Divergent series summation in Hamilton Jacobi equation (resonances)

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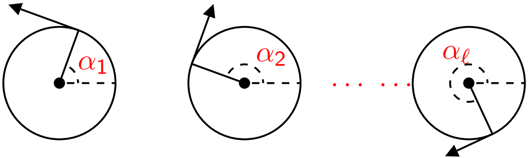
Eq. Motion: $\ddot{\boldsymbol{\alpha}} = -\varepsilon \partial_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha})$



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Representation of phase space in terms of ℓ rotators.

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{T}^\ell$$

$f(\boldsymbol{\alpha})$ analytic or

$$f(\boldsymbol{\alpha}) = \sum_{\boldsymbol{\nu} \in \mathbb{Z}^\ell} f_{\boldsymbol{\nu}}\, e^{i\boldsymbol{\nu}\cdot\boldsymbol{\alpha}}, \quad f_{\boldsymbol{\nu}} \equiv 0 \text{ if } |\boldsymbol{\nu}| > N$$

Equations of motions $\quad \ddot{\boldsymbol{\alpha}} = -\varepsilon\, \partial_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha})$

3

Resonance of order s with frequencies $\boldsymbol{\omega}_0 \in \mathbb{R}^r$ (*unperturbed*)
$\equiv$ motions with rotation $\boldsymbol{\omega} = (\boldsymbol{\omega}_0, \mathbf{0}) \in \mathbb{R}^r \times \mathbb{R}^s, \quad \ell = r + s$

$$|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}| > \frac{1}{C|\boldsymbol{\nu}|^\tau}, \qquad \forall \quad \mathbf{0} \neq \boldsymbol{\nu} \in \mathbb{Z}^r$$

$$\boldsymbol{\alpha} \stackrel{def}{=} (\boldsymbol{\gamma}, \boldsymbol{\beta}) \in \mathbb{T}^r \times \mathbb{T}^s, \quad t \to (\boldsymbol{\gamma} + \boldsymbol{\omega}_0 t, \boldsymbol{\beta})$$

$\boldsymbol{\gamma}=$ "fast angles", $\boldsymbol{\beta}=$"slow angles"

Hamilton-Jacobi: Find $\mathbf{h}(\psi) \stackrel{def}{=} \begin{pmatrix} \mathbf{g}(\psi) \\ \mathbf{k}(\psi) \end{pmatrix}$, $\psi \in \mathbb{T}^r$, $\beta_0 \in \mathbb{T}^s$ with

$$\mathbf{g}(\psi), \mathbf{k}(\psi) \in \mathbb{R}^r \times \mathbb{R}^s$$

so that $\ddot{\alpha} = -\varepsilon \partial_{\alpha} f(\alpha)$, $B\alpha \equiv (\gamma, \beta)$, is solved by

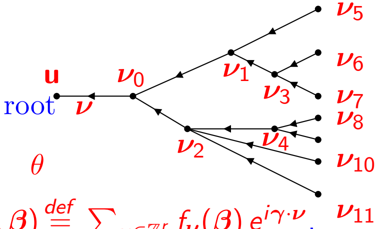
$$\gamma = \psi + \mathbf{g}(\psi), \qquad \beta = \beta_0 + \mathbf{k}(\psi), \qquad \psi \to \psi + \omega_0 t$$

This means

$$(\omega_0 \cdot \partial_\psi)^2 \begin{pmatrix} \mathbf{g}(\psi) \\ \mathbf{k}(\psi) \end{pmatrix} = -\varepsilon\, \partial_{\alpha} f\big(\psi + \mathbf{g}(\psi), \beta_0 + \mathbf{k}(\psi)\big)$$

Resonance $\Rightarrow$ dimensionality drop from ℓ to $r \Rightarrow \partial_{\beta} \overline{f}(\beta_0) = \mathbf{0}$,
Let $\overline{f}(\beta) \stackrel{def}{=} \int \frac{d\gamma}{(2\pi)^r} f(\gamma, \beta)$. **Condition** $\det \partial^2_{\beta\beta} \overline{f}(\beta_0) \neq 0$

Proposition: $\exists$ power series solution *(elementary)*



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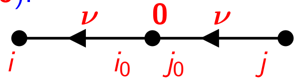
Notation: $f(\boldsymbol{\gamma},\boldsymbol{\beta}) \stackrel{def}{=} \sum_{\boldsymbol{\nu}\in\mathbb{Z}^r} f_{\boldsymbol{\nu}}(\boldsymbol{\beta})\, e^{i\boldsymbol{\gamma}\cdot\boldsymbol{\nu}}$,

$$\partial_j f_\nu(\boldsymbol{\beta}_0) \stackrel{def}{=} i\nu_j f_{\boldsymbol{\nu}}(\beta_0), \qquad \partial_j f_\nu(\boldsymbol{\beta}_0) \stackrel{def}{=} \partial_{\boldsymbol{\beta}_j} f_\nu(\boldsymbol{\beta}_0),$$

$$\partial_J f_{\boldsymbol{\nu}}(\boldsymbol{\beta}_0) \stackrel{def}{=} \partial_{j_0,\ldots,j_p} f_{\boldsymbol{\nu}}(\boldsymbol{\beta}_0), \quad J = (j_0,\ldots,j_p)$$

trivial node v $\stackrel{def}{=}$ "one entering line and $\mathbf{0}$ harmonic label" ($\boldsymbol{\nu}_v = \mathbf{0}$).

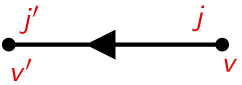


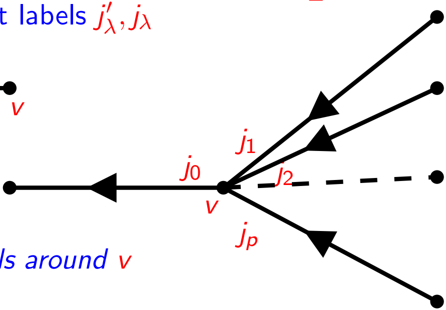
Let $M_{0;i_0j_0} \stackrel{def}{=} \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon\partial^2\overline{f}(\boldsymbol{\beta}_0)\end{pmatrix}$ 5

(1) To node v attach a *harmonic* $\boldsymbol{\nu}_v \in \mathbb{Z}^r$

(2) To line $\lambda \equiv v'v$ attach *current* $\boldsymbol{\nu}(\boldsymbol{\lambda}) = \sum_{w \le v} \boldsymbol{\nu}_w$

(3) and two component labels j'_λ, j_λ





Component labels around v

$v \to J_v = (j_0, \ldots, j_p)$ and $\partial_{J_v} f_{\nu_v}(\boldsymbol{\beta}_0)$ are defined.

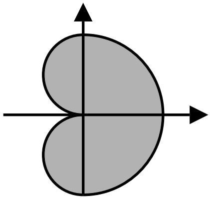
(4) *Value* : $\mathrm{Val}(\theta) = \frac{1}{k!} \int \Big(\prod_v \varepsilon \partial_{J_v} f_{\boldsymbol{\nu}_v}(\beta_0)\Big) \Big(\prod_{lines\,\boldsymbol{\lambda}} g_{j_{\boldsymbol{\lambda}} j'_{\boldsymbol{\lambda}}}\Big)$

$g_{ij} \stackrel{def}{=} \frac{\delta_{ij}}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu}(\boldsymbol{\lambda}))^2}$, or $g_{ij} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (\varepsilon \partial^2_{\boldsymbol{\beta}} \overline{f}(\boldsymbol{\beta}_0))^{-1} \end{pmatrix}$ if $\boldsymbol{\nu}(\boldsymbol{\lambda}) = \mathbf{0}$

$\mathbf{h}_{\boldsymbol{\nu}} \equiv \sum^*_{\boldsymbol{\theta}} \mathrm{Val}(\theta)$: $\ast \longleftrightarrow$ *no trivial node with* $\mathbf{0}$ *incoming current*

Estimate: $|h^{(k)}_{\boldsymbol{\nu}}| \le bB^k \varepsilon^{/2} kk!^{2\tau} \to$!! Results:

Theorem: **The tree series can be rearranged to yield a convergent series representation of h, hence its existence, in**



$\varepsilon > 0$, hyperbolic case
$(\partial^2_{\beta\beta}\overline{f}(\beta_0) < 0)$
analyticity region
common to all $\varepsilon > 0$
estimate $k!^{2\tau}$ at $\varepsilon = 0$

4

$\mathcal{E} \subset (-\varepsilon_0, 0]$ has open dense complement but 0 is a (Lebesgue) density point

ε-plane

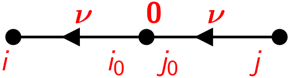
Fig.4: $\varepsilon \in \mathcal{E}$; $\mathcal{E}$ dense at 0. The figure illustrates the analyticity domain $\mathcal{D}(\varepsilon)$ associated with a *single* $\varepsilon \in \mathcal{E}$, represented by the dot, ("elliptic case"). The cusp at the origin is quadratic.

4

Need $k!^2$ for Borel summability ($r = 2$?).

Question: *is there uniqueness* ? Are others' results the same ?
(Delshams, Llave, Zhou $\ell = 3, r = 2$, Treshev $\varepsilon > 0$ only)

Inserting a "trivial" node with $\mathbf{0}$ harmonic ($\Rightarrow \nu \neq \mathbf{0}$)



Let $M_{0;i_0j_0} \stackrel{def}{=} \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon\partial^2\overline{f}(\beta_0) \end{pmatrix}$ 5

get $\frac{\delta_{ij}}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \to \frac{\delta_{ii_0}}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\left(M_{0;i_0j_0}\frac{\delta_{j_0j}}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\right) \Rightarrow$ propagator modification

$$M_{0;i_0j_0} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \varepsilon\partial^2_{i_0j_0} f_{\mathbf{0}}(\beta_0) \end{pmatrix}, \qquad f_{\mathbf{0}}(\beta_0) \equiv \overline{f}(\beta_0)$$

Can form *chains* of trivial nodes (large values $k!^{2\tau}$)

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$$\frac{\delta_{ij}}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \;\to\; \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\Big(M_0\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\Big)^k_{ij}$$

.

"Simplify": NO trivial nodes; price :

$$\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \Rightarrow \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\sum_{k=0}^{\infty}\Big(M_0\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}\Big)^k \equiv \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2 - M_0}$$

BUT $z = M_0\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} < 1$? NO **!**

so we are using $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$, $z \neq 1$, *e.g.*

$$\sum_{k=0}^{\infty} 2^k = 1 + 2 + 4 + 8 + 16 + \cdots = -\mathbf{1}$$

dont mind !

$\Rightarrow$ *Eventually*: **Check that the $\mathbf{h}$ is solution must be done**

If accepted $\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \Rightarrow \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \sum_{k=0}^{\infty} \left(M_0 \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2} \right)^k \equiv \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2 - M_0}$

$M_0 = \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon \partial_{\boldsymbol{\beta}}^2 \overline{f}(\boldsymbol{\beta}_0) \end{pmatrix}$ gives $\varepsilon > 0$ “easier” than $\varepsilon < 0$.

For $\varepsilon < 0$ expect to exclude ε s.t $|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| = \pm\sqrt{-\varepsilon \mu_j}$.

PROBLEM: there are LOTS of other chains ! They cause values $k!^{\eta}$, $\eta > 0$

IDEA: eliminate them "by resummation": left with convergent series

KEY: Siegel's theorem

Given a tree θ let $\mathcal{N}_n$ be the number of lines of scale n: i.e. s.t.

$$2^{-n} < C|\boldsymbol{\omega}\cdot\boldsymbol{\nu}| \le 2^{-n+1},\ n \le -1, \qquad 1 < C|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|,\ n = 0$$

$n = 0, 1, \ldots$. *IF no pair lines $\lambda' < \lambda$ with $\boldsymbol{\nu}(\lambda') = \nu(\lambda)$ with only lower scale intermediates THEN*

$$\mathcal{N}_n \le 4N2^{-n/\tau}\, k$$

$$\mathcal{N}_n \le 4N 2^{-n/\tau}\, k \Rightarrow$$

$$\Rightarrow \prod_v \frac{1}{(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu})^2} \le C^{2k} \prod_{n=0}^{\infty} 2^{2n\mathcal{N}_n} \le C^{2k} \Big(\prod_{n=0}^{\infty} 2^{2n(4N2^{-n/\tau})} \Big)^k$$

Trivial bound ($\varepsilon > 0$):

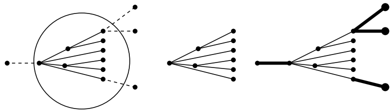
$$\prod_v |\partial_{J^v} f_{\boldsymbol{\nu}_v}(\beta_0)| \le \prod_v N^{|J^v|} F^k \le N^{2k} F^k$$

number of harmonics $\boldsymbol{\nu}$: $\le (2N+1)^k$, number of trees $\le k^{k-1}$
Convergence:

$$|\varepsilon| < \big(N^2 \cdot C^2 \cdot (2N+1)^{\ell} \cdot 3 \cdot F \cdot 2^{8N \sum_n n 2^{-n/\tau}} \big)^{-1}$$

Multiscale analysis: Organize the lines of θ into *clusters*

Definition: A cluster of scale n is a maximal connected set of lines of θ with scales $p \le n$ and with one line at least of scale n.



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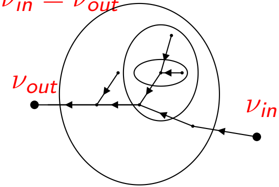
Self energy clusters $\nu_{in} = \nu_{out}$



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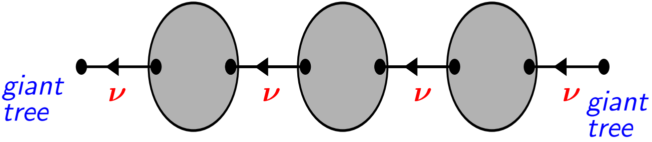
Eliminate self energy clusters by resummations

Necessary multiscale analysis to avoid "overlapping divergences"
First identify the self energy clusters of scale [0]
i.e. with $x \stackrel{def}{=} C\omega\cdot\nu$ with $1 \le x^2$ and resum all chains



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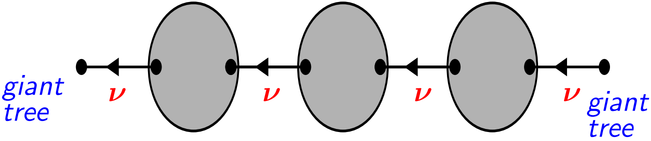
Key remark: *each s.e. cluster does not contain s.e. clusters*

Summing over the contents of each s.e. cluster $\Rightarrow$ convergent sum by Siegel's lemma.

16 Summing over s.e. clusters of scale [0] leads to modify propagators of lines on scale [≥ 1]



$$g^{[\geq 1]}(x) = \frac{1}{x^2 - M_0} \to \frac{1}{x^2 - M_0} \sum_{n=0}^{\infty} (M_1 \frac{1}{x^2 - M_0})^n$$

which becomes

$$\frac{1}{x^2 - M_0 - M_1}$$

and graphs simplify with no s.e. subgraphs of scale [0].

Iterate! at every step only graphs with no s.e. subgraphs have to be considered; $\Rightarrow$ convergent additions made on propagators

In the hyperbolic case no real new problems arise.

In the elliptic the situation is very different.
As the scale decreases the scale $2^{-n} \simeq \varepsilon$ is reached and $x^2 - M_0 - M_1 - \ldots - M_{n-1}$ can vanish $\Rightarrow$

(a) More values of ε excluded

(b)The successive scales must be measured by the size of $x^2 - M_0 - M_1 - \ldots$; analysis becomes delicate:

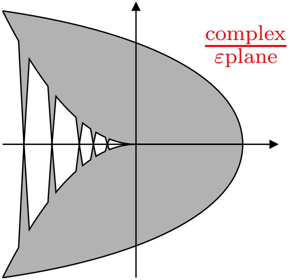
DIFFICULTY: even in the hyperbolic case it is necessary to check that the renormalized propagators have the same size as the bare ones

Siegel's lemma applies only to graphs in which the propagators size is of the order of $x^2 \equiv (\boldsymbol{\omega} \cdot \boldsymbol{\nu})^{-2}$.

Not automatic but checked via the cancellation mechanism of the KAM theory: this time the cancellations are only partial but still enough.

OPEN PROBLEM:

uniqueness (and relation with alternative existence results)



O. Costin, G. Gentile, A. Giuliani, GG
For $r = 2$ possible $\tau = 1$. In this case Borel summability for $\varepsilon > 0$.
Uniqueness, *within the framework of multiscale analysis*, is proved.
F admits a Taylor exp. at 0 $F(\eta) \sim \sum_{k=2}^{\infty} F_k \eta^k$ then

$$F_B(p) \sim \sum_{k=2}^{\infty} F_k \frac{p^{k-1}}{(k-1)!}.$$

If series for F converges (*i.e.* analytic at 0) then

$$F(\eta) = \int_0^{+\infty} e^{-p/\eta} F_B(p) \, \mathrm{d}p$$

Definition: *$\eta \to F(\eta)$ be (i) analytic in a disk tangent at 0 with admits asymptotic Taylor series at 0*
(ii) $F_B(p)$ real analytic for $p > 0$, and growing at most exp. as $p \to +\infty$,
(iii) it can be expressed, for $\eta > 0$ small enough, as

$$F(\eta) = \int_0^{+\infty} e^{-p/\eta} F_B(p)\, \mathrm{d}p.$$

Theorem: *If $\eta = \sqrt{\varepsilon}$, $\mathbf{h}(\psi)$ is Borel summable in η for $\varepsilon > 0$. The $\mathbf{h}$ constructed is the unique possible within the class of Borel summable functions.*
BUT this does not exclude existence of other less regular quasi–periodic motions.

Regularize $\mathbf{h}(\psi)$: $\mathbf{h}^{(N)}$ defined as $\mathbf{h}$ with sum to trees with only lines of scale $n \le N$.

BT of $\mathbf{h}^{(N)}(\psi,\eta)$ is entire function that can be written for p real and positive as

$$(\mathbf{h}^{(N)})_B(\psi,p) = \mathcal{L}^{-1}\mathbf{h}^{(N)}(\psi,p) = \int_{\eta_N^{-1}-i\infty}^{\eta_N^{-1}+i\infty} \mathrm{e}^{pz}\,\mathbf{h}^{(N)}(\psi,\frac{1}{z})\frac{\mathrm{d}z}{2\pi i},$$

where η_N is the convergence radius of $\mathbf{h}^{(N)}(\psi,\eta)$.

Key: $\frac{1}{\omega_0\cdot\nu^2-\varepsilon M_0}$ is the *paradigm* of a Borel summable function.

Inductively: $\mathbf{h}^{(N)}(\psi,\frac{1}{z})$ is analytic for $|z| > 2\eta_0^{-1}$.

Integral can be shifted to a contour on the vertical line with abscissa $\overline{\rho} > 2\eta_0^{-1}$, *N-independent*.

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Precise formulation in HJ (*fixed* ω (C,τ)-*Diophant.*)

$$F(\mathbf{A}',\boldsymbol{\alpha}) \stackrel{def}{=} \frac{1}{2}\big(\mathbf{A}' + \partial_{\boldsymbol{\alpha}}\Phi(\mathbf{A}',\boldsymbol{\alpha})\big)^2 + \varepsilon f(\boldsymbol{\alpha})$$

$\exists\ \mathbf{A}'_n \xrightarrow[n\to\infty]{} \mathbf{A}'_\infty$ and ρ_n, ξ such that in $S_{\rho_n}(\mathbf{A}'_n) \times (\mathbb{T}^\ell)_\xi$

$$\Phi_n(\mathbf{A}'_n,\boldsymbol{\alpha}) \Rightarrow \begin{cases} \partial_{\boldsymbol{\alpha}}\Phi_n \xrightarrow[n\to\infty]{} \widetilde{H}(\boldsymbol{\alpha}),\ \partial_{\mathbf{A}'}\Phi_n \xrightarrow[n\to\infty]{} \widetilde{\mathbf{h}}(\boldsymbol{\alpha}) \\ \partial^2_{\boldsymbol{\alpha}}\Phi_n \xrightarrow[n\to\infty]{} \widetilde{H}'(\boldsymbol{\alpha}),\ \partial^2_{\boldsymbol{\alpha},\mathbf{A}'}\Phi_n \xrightarrow[n\to\infty]{} \widetilde{H}''(\boldsymbol{\alpha}) \end{cases}$$

$$\Rightarrow \begin{cases} \frac{1}{2}(\mathbf{A}'_\infty + \widetilde{\mathbf{H}}(\boldsymbol{\alpha}))^2 + \varepsilon\, f(\boldsymbol{\alpha}) = E = \boldsymbol{\alpha} - \mathrm{indep.} \\ \partial_{\mathbf{A}'} F(\mathbf{A}'_n,\boldsymbol{\alpha}) \xrightarrow[n\to\infty]{} \boldsymbol{\omega} \\ \boldsymbol{\psi} = \boldsymbol{\alpha} + \widetilde{h}(\boldsymbol{\alpha}) \longleftrightarrow \boldsymbol{\alpha} = \boldsymbol{\psi} + \mathbf{h}(\boldsymbol{\psi}) \\ \boldsymbol{\psi}(t) = \boldsymbol{\psi} + \boldsymbol{\omega}\, t \text{ is solution} \end{cases}$$