1: Reservoirs and entropy creation

$$m_i \ddot{\mathbf{x}}_i = -\partial_{\mathbf{x}_i} V(\mathbf{X}) + \mathbf{F}_i(\mathbf{X};\boldsymbol{\Phi}) - \boldsymbol{\vartheta}_i, \qquad i = 1, \ldots, n$$

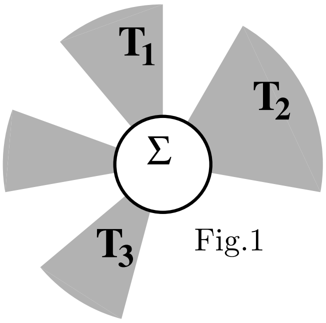


Fig.1

Reservoirs occupy infinite regions *e.g.* sectors $C_a \subset R^3$, $a = 1, 2 \ldots$. Their particles are in a configuration typical of an equilibrium state at temperature T_a. The *empirical* probability of configurations in each C_a is Gibbsian with some temperature T_a.

$$\underset{\mathbf{r}+\Lambda\subset C_a}{\text{average}}\,(f_{\Lambda+\mathbf{r}}[(\dot{\mathbf{Y}}, \mathbf{Y}+\mathbf{r}); \dot{\mathbf{W}}, \mathbf{W}+\mathbf{r}]) = \frac{e^{-\beta_a\left(\frac{1}{2m_a}|\dot{\mathbf{Y}}|^2 + V_a(\mathbf{Y}|\mathbf{W})\right)}}{\text{normalization}}$$

In finite thermostats temperature is enforced by a constraint, *e.g.* Gaussian

$$K_a = \mathit{constant} = \frac{3}{2} N_a k_B T_a \equiv \frac{3}{2} N_a \beta_a^{-1}$$

Amount of heat $\dot{Q}$ produced while in a stationary state identified with the work that the thermostat forces $\boldsymbol{\vartheta}$ perform per unit time

$$\dot{Q} = \sum_i \boldsymbol{\vartheta}_i \cdot \dot{\mathbf{x}}_i$$

Often $\boldsymbol{\vartheta} = \sum_{a=1}^{m} \boldsymbol{\vartheta}^{(a)}(\dot{\mathbf{X}}, \mathbf{X}) \;\Rightarrow\; \sigma^{(a)}(\dot{\mathbf{X}}, \mathbf{X}) \stackrel{def}{=} \sum_j \partial_{\dot{\mathbf{x}}_j} \cdot \boldsymbol{\vartheta}_j^{(a)}(\dot{\mathbf{X}}, \mathbf{X})$ then

$\sigma_+^{(a)} \stackrel{def}{=} \langle \sigma^{(a)}(\dot{\mathbf{X}}, \mathbf{X}) \rangle, \qquad \dot{Q}_a \stackrel{def}{=} \sum_i \boldsymbol{\vartheta}_i^{(a)} \cdot \dot{\mathbf{x}}_i$
Temperature *defined* by $T_a = \frac{\langle \dot{Q}_a \rangle}{k_B \sigma_+^{(a)}}$. Is it > 0?

A class of thermostats with $T_a > 0$

N particles in C_0 interact via a potential $V_0 = \sum_{i<j} \varphi(\mathbf{q}_i - \mathbf{q}_j) + \sum_j V'(\mathbf{q}_j)$ (V' models external conservative forces like obstacles, walls, gravity, ...) and interact with M systems Σ_a, of N_a particles of mass m_a, in containers C_a contiguous to C_0: "the M parts of the system in contact with thermostats at T_a, $a = 1, \ldots, M$".

$$m\,\ddot{\mathbf{q}}_j = -\partial_{\mathbf{q}_j}\big(V_0(\mathbf{Q}) + \sum_{a=1}^{N_a} W_a(\mathbf{Q}, \mathbf{x}^a)\big) \text{ ("conservative")}$$

$$m_a\,\ddot{\mathbf{x}}^a_j = -\partial_{\mathbf{x}^a_j}\big(V_a(\mathbf{x}^a) + W_a(\mathbf{Q}, \mathbf{x}^a)\big) - \boldsymbol{\vartheta}^a_j \text{ "(non conservative)"}$$

$\boldsymbol{\vartheta}^a$ via Gauss' principle: $\Rightarrow \boldsymbol{\vartheta}^a_j = \frac{L_a - \dot{V}_a}{3N_a k_B T_a}\, \dot{\mathbf{x}}^a_j \stackrel{def}{=} \alpha^a\, \dot{\mathbf{x}}^a_j$ where L_a is the work per unit time done by the particles in C_0 on the particles of Σ_a and V_a is their potential. Partial divergence $\sigma^a = 3N_a\alpha^a = \frac{L_a}{k_B T_a} - \frac{\dot{V}_a}{k_B T_a} \Rightarrow T_a > 0$ because L_a can be naturally interpreted as *heat* Q_a ceded, per unit time, by the particles in C_0 to the subsystem Σ_a (hence to the a-th thermostat because the temperature of Σ_a is constant), while the derivative of V_a *will not contribute to the value of* σ^a_+. *Apart from the total derivative terms*, ("true" $\longleftrightarrow$ "up to an additive total derivative")

$$\sigma_{true}(\dot{\mathbf{X}}, \mathbf{X}) = \sum_{a=1}^{N_a} \frac{\dot{Q}_a}{k_B T_a}$$

$\to$ interpretation of σ as *entropy creation rate.*
Another viewpoint: the system only consist of N particles in C_0 and the Σ_a are thermostats $\Rightarrow$ model of system subject to thermostats.

This is a conservative system interacting with thermostats. *Instead of the divergence* interesting is $\sigma_{true} \Rightarrow$ general "entropy creation" in a subsystem $\equiv$ amounts of work done on the external particles divided by the temperatures.

Note that σ_{true} **will satisfy FR**: the large deviations of $p \stackrel{def}{=} \frac{1}{T}\int_0^T \frac{\sigma_{true}(t)}{\langle\sigma_{true}\rangle_{SRB}}$ have rate function

$$\zeta(-p) = \zeta(p) - p\sigma_+$$

2. How irreversibile is an irreversible trasformation?

Let $E(t)$ be a parameter varying from E_0 to $E_\infty = E + 0 + \Delta E$.
System initially in SRB state μ_0 and equations of motion

$$\dot{\mathbf{X}} = \mathbf{F}_{E(t)}(\mathbf{X}, \dot{\mathbf{X}})$$

(thermostatted system under variable forcing).
Let μ_t be the distribution into which μ_0 evolves. Let $\mu_{E(t)}$ be the SRB distribution corresponding to a "frozen" value $E(t)$. The quantity (τ=time scale of $E(t)$)

$$I = \tau \int_0^\infty \left(\langle \sigma_t \rangle_{\mu_t} - \langle \sigma_t \rangle_{\mu_{E(t)}} \right)^2 dt$$

can be regarded a quantitative indicator of irreversibility degree. If $E(t) = E_0 + (1 - e^{-\gamma\kappa t})\Delta E$ then $I(\gamma) \xrightarrow[\gamma\to 0]{} 0$: *quasi static evolution* ($\tau = (\gamma\kappa)^{-1}$) does not create entropy and has 0 "irreversibility".

3. Navier Stokes: equivalence and barometric formula

Application to **NS** (incompressible $\partial\cdot\mathbf{u}=0$)

$$\dot{\mathbf{u}}+\underset{\sim}{\mathbf{u}}\cdot\underset{\sim}{\partial}\,\mathbf{u}=\nu\Delta\mathbf{u}-\partial\,p+f\mathbf{g},\qquad R=\frac{\sqrt{fL}}{\nu}$$

Actually think of : cut off at $|\mathbf{k}|\le K_k=L^1R^{\frac34}$, $N\simeq R^{\frac94}$, *i.e.* OK41 is assumed.
To apply the chaotic hyp. need
(1) *chaos* (yes, if R large).
(2) *reversibility* (no)
(3) *pairing* (mechanism to recover reversibility when the attractor is very small)

(1) Equivalence with reversible equations "Gaussian NS eq."

$$\dot{\mathbf{u}}+\underset{\sim}{\mathbf{u}}\cdot\underset{\sim}{\partial}\,\mathbf{u}=\alpha(\mathbf{u})\Delta\mathbf{u}-\partial\,p+f\mathbf{g},\qquad \alpha=\frac{\int\mathbf{u}\cdot f\,\mathbf{g}}{\int(\partial\,\mathbf{u})^2}\ \Rightarrow\ \int\mathbf{u}^2=\mathcal{E}=const$$

Same statistics for "local observables": F local $\Rightarrow$ F depends on finitely many Fourier components of $\mathbf{u}$. **Same statistics** as $R\to\infty$ if $\mathcal{E}$ is chosen $=\langle\int\mathbf{u}^2\rangle_{\mu_\nu}$ (equivalence)
Consequence $\langle\alpha\rangle/\nu\to1$: **only numerical tests in strongly cut off equations and** $d=2$ (*Rondoni, Segre*).

Earlier *She, Jackson*: large numerical simulations (different reversible equation)

Other tests: are Lyapunov spectra also identical? (Rondoni, Segre, G.). Here are a few graphs in highly truncated equations ($d = 2$)
Also the linear FR relation comes out within the precision: the approximate pairing that can be observed leads to test the slope $(1 - \frac{2M}{2N})\sigma_+$ in the GNS equations: from the theory it is expected a slope $< \sigma_+$ by the ratio of the number of negative pairs to the nuber of total pairs.

Barometric formula:
Consider the equations (incompressible NS and ED)

$$\dot{\mathbf{u}} + \underset{\sim}{\mathbf{u}} \cdot \underset{\sim}{\partial}\mathbf{u} = \nu\Delta\mathbf{u} - \partial\, p + f\mathbf{g}, \qquad \dot{\mathbf{u}} + \underset{\sim}{\mathbf{u}} \cdot \underset{\sim}{\partial}\mathbf{u} = -\chi\mathbf{u} - \partial\, p + f\mathbf{g},$$

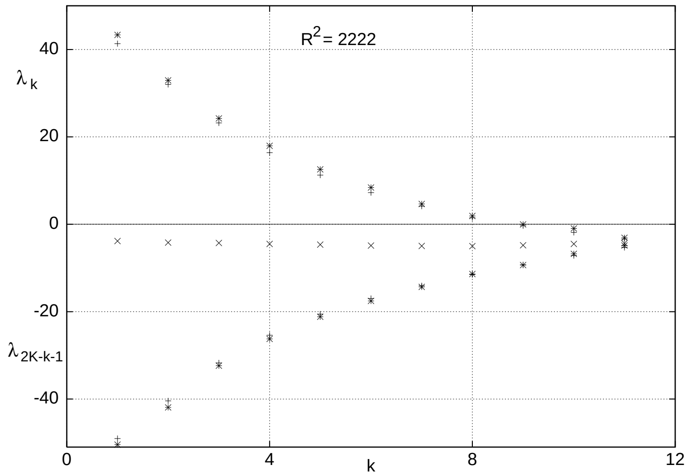
here $\mathbf{u} = \sum_{\mathbf{k}} \boldsymbol{\gamma}_{\mathbf{k}}\, e^{i\mathbf{k}\cdot\mathbf{x}}$, $\mathcal{E} = L^3 \sum_{\mathbf{k}} |\boldsymbol{\gamma}_{\mathbf{k}}|^2$.
The equivalence idea leads to think that although the statistics of the two equations are certainly different *nevertheless they might coincide on an appropriate scale.* The friction in NS varies with the scale $\mathbf{k}$ and at some scale it might match that of ED.
By OK41 $v_k^3 k = \mathit{constant} = \eta\nu$ in NS: OK41 does not hold for ED: to fix ideas asume that at fixed cut off k_χ there is *equipartition* between the modes. Then $\langle|\boldsymbol{\gamma}_{\mathbf{k}}|^2\rangle \equiv \gamma^2$

$$\frac{4\pi}{3}\gamma^2\,(k_\chi\frac{L}{2\pi})^3 = \varepsilon, \qquad \textit{energy density at equipartition}$$

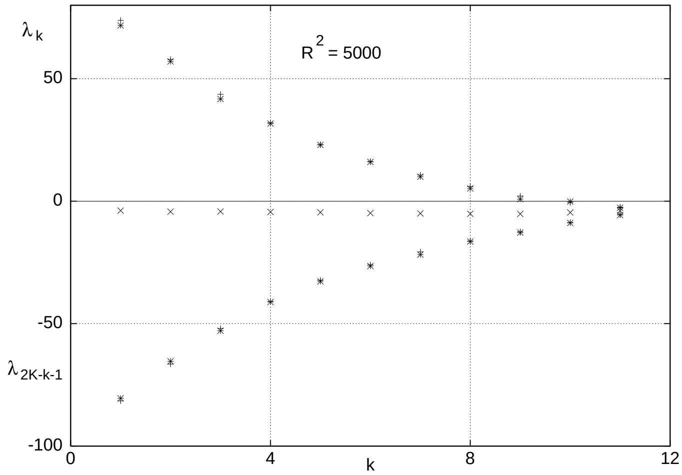
$$K_E(k) = \frac{3\varepsilon}{4\pi}\frac{k^2}{k_\chi^3}, \qquad \textit{energy density between } k \textit{ and } k + dk$$

$$v_k^3\, k \;=\; \big((kL)^3\gamma^2\big)^{\frac{3}{2}} k = \varepsilon^{\frac{3}{2}} k_\chi\,\big(\frac{k}{k_\chi}\big)^{\frac{11}{2}}, \qquad \textit{dimensionless dissipation on scale } k$$

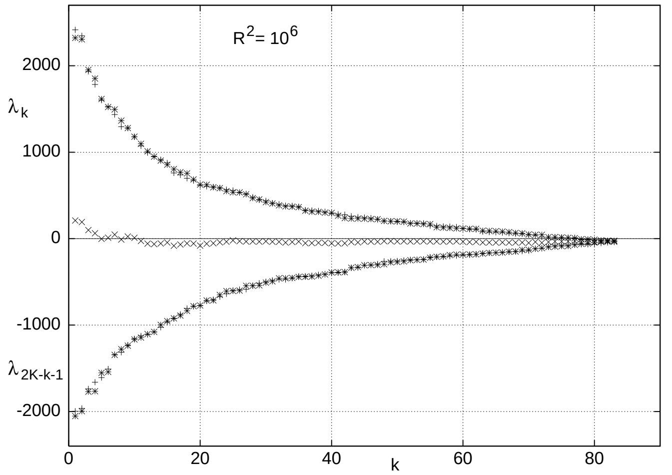
If $\varepsilon^{\frac{3}{2}} k_\chi\,\big(\frac{k}{k_\chi}\big)^{\frac{11}{2}} = \eta\nu$ then NS equation and ED equations have the same statistics on scale k: doubling the dissipation in NS the statistics of the two equations agree on scale 1.3 higher.
The k, or better $\log\frac{k}{k_\chi}$, is the analog of the height and the dim.less dissipation is the analogue of the pressure.



R2= 2222
λk
λ2K-k-1
40
20
0
-20
-40
0
4
8
12
k



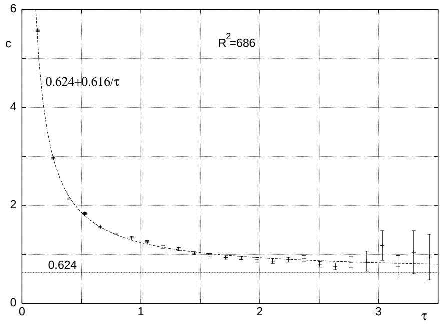
R2 = 5000
λk
λ2K-k-1
50
0
-50
-100
0
4
8
12
k



R2= 106
λk
λ2K-k-1
2000
1000
0
-1000
-2000
0
20
40
60
80
k

R^2	$\delta Q_0/\langle Q_0\rangle_{NS}$	$\triangle\alpha$	$\triangle Q_1$	$o(M)/M$
800	0.005	0.030	0.053	0.068
1250	0.020	0.018	0.062	0.057
2222	0.002	0.039	0.058	0.077
4444	0.050	0.021	0.093	0.059
5000	0.010	0.008	0.058	0.033

Equivalence NS-GNS dynamics at different Reynolds numbers, column $\Delta\alpha$ to be compared with 1, cfr. [RS99]).



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Evolution towards limiting slope as τ increases