# Entropy, Thermostats and Chaotic Hypothesis

**Problem**: establish relations between time averages of a few observables associated with a system of particles subject to work-performing external forces and to thermostat-forces that keep the energy from building up ($\rightarrow$ stationary state)

`mp_arc 06-190; cond-mat/0606690;`
`http://ipparco.roma1.infn.it`

References: listed in the notes

Stationary state = probability distribution $\mu$ on phase space $\mathcal{F}$

$$\frac{1}{\tau}\sum_{j=0}^{\tau-1} F(S^j x) \xrightarrow[\tau\to\infty]{} \int_{\mathcal{F}} F(y)\,\mu(dy) \qquad \text{(map)}$$

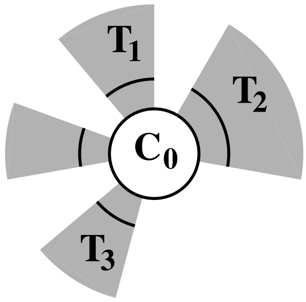
$$\frac{1}{\tau}\int_0^{\tau} F(S_t x)\,dt \xrightarrow[\tau\to\infty]{} \int_{\mathcal{F}} F(y)\,\mu(dy) \qquad \text{(flow)}$$

a.a. $x$. Flow: differential equation on $\mathcal{F}$:

$$\dot{x} = \mathbf{f}_{\mathbf{E}}(x)$$

with $\mathbf{f}_{\mathbf{E}}$ internal, external and thermostats forces. Divergence

$$\sigma(x) = -\sum_j \partial_{x_j} f_{\mathbf{E},j}(x) \neq 0$$




$\mathbf{X}_0$: $N_0$ part. in $\mathcal{C}_0$ $\mathbf{X}_i, \mathbf{X}'_i$: $N_i, N'_i$ part in $\mathcal{C}_i$ and $\mathcal{C}'_i$, $i = 1, \ldots, n$,
Equations of motion are, for $i = 0$ and $i > 0$ respectively,

$$\ddot{\mathbf{X}}_0 = -\partial_{\mathbf{X}_0}\Big(U_0(\mathbf{X}_0) + \sum\nolimits_{i>0} W_{0i}(\mathbf{X}_0, \mathbf{X}_i)\Big) + \mathbf{E}(\mathbf{X}_0)$$
$$\ddot{\mathbf{X}}_i = -\partial_{\mathbf{X}_i}\Big(U_i(\mathbf{X}_i) + W_{0i}(\mathbf{X}_0, \mathbf{X}_i) + W_{i,i'}(\mathbf{X}_i, \mathbf{X}_{i'})\Big)$$
$$\ddot{\mathbf{X}}'_i = -\partial_{\mathbf{X}'_i}\Big(U'_i(\mathbf{X}'_i) + W_{i,i'}(\mathbf{X}_i, \mathbf{X}'_i)\Big) - \alpha_i\,\dot{\mathbf{X}}'_i$$

Thermostat condition ("fixed temperatures")

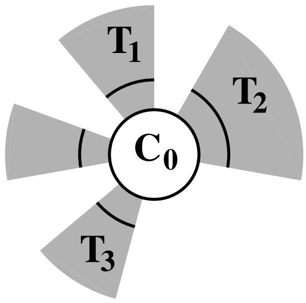
$$\frac{1}{2}\dot{\mathbf{X}}_i'^2 = \frac{3}{2} N_i' k_B T_i \qquad \Rightarrow \qquad \alpha_i = \frac{L_i - \dot{U}_i'}{3\, N_i'\, k_B T_i}$$

$L_i$ = work on $\mathbf{X}_i' \in \mathcal{C}_i'$, *i.e.* on thermostats.

$$L_i = -\dot{\mathbf{X}}_i' \cdot \partial_{\mathbf{X}_i'} W_{i,i'}$$

Other thermostats: structure no influence on statistics in $\mathcal{C}_0$.

E.g. $\mathcal{C}_i'$ could be *infinite* in equilibrium at temperature $T_i$.



Divergence of the equations **easily** is $-\sigma(x)$ with

$$\sigma(x) = \sum_{i>0} \Big(\frac{L_i}{k_B T_i} - \frac{\dot U'_i}{k_B T_i}\Big)\frac{3N'_i - 1}{3N'_i} \stackrel{def}{=} \sum_{i>0} \frac{L_i}{k_B T_i} + \dot\Phi$$

But $L_i = -\dot X'_i \cdot \partial_{X'_i} W_{i,i'} \equiv +\dot X_i \cdot \partial_{X_i} W_{i,i'} - \dot W_{i,i'} = -L'_i - \dot W_{i,i'}$

$$\textit{Energy conservation} \text{ and } L_{0i} \stackrel{def}{=} -\dot{\mathbf{X}}_i \cdot \partial_{\mathbf{X}_i} W_{0i} \to$$

$$L'_i \equiv -L_{0i} + \frac{d}{dt}\Big(\frac12 \dot X_i^2 + U_i\Big) = -Q_i + \dot\Phi'$$

$L_{0i} \equiv Q_i$ work $\mathcal{C}_0$ on $\mathcal{C}_i$: $\equiv$ *heat* from $\mathcal{C}_0 \to \mathcal{C}_i \to$ thermostats $\mathcal{C}'_i$.

$$\varepsilon(x) \stackrel{def}{=} \sum_{i>0} \frac{Q_i}{k_B T_i} \quad \Rightarrow \quad \sigma(x) = \varepsilon(x) + \dot R$$

where $R(x) = -\sum_i \frac{(W_{i,i'} + U'_i + U_i + \frac12 \dot X_i{}^2)}{k_B T_i}$.

$\sigma(x)$ is *entropy creation*: large time same as $\varepsilon(x) = \sum_{i>0} \frac{Q_i}{k_B T_i}$

Difference $\frac{1}{\tau}(R(S_\tau x) - R(x)) \xrightarrow[\tau\to\infty]{}$, IF $R$ is bounded.

Strong assumption.
"Efficiency"; violation interesting, [CvZ02,BGGZ05,Ga06,GG06].

**Chaotic Hypothesis:** *Motions developing on the attracting set of a chaotic system can be regarded as a transitive hyperbolic system.*

General: statistics $\mu$ uniquely determined probability.
*Entropy creation* rate to be

$$\varepsilon_+ = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau \sigma(S_t x)\, dt = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau \varepsilon(S_t x)\, dt$$

Assume *dissipativity*: $\varepsilon_+ > 0$

$$p \stackrel{def}{=} \frac{1}{\tau}\int_0^\tau \frac{\varepsilon(S_t x)}{\varepsilon_+}\,dt$$

"dim.less phase space contraction".

A general property $a = \frac{1}{\tau}\int_0^\tau F(S_t x)\,dt$

$$P_\mu(a\in\Delta) = \exp(\tau \max_{a\in\Delta}\zeta_F(a) + O(1))$$

for $\Delta\subset(a_-,a_+)$, where $\zeta_F(a)$ defined, analytic and convex.
$\zeta_F(a)=-\infty$ naturally for $a\notin[a_-,a_+]$: *large deviations rate*

*Reversible* $\Rightarrow$ isometry $I$ s.t. $IS_t = S_{-t}I$ or $IS = S^{-1}I$.
$F$ odd, *i.e.* $F(Ix) = -F(x)$: fluctuation interval $[-a^*, a^*]$.

Fluctuation $\sigma(x)/\varepsilon_+$ and of $\varepsilon(x)/\varepsilon_+$ is **same** and, [CG95],[Ge98],

**Fluctuation theorem:** $\zeta(-p) = \zeta(p) - p\sigma_+$**,** $|p| < p^*$**.**

By CH FT holds for dimless contraction of any reversible chaotic motion with a dense attractor (call FR)

In particular to the model: where $\varepsilon(x) \stackrel{def}{=} \sum_{a>0} \frac{Q_a}{k_B T_a}$ has a physical meaning and can be measured in experiments even without equations of motion

The quantity $\varepsilon(x)$ is *local*; depends only on the microscopic conf. of $\mathcal{C}_0$ and of the walls $\mathcal{C}_i$. One can also imagine FR remains valid with $\infty$-thermostats

Quite a few tests of CH and FR [BGGZ06] (granular materials).

**Extending Onsager-Machlup's theory and Reciprocity + Green-Kubo**

*Fluctuation patterns*: probability that the successive $F(S_t x)$ follow, for $t \in [-\tau, \tau]$, a preassigned *pattern* $\varphi(t)$, [Ga97].

In a reversible hyperbolic and transitive system consider $n$ observables $F_1, \ldots, F_n$ with $F_j(Ix) = \pm F_j(x)$. Given $n$ functions $\varphi_j(t)$, $t \in [-\frac{\tau}{2}, \frac{\tau}{2}]$:

*probability that $F_j(S_t x) \sim \varphi_j(t)$ for $t \in [-\frac{\tau}{2}, \frac{\tau}{2}]$?* FPT theorem:

**Fluctuation Patterns Theorem:** *Under assumptions of FT given $F_j, \varphi_j$, and given $\varepsilon > 0$ and an interval $\Delta \subset (-p^*, p^*)$ SRB-probability*

$$\frac{P_\mu(|F_j(S_t x) - \varphi_j(t)|_{j=1,\ldots,n} < \varepsilon, p \in \Delta)}{P_\mu(|F_j(S_t x) \mp \varphi_j(-t)|_{j=1,\ldots,n} < \varepsilon, -p \in \Delta)} =$$
$$= \exp(\tau \max_{p \in \Delta} p\, \sigma_+ + O(1))$$

FPT means "all that has to be done to change the time arrow is to change the sign of the entropy production", *i.e.* the *time reversed processes occur with equal likelyhood as the direct processes if conditioned to the opposite entropy creation.*

Application, [Ga96a,Ga96] with $j_j(x) = \partial_{E_j}\sigma(x)$ is *Green–Kubo formulae* hence Onsager recip.

**Jarzinsky, Bonetto and Fluctuation relations**

*Bonetto's forml.*, BF, [Ga00]: FT or FR imply $\langle e^{-\int_0^\tau \varepsilon(S_t x)\,dt}\rangle_{SRB}$ **bounded** as $\tau \to \infty$: *would be exactly* 1 if FR true finite $\tau$.

*Jarzinsky's forml.*, JF: Let $\gamma$ be a function $H(,p,q,t)$ interpolating between $H_0(p,q)$ and $H_1(p,q)$, called a "protocol $\gamma$". Let $S_{0,t}(p,q)$ be the time evolution of $(p,q)$ under the time dependent protocol. Let $(p',q') = S_{0,1}(p,q)$ and $W(p',q') = H_1(S_{0,1}(p,q)) - H_0(p,q)$:

$$\frac{Z_0}{Z_1} e^{-\beta W(p',q')} \frac{e^{-\beta H_0(p,q)}}{Z_0} dpdq \equiv \frac{e^{-\beta H_1(p',q')} dp'dq'}{Z_1}, \ \textit{hence}$$

$$\langle e^{-\beta W}\rangle_{\mu_0} = \frac{Z_1}{Z_0} = e^{-\beta \Delta F(\beta)}$$

$\Delta F$ = free energy variation between equilibria $H_1$ and $H_0$.

JF is *exact*, an instance of the Monte Carlo method. It can be implemented *without actually knowing* neither $H_0$ nor $H_1$ nor the *protocol* $H(p,q,t)$ to evaluate $\Delta F$. *But* BF and JF quite different:

(1) $\int_0^\tau \sigma(S_t x)\,dt$ is entropy creation not energy variation $W$ ($W$ "work done by the machines implementing the protocol").

(2) average is SRB of a stationary state, out of equilibrium, not a canonical equilibrium state.

(3) the BF says that $\langle e^{-\int_0^\tau \varepsilon(S_t x)\,dt}\rangle_{SRB}$ is bounded as $\tau \to \infty$ rather than being 1 exactly.

JF: very useful in equilibrium problems.

In a steady state with entropy at rate $\varepsilon_+$ (*e.g.* **a living organism peacefully feeding on a background**) CH $\to$ FR $\to$ BF gives informations on heat produced: could check that all relevant heat transfers have been properly taken into account.