated one gets a factor D_6 and an expression similar to the one to be bounded but for a simpler tree. Since in $m-1$ steps at most one reaches the trivial tree (19.8) is proved.

Appendix A6. Estimate of the number of Feynman graphs compatible with a tree

This appendix is due to Giovanni Felder, Zürich, who proves the following

Lemma: Let G be an unlabeled Feynman graph with n vertices and let]g be a tree with n end points. Then the number $N(G,\gamma,\{n_v^e\}_{v\in\gamma})$ of labelings of G compatible with]g and such that for all vertices v the subgraph of G corresponding toMv has n_v^e external lines, is bounded above by $C_v^n n(\sigma)e^{\varepsilon\sum_v n_v^e}$, for all $\varepsilon>0$ and some constant C_ε, if σ is the shape of the tree γ.

Proof: Consider $G,\gamma,\{n_v^e\}_{v\in\gamma}$ fixed. Let γ_v be the subtree of γ with root v, and $N_v(j)$ the number of ways of choosing and labeling a subgraph of G compatible with γ_v and having an external line connected to the vertex j of G. Furthermore, let $v_1,\ldots,v_{s_v}$ be the vertices following v in γ. Since the subgraphs $G_{v_1},\ldots,G_{v_{s_v}}$ corresponding to $v_1,\ldots,v_{s_v}$ have to be connected together, there exists at least one tree diagram T_v with vertices $v_1,\ldots,v_{s_v}$ whose lines correspond to propagators connecting $G_{v_1},\ldots,G_{v_{s_v}}$. Let d_{v_i} be the number of lines of T_v emerging from v_i. We have the estimate

$$N_v(j)\le\Big(\prod_{i=1}^{s_v}\max_{j'\in G}N_{v_i(j')}\Big)\cdot \qquad (A6.1)$$

$$\cdot\sum_{\substack{d_{v_1},\ldots d_{v_{s_v}}\ge 1\\ \sum_i(d_{v_i}-1)=s_v-2}}\Big(\prod_{i=1}^{s_v}(n_{v_i}^e)^{d_{v_i}-1}\Big)\frac{s_v\,(s_v-2)!}{\prod_{i=1}^{s_v}(d_{v_i}-1)!},$$

where the last ration is the Cayley formula for the number of roted trees T_v with fixed coordination numbers (see J.W. Moon, *Enumerating labelled graphs*, in *Graph theory and theoretical Physics*, edited by F.Harari, Academic Press, 1967) and $\prod_{i=1}^{s_v}(n_{v_i}^e)^{d_{v_i}-1}$ is a bound on the number of ways of choosing external lines of G_{v_i} corresponding to the lines of T_v. The sum over d_{v_i} can be performed explicitly:

$$N_v(j)\le\Big(\prod_{i=1}^{s_v}\max_{j'\in G}N_{v_i(j')}\Big)\,s_v\,(\sum_i n_{v_i}^e)^{s_v-2}, \qquad (A6.2)$$

and using $x^u\le k!\varepsilon^{-k}e^{\varepsilon k}$, $\sum_v(s_v-1)=n-1$, we get

$$N(\gamma,G,\{n_v^e\}_{v\in\gamma})\le\overline{C}_\varepsilon\,(\prod_{v\in\gamma}s_v|)e^{\varepsilon\sum_{v\in\gamma}n_v^e} \qquad (22.3)$$

But $\frac{\prod_{v\in G}s_v!}{n(\sigma)}=\prod_{v\in\gamma}\frac{s_v!}{\prod_i t_{i,v}!}$ (where $t_{i,v}$ are the multiplicities of the different tree shapes that start from v) is just the number of ways of drawing the shape σ by choosing at each vertex how t order the trees starting from it: this number is bounded by the number of ways of drawing all the trees with n end points, which, by the same argument used to count the trees, is bounded by C^n for soome constant C.

Appendix A7. Applications to the hierarchical model

A very simple and particularly interesting example of field theory is the $]f^4$ hhierarchical model.

Tis model is defined by an interaction like (5.6), i.e. "pure φ^4". but with a different interpretation of the free field $\varphi^{(\le N)}$ and with d being an integer ≤ 4.

In this appendix I discuss the minor changes necessary to treat this new case; in fact it will be a useful exercise for the reader to check the statements made below, while reading various oparts of this paper.

To define the free fields $\varphi^{(\le N)}$ with cutoffs at scales γ^{-N} one introduces a sequence $\mathcal{Q}_0, \mathcal{Q}_1, \ldots$ of compatible pavements of the unit cube Λ: the pavement Q_j is built with cubes of side size γ^{-j}, where $\gamma > 1$ is the "scale parameter" (an integer).

Each point ξ is in one cube $\Delta \in \mathcal{Q}_j$, for $j = 0, 1, \ldots$. with the obvious (and trivial) exception of the points in the boundaries of the cubes. Then one defines

$$\varphi_\xi^{(\le k)} = \sum_{j=1}^{k} \gamma^{\frac{d-2}{2} j} z_{\Delta_j} \equiv \sum_{\substack{\Delta \ni \xi \\ |\Delta| \ge \gamma^{-h\,d}}} |\Delta|^{\frac{d-2}{2d}} z_\Delta \qquad (A7.1)$$

where Δ_j is the cube in $\mathcal{Q}_j$ containing ξ and the ζ_Δ are Gaussian independent variables with covariance $\frac{1}{2}$ except forone among them, z_{Δ_0} corresponding to $\Delta \equiv \Lambda \in \mathcal{Q}_0$, which will be assumed to have covariance $\frac{1}{2(1-\gamma^{-(d-2)})}$

The fields $\varphi^{(\le N)}$ behave roughly as the Euclidean free field with cutoff at γ^{-N}.

Hierarchical models in field theory were introduced in the papers of Wilson, (Wilson, 1971, 1972)[3] and (Wilson and Kogut, 1973), as approximations to the Euclidean theory and the equations to which they lead are, therefore, called "approximate recursion formulae". In statistical mechanics they were introduced by (Dyson, 1969, 1971) and studied also by (Bleher and Sinai, 1973, 1975) and by (Collet and Eckmann, 1978).

The mode (A7.1) is not the one studied in the above--mentioned papers; its relevance and importance for field theory were pointed out in (Gallavotti, 1978, 1979a,b), and it was applies to constructive field theory for Euclidean fields in (Benfatto *et al.*, 1978, 1980a). It is mentioned earlier in (Wilson and Kogut, 1973), p.120, line 11, see (A7.3) below for comparison, without comments except perhaps the implication that it may be not too relevant. Many of the results that follow would apply as well to the hierarchical models considered in the above-mentioned papers after some obvious changes; for papers on such "classical" hierarchical models see (Gawedski and Kupiainen, 1980) and the references therein.

The theory of the φ^4 fields with interaction given by (5.6) for $d = 4$ can be pursued exactly as in Secs. 16-19 with a few remarkable simplifications: the results, and the simplifications just mentioned , are listed below. The reader who has followed Secs.5-9 and 16-20 will find them very easy to prove; their proof is however very instructive, as it shows the true nature of the problems of perturbative field theory deprived of most technical complications which accompany them.

(1) Classifying the divergences leads to the same results of Sec.16, provided one sets everywhere $m_{2',v} = 0$, $n_{1,v} = 0$, thus disregarding the $\partial\varphi$ fields (which are not defined in this model and which are absent from the interaction).

(2) The renormalization is also done along the same lines. It is, however, much easier in practice because the effective interaction (very peculiarly for this model) remains "purely local" on each scale: i.e. the effective potential on scale k has the form

$$\begin{aligned} V^{(\le k)} &= \sum_{n=1}^{\infty} \omega(k,n) \int_\Lambda :\varphi^{(\le k)\,2n}:\, dx \stackrel{def}{=} \\ &\stackrel{def}{=} \sum_{\Delta \in \mathcal{Q}_k} \Omega_k(X_\Delta) \end{aligned} \qquad (A7.2)$$

where $X_\Delta \stackrel{def}{=} \frac{\varphi_x^{(\le k)}}{2(\mathcal{E}\varphi_x^{(\le k)\,2})^{\frac{1}{2}}}$ if $x \in \Delta$, and in the second step use has been made of the fact that $\varphi_x^{(\le k)}$ is condtant over boxes Δ of side size γ^{-k} The function $\Omega_k(x)$ and the coefficients $\omega(k,n)$ are defined implicitly by (A7.2). The normalized field is introduced for convenience.

(3) In fact one can see, indpendently of perturbation theory, that the functions $|O_k$ are related by a recursion formula; namely it is $\Omega_k = T\Omega_{k+1}$, $k > 0$, where T is, (Gallavotti, 1979b), for $d > 2$

$$(T\Omega)(x) = \gamma^d \log \int e^{\Omega(\alpha z + \beta x)} e^{-z^2} \frac{dz}{\sqrt{\pi}} \qquad (A7.3)$$

with $\alpha = (1-\beta^2)^{\frac{1}{2}}, \beta = \gamma^{-\frac{d-2}{2}}$. *The case* $d = 2$ *is slightly different* (and easier) *because* $d - 2 = 0$ *and the (A7.3) needs to be reinterpreted appropriately, (Gallavotti, 1978, 1979a): for this reason we do not discuss it.*

Therefore $\Omega_k = T^{N-k}\Omega_N$ if $k > 0$, and a simple calculation shows that the interaction $-\int_\Lambda (\lambda_N : \varphi_x^{(\le N)\,4} : +\mu_N \varphi_x^{(\le N)\,2} + \nu_N) dx$ can be written as (A7.2) with

[3] See in particular footnote 8. This paper introduces a hierarchical model and deals mainly with φ_3^6; other similar hierarchical models had been introduced earlier in (Dyson, 1969, 1971) and later in (Baker, 1973) in statistical mechnaics and in (Gallavotti, 1978) in field theory. A general theory of the recursion relations associated with certain hierarchical models can be found in (Collet and Eckmann, 1978), where the work initiated by (Bleher and Sinai, 1973, 1975) is extended

$$\Omega_N(x) = -\,(\lambda_N \gamma^{(4-d)N} C^4 H_4(x) + \\ + \mu_N \gamma^{-2N} C^2 H_2(x) + \nu_N \gamma^{-dN}), \tag{A7.4}$$

where $C = (1-\gamma^{-\frac{d-2}{2}}$. The reason why $k=0$ is special is that z_Δ, for $\Delta \in \mathcal{Q}_)$, has a slightly different covariance. If z_Δ, $\Delta \in \mathcal{Q}_0$ had been taken with ovariance $\frac{1}{2}$, too, then (α,β) in (A7.3) would, however, have turned out slightly k dependent.
(4) Because of remark (2) the $\mathcal{L}$ and R operations of Sec.18 need only to be defined for $1, :\varphi_1^2:, :\varphi_1^4:$ and are simply

$$\begin{aligned} &\mathcal{L}1 = 1,\ \mathcal{L}: \varphi_1^2 :=: \varphi_1^2:,\ \mathcal{L}: \varphi_1^4 :=: \varphi_1^4 : \\ &\mathcal{L}: \varphi^n := 0 \text{ if } n > 4 \\ &R1 = 0,\ R: \varphi^2 := 0,\ R': \varphi^4 := 0 \\ &R: \varphi_1^n :=: \varphi_1^n : \text{ if } n > 4 \end{aligned} \tag{A7.5}$$

No D_{xy} fields arise: because of the locality remark (2) above x would be equal to y so that $D_{xy} = 0$. Since the D_{xy} fields vanish there is no need to increase the order of subtraction, because D_{xy} "has clearly a zero of infinite order".

Therefore the above theory is renormalizable, in spite of the absence of $(\partial\varphi)^2$ terms in the interaction. This is a proof of a theorem by Wilson, see (Wilson, 1972) line 26 from bottom at p. 424, who proved this result (just by stating it)in φ_3^6 theory (and hence in φ_4^4 theory also, the argument being the same in the two cases). It seems that this deep result went almost unnoticed, probably because he failed to stress its interest, very high in my opinion. The difference between the models used here is irrelevant and the above proof can be repeated verbatim for the "classical" hierarchical models.
(5) Finally consider the resummations. The equation for the form factors, see Fig.(32), and formulae (9.9) and (20.8) can be written in terms of the "beta functional"

$$(\mathcal{B}\boldsymbol{\lambda})^{(\alpha)}(k) = \sum_\sigma \sum_{\mathbf{h},\, h_r,\, \ge k+1} \beta_\sigma^{(\alpha)}(k+1; \mathbf{h}, \boldsymbol{\alpha}')\cdot \\ \cdot \prod_{\text{end point of } \sigma} \lambda^{(\alpha_i')}(h_i), \tag{A7.6}$$

where we have explicitly exhibited the decomposition of the]b-coefficients of (20.8) in terms of the contributions from the various trees. Then as the reader can easily check, the bounds on β are, if $v_0 =$ first vertex of σ,

$$|\beta_\sigma^{(\alpha)}(k+1; \mathbf{h}, \boldsymbol{\alpha}')| \le C_0^n n! \gamma^{-\sum_{v>v_0} \widehat{\rho}} \\ \prod_{v>v_0} \gamma^{\widehat{\rho}(h_v - h_{v'} - 1)} \tag{A7.7}$$

with the usual notations and with $C_0 > 0$, $\widehat{\rho} = -d + (6-\varepsilon)\frac{d-2}{2}$, $\varepsilon > 0$. The 6 in $\widehat{\rho}$ is explained by the fact that the vertices of σ carrying a superscript R generate (because of (A7.5)) only Feynman graphs with at least six external lines (in fact the R operation just deletes the contributions from nontrivial Feynman graphs with two or four external lines emerging out of clusters generated by the vertices of the tree σ).

Expression (A7.7) suggests that the hierarchical model may have a "γ^{-1} expansion" for the beta function $\mathcal{B}$ in (A7.6). One sees that the *r.h.s.* of (A7.7) is of order $O(1)$ as $\gamma \to \infty$ only for the trees which have only one nontrivial vertex v_0, which we can call "simple" trees. For the other trees the bound (A7.7) contains terms of order $O(\gamma^{-\widetilde{\rho}})$.

Therefore it might be of some interest to analyze the equation for the dimensionless form factors (A7.6) in the approximation in which only "simple" trees are considered in the *r.h.s.* of (A7.6). This approximation is *not* equivalent to taking an order by order dominant term in the γ^{-1} expansion of the β_σ for the σ of given order n, because even for simple trees σ the β_σ depend on γ and have subleading corrections in γ^{-1}.

Hence the approximation has the same character of the "leading log" or "most divergent graphs" resummations discussed in Secs.9 and 20. However it in some sense to be clarified (one hopes) a deeper resummation as, unlike the case of the "most divergent graphs" or the "planar graphs" resummations, its beta function has an asymptotic expansion which has zero-radius of convergence: the contribution from the tree with n end points being proportional to $n!$, see (A7.7) and (A7.8) below.

But the really interesting aspect of the above resummations is that the beta function can be computed "exactly". In fact from the graphical interpretation of (5.13) and (5.14) in terms of "simple" trees one can easily recognize that the contribution of the simple trees to the *r.h.s.* of (A7.6) is just the power series expansion $\mathcal{B}_g(\boldsymbol{\lambda}(k+1))$ in formal powers of $\boldsymbol{\lambda}(k+1)$ and $B_g^{(\alpha)}(\lambda^{(4)}, \lambda^{(2)})$ is

$$B_g^\theta(\lambda^{(4)}, \lambda^{(2)}) = \gamma^d \frac{2^\theta}{\theta!} C^{-\theta} \int H_\theta(x) e^{-x^2} \frac{dx}{\sqrt{\pi}}\cdot \\ \cdot \Big(\int e^{-z^2} \frac{dz}{\sqrt{\pi}} e^{\sum_{\theta'} \lambda^{|\theta'|} C^{\theta'} H_{\theta'}(\alpha z + \beta x)}\Big) \tag{A7.8}$$

where $\theta = 2,4$, $C = (1-\gamma^{\frac{d-2}{2}})^{-\frac{1}{2}}$, $\alpha = (1-\nu^2)^{\frac{1}{2}}$, $\beta = \gamma^{-\frac{d-2}{2}}$.

Thus if we set$\lambda = -\lambda^{(4)}(k+1)\, C^4$, $\mu = -\lambda^{(2)}(k+1)\, C^2$, $\lambda' = -\lambda^{(4)}(k) C^4$, $\mu' = -\lambda^{(2)}(k)\, C^2$, it follows that the dimensionless form factors of scale k are expressed in terms of this on scale $k+1$ simply by

$$\lambda' = -\gamma^d \frac{2^4}{4!} \int H_4(x) e^{-x^2} \frac{dx}{\sqrt{\pi}} \Big(\log \int e^{-z^2} \frac{dz}{\sqrt{\pi}} \cdot$$
$$\cdot\, e^{-\lambda H_4(\alpha z + \beta x) - \mu H_2(\alpha z + \beta x)} \Big)$$
$$\mu' = -\gamma^d \frac{2^2}{2!} \int H_2(x) e^{-x^2} \frac{dx}{\sqrt{\pi}} \Big(\log \int e^{-z^2} \frac{dz}{\sqrt{\pi}} \cdot$$
$$\cdot\, e^{-\lambda H_4(\alpha z + \beta x) - \mu H_2(\alpha z + \beta x)} \Big) \tag{A7.9}$$

This formula (which I derived, with some algebraic errors later corrected by Nicoló, from the remark that (5.13) and (5.14) imply that $\mathcal{B}_g$ can be summed explicitly) gives a recursion relation somewhat interesting in itself. But more interesting would be to see in what sense (if at all) the above resummation provides a good resummation rule up to $O(\gamma^{-1})$. Such problems have not been investigated yet.

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Piscataway, NJ, July 2004.