Fluctuations and Symmetries in Dynamical Systems

The time reversal symmetry in classical systems is simply velocity reversal

$$I(\mathbf{q}_n, \mathbf{p}_n) = (\mathbf{q}_n, -\mathbf{p}_n)$$

It is an isometry which anticommmutes with evolution: if $x \to x(t) = S_t x$ is the map solution of the equations of motion then

$$I S_t x = S_{-t} I x$$

Its consequences have been studied from many aspects.
Consider in general a dynamical system described by a differential equation

$$\dot{x} = f(x), \qquad x \in M$$

where M is a smooth bounded manifold.
In applications to nonequilibrium statistical mechanics motion, in many models, is not volume preserving because it is not Hamiltonian although time reversal is still preserved as an isometric symmetry $iS_t = S_{-t} i$

It has been remarked that time reversal puts some constraints on fluctuations in systems that evolve towards non equilibrium starting from an equilibrium state μ_0. Namely if

$$\sigma(x) = -divergence = -\partial \cdot f(x), \qquad \dot{x} = f_0(x) + Eg(x)$$

with $\dot{x} = f_0(x)$ a volume preserving evolution then asking the question

which is the probability that in time t the volume contracts by the amount $A = \int_0^t \sigma(S_t x)dt$, compared to that of the opposite event $-A$?

answer :

$\mathcal{E}_A$ = set of points with contraction A;

at time t becomes $S_t\mathcal{E}_A$ with $\mu_0(S_t\mathcal{E}_A) = e^{-A}\mu_0(\mathcal{E}_A)$, by definition.

Then $\mathcal{E}_A^- \stackrel{def}{=} iS_t\mathcal{E}_A$ is the set of points which contract by $-A$

$$e^{-\int_0^\tau \sigma(S_\tau iS_t x)d\tau} \equiv e^{-\int_0^t \sigma(S_\tau S_{-t} ix)d\tau} \equiv e^{-\int_0^t \sigma(iS_{-\tau} S_t x)d\tau}$$

$$\equiv e^{+\int_0^\tau \sigma(iS_{t-\tau} x)d\tau} \equiv e^A$$

In other words the set $\mathcal{E}_A$ of points which contract by A in time t becomes the set of points whose time reversed images is the set $\mathcal{E}_A^- \stackrel{def}{=} iS_t\mathcal{E}_A$ which contract by A. The measures of such sets are $\mu_0(\mathcal{E}_A)$ and $\mu_0(iS_t\mathcal{E}_A) \equiv \mu_0(\mathcal{E}_A)e^{-A} \equiv \mu_0(\mathcal{E}_A^-)$

$$\frac{\mu_0(\mathcal{E}_A)}{\mu_0(\mathcal{E}_A^-)} \equiv e^A$$

for any A (as long as it is “possible”. (Evans-Searles 994).

This has been called “transient fluctuation theorem”. It is extremely general and does not depend on any chaoticity assumption. Just reversibility and time reversal symmetry.

A similar result is “Jarzinsky relation”: this deals with a “protocol” i.e. a fixed procedure to act on a system that is initially in a Gibbs equilibrium μ_0 with Hamiltonian H_0 so that during the protocol the evolution is governed by a Hamiltonian H_t which at the end of the process is H_T.

Select initial data x with distribution μ_0 and follow evolution up to time T: sampling many times the initial data $x = (p, q)$ get many final data $S_T x = (p', q')$.

Study their statistics (i.e. $\frac{1}{Z_0} e^{-H_0(S_{-T}x)} dx$).

Then if $W(p', q') \stackrel{def}{=} (H_T(p', q') - H_0(p, q)$,

$$\frac{1}{Z_T} e^{-\beta H_T(p',q')} dp' dq' \equiv \frac{Z_0}{Z_T} e^{-\beta W(p',q')} \frac{1}{Z_0} e^{-\beta H_0(p,q)} dp' dq'$$

This is an instance of Monte Carlo method: in particular it yields

$$e^{-\beta(F_T - F_0)} = \langle\, e^{-\beta W} \,\rangle_{\mu_0}$$

The similarity is that also in this case the results are properties of the equilibrium states and requiire “no assumptions”.

What about the stationary states out of equilibrium?

In this case stronger assumptions are needed $\Rightarrow$ strong chaos.

For instance (*discrete time case:* $t = n$) one can assume that the system is smooth hyperbolic and transitive

This means that points in phase space can be coded into sequences of symbols $\underline{\gamma} = (\gamma_i)_{i=-\infty}^{\infty}$ adapted to the dymamics

$$x \longleftrightarrow (\gamma_i)_{i=-\infty}^{\infty} \;\equiv\; Sx \longleftrightarrow (\gamma_{i+1})_{i=-\infty}^{\infty}$$

"Markovian": there is a "compatibility matrix" $M_{a,b} = 0, 1$ such that the codes $\underline{\gamma}$ of points $\longleftrightarrow$ $M_{\gamma_i,\gamma_{i+1}} \equiv 1$

Smoothness $\Rightarrow$ invariant distribution giving the statistics of motions starting with random data chosen with a distribution $\rho(x)dx$ (any ρ), called
SRB distribution,
is a short range Gibbs state over the space of symbols; transitivity $\Rightarrow$ it is unique.

Furthermore time reversal is $\gamma_k \to \gamma_{-k}$ and the Gibbs state has an interaction potential related to the expansion rates

Precisely this means that it is given formally by the energy function

$$H(\underline{\gamma}) \equiv H((\gamma_i)_{-\infty}^{\infty}) = \sum_k \Lambda_+(\tau^k \gamma)$$

just like in the Ising model the Gibbs state is defined by a formal energy

$$H = \sum_k J(\tau^k \gamma), \quad J(\gamma) = \gamma_0 \gamma_1 \qquad \Rightarrow \qquad H = \sum_k g_k \gamma_{k+1}$$

and $e^{\Lambda_+(\gamma)}$ is the expansion of a surface element in the expanding plane through the point x corresponding to γ.
Finally

$$e^{-\sigma(x)} = e^{\Lambda_+(\gamma)+\Lambda_-(\gamma)}$$

Then the following fluctuation theorem can be proved as a consequence of the Gibbsian nature of the SRB distribution μ_{SRB}

If $\sigma_+ \stackrel{def}{=} \langle\, \sigma \,\rangle_{SRB} > 0$, the system is time reversible, hyperbolic, transitive

$$p(x) \stackrel{def}{=} \frac{1}{T} \int_0^T \frac{\sigma(S_t x)}{\sigma_+} dt$$

then ($p^* \geq 1$) (Cohen, G discrete time, Gentile continuous time)

$$\frac{P_{SRB}(p(x) \sim p)}{P_{SRB}(p(x) \sim -p)} \xrightarrow[T\to\infty]{} e^{Tp\sigma_+}, \qquad |p| \leq p^*$$

Setting $A \stackrel{def}{=} p(x)\sigma_+ T$ this becomes

$$\frac{P_{SRB}(A)}{P_{SRB}(-A)} = e^A$$

looks similar to transient fluctuation th.: but very different because it deals with the SRB distribution (which is singular) rather than with (non singular) equilibrium distributions.

There are many simulations done to test the FR: however I know of no experiment testing it. All experiments I know of test the transient fluctuation.

Of course it makes no sense to make an experiment to test a theorem.

But assumptions for theorem are too strong; doubt hey are fulfilled in any concrete application other that a mathematical.

The Chaotic hypothesis (Cohen, G) says: a system empirically chaotic can be regarded as transitive hyperbolic: therefore if reversibility if fulfilled the phase space contraction satisfies a Fluctuation relation: this is not a theorem but a physical principle and therefore it should be tested.

Chaotic hypothesis (CH) *Motions developing on the attracting set for map S representing the evolution of a chaotic system of particles, observed in discrete time via a choice of timing events* Σ, *may be regarded as motions of transitive hyperbolic system.*

The FT is interesting because it is related to other results

1) If the motion is $\dot{x} = f_0(x) + \mathbf{E}\cdot\mathbf{g}(x)$, reversible transitive hyperboic then $\mathbf{J}(x) \stackrel{def}{=} \partial_{\mathbf{E}}\sigma(x)$ then FT implies

$$\partial_{E_i}\langle\, J_{\mathbf{E}}\,\rangle|_{\mathbf{E}=0} \stackrel{def}{=} L_{ij} = L_{ji} = \frac{1}{2}\int_{-\infty}^{\infty}\langle\, J_i(S_t x) J_j(x)\,\rangle_0 dt$$

that can be interpreted as *Onsager reciprocity* and *Green-Kubo formula*.

(2) **Fluctuation Patt2erns:** *Under the FT assumptions, given*
a) F_j *= observable*
b) pattern: $\varphi_j(t), \tau \in [-\tau, \tau]$
joint SRB probabilities that $F_j(S_t x)$ *follows pattern* $\varphi_j(t)$ *or "time reversed pattern"* $\pm\varphi_j(-t)$ *(the sign depending on the parity of* F_j*) are related by*

$$\frac{P_{SRB}(|F_j(S_t x) - \varphi_j(t)|_{j=1,\ldots,n}, p)}{P_{SRB}(|F_j(S_t x) \mp \varphi_j(-t)|_{j=1,\ldots,n} < \varepsilon, -p)} = \exp(\tau\, p\, \sigma_+)$$

where sign choice $\mp$ *is opposite to the parity of* F_j *and* $p \overset{def}{=} \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \frac{\sigma(S_t x)}{\sigma_+}\, dt$.

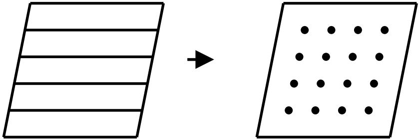
The relation holds for patterns which can be realized with a probability that does not vanish faster than exponentially in time. and for all F's.

It is worth pointing out that the SRB distribution can be interpreted as a uniform distribution over the attractor.

The coding into sequences makes it possible to divide into cells the phase space so that each cell $C_{\underline{\gamma}_N}$ consists of the points that are coded int $\gamma_{-N}, \ldots, \gamma_N$ between time $-N, N$.

Time evolution is is a non invertible map of the discretized points.

Time evolution map in the discrete representation $\mathcal{A}$ appears, in each coarse cell, as a family of points regularly arranged on a finite number of unstable manifolds



The attractor $\mathcal{A}$ is the set of points over which it is invertible. Transitivity insures that the the number $\mathcal{N}(\underline{\gamma})$ of points of $\mathcal{A}$ in the cell $C_{\underline{\gamma}}$.

The number is subject to a strong compatibility constraint: it is proportional to the inverse of the expansion rate along the unstable manifold $e^{-\Lambda_+(\underline{\gamma})}$.

Then averages are computed with a uniform distribution! over the points on the attractor $\mathcal{A}$

$$\langle F \rangle = \frac{\sum_{\underline{\gamma}} \mathcal{N}(\underline{\gamma}) F(\underline{\gamma})}{\sum_{\underline{\gamma}} \mathcal{N}(\underline{\gamma})}$$

hence if $\lambda_i(\underline{\gamma}) \stackrel{def}{=} \log \Lambda_i(\underline{\gamma})$

$$\langle F \rangle = \frac{\sum_{\underline{\gamma}} e^{-\lambda_i(\underline{\gamma})} F(\underline{\gamma})}{\sum_{\underline{\gamma}} e^{-\lambda_i(\underline{\gamma})}}$$

espressing the SRB distribution.

If evolution is reversible (as in above models) $\exists I$ such that $I^2 = 1$, $IS = S^{-1} I$. Then $\lambda_i(I\underline{\gamma}) = -\lambda_s(\underline{\gamma})$

Hence if $p = \frac{1}{\tau} \sum_{j=0}^{\tau-1} \frac{\sigma(S^j x)}{\sigma_+}$, $\sigma_+ = \langle \sigma \rangle > 0$

$$\frac{P_\tau(p)}{P_\tau(-p)} = \frac{\sum_{\underline{\gamma}, p \text{ fixed}} e^{-\lambda_i(\underline{\gamma})}}{\sum_{\underline{\gamma}, -p \text{ fixed}} e^{-\lambda_i(\underline{\gamma})}} = \frac{\sum_{\underline{\gamma}, p \text{ fixed}} e^{-\lambda_i(\underline{\gamma})}}{\sum_{\underline{\gamma}, p \text{ fixed}} e^{-\lambda_i(I\underline{\gamma})}} = \frac{\sum_{\underline{\gamma}, p \text{ fixed}} e^{-\lambda_i(\underline{\gamma})}}{\sum_{\underline{\gamma}, p \text{ fixed}} e^{\lambda_s(\underline{\gamma})}} = e^{\tau p \sigma_+}$$

because $-\lambda_i(\underline{\gamma}) - \lambda_s(\underline{\gamma}) = p \sigma_+ \tau$.

no parameters, model independent (provided reversible).