Methods and variations in KAM theory

I: Siegel's Theorem in $\mathcal{C}$

(1) $z' = e^{i\omega} z + P(z)$ P polyn. min. deg. ≥ 2

Question: linearizable? or does $\exists$ Γ s.t.

$z = \zeta + \Gamma(\zeta)$ & $z' = \zeta' + \Gamma(\zeta')$ $\longleftrightarrow$ $\zeta' = e^{i\omega}\zeta$??

$$\Gamma(e^{i\omega}\zeta) - e^{i\omega}\Gamma(\zeta) = P(\zeta + \Gamma(\zeta)$$

Look for $\Gamma(\zeta) = \sum_{k=2}^{\infty} \gamma_k \ z^k$

$$\Gamma(e^{i\omega}\zeta) - e^{i\omega}\Gamma(\zeta) = P(\zeta + \Gamma(\zeta))$$

$\Gamma(\zeta) = \sum_{k=2}^{\infty} \gamma_k \ z^k$ means ($\gamma_1 \stackrel{def}{=} 1$):

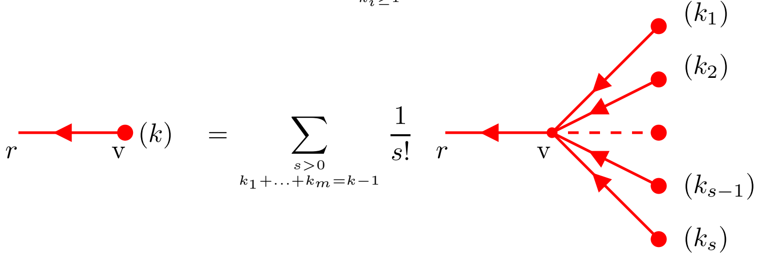
$$\sum_{k=2}^{\infty}(e^{i\omega\, k} - e^{i\omega})\,\zeta^k\,\gamma_k \;=\; \sum_{s=2}^{\infty}\frac{P_s}{s!}\,\prod_{i=1}^{s}(\sum_{k_i=1}^{\infty}\zeta^{k_i}\gamma_{k_i})$$

Need: $|\Delta_k| \stackrel{def}{=} |e^{i\omega k} - e^{i\omega}| \geq \frac{1}{C|k|^\tau}$ $\qquad k \neq 1$

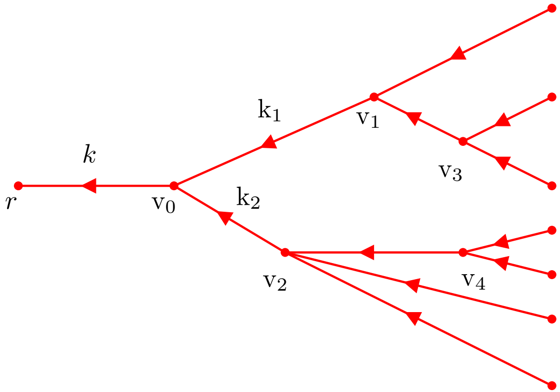
$$\longleftrightarrow \qquad \gamma_k = \sum_{s=2}^{\infty}\frac{1}{\Delta_k}\frac{P_s}{s!}\sum_{\substack{k_1+\ldots+k_s=k\\ k_i\geq 1,\ s\geq 2}}\gamma_{k_1}\cdots\gamma_{k_s}$$

Graphical representation

$$\gamma_k = \sum_{s=2}^{\infty} \frac{1}{\Delta_k} \frac{P_s}{s!} \sum_{\substack{k_1+\ldots+k_s=k \\ k_i \geq 1}} \gamma_{k_1} \cdots \gamma_{k_s}$$



Diagrammatic interpretation. Each line λ represents γ_{k_λ}. Iterate.



labels k_{v_j} on each non-top line λ count (# of preceding lines).

$$\mathrm{Val}(\theta) \stackrel{def}{=} \Big(\prod_{\substack{lines\ \lambda \\ k_\lambda > 1}} \frac{1}{\Delta_\lambda}\Big)\Big(\prod_{nodes\ \mathrm{v}} \frac{P_{s_v}}{s_v!}\Big)$$

$\Delta_\lambda \stackrel{def}{=} (e^{i\omega k_\lambda} - e^{i\omega})$ *line propagator*, $\quad k(\theta) = k = \#$ non-top lines

$$\mathrm{Val}(\theta) \stackrel{def}{=} \Big(\prod_{lines\ \lambda} \frac{1}{\Delta_\lambda}\Big)\Big(\prod_{nodes\ \mathrm{v}} \frac{P_{s_v}}{s_v!}\Big)$$

is bounded by the maximum $\overline{P}^k$ of $|P_s|^k$ times the number of trees:

$$\text{number of trees } \le \ 2^{2k}$$

times product of propagators $|\Delta_\lambda|^{-1}$.

Define *scale* of a propagator line to be $n = 1, 0, -1, -2, \ldots$

$$\begin{array}{lll} 1 & \text{if } C|\Delta_\lambda| \ge 1 & \text{for n=1} \\ n & \text{if } 2^{n-1} < C|\Delta_\lambda| \le 2^n & \text{for } n \le 0 \end{array}$$

Therefore

$$\sum_{\theta \text{ with } k \text{ lines}} |\mathrm{Val}(\theta)| \le \overline{P}^k C^k 2^{2k} \prod_{n\le 0} 2^{-n\mathcal{N}_n}$$

$\mathcal{N}_n$ = max number of lines of scale n in a tree with k lines

$$\overline{P}^k C^k 2^{2k} \prod_{n\le 0} 2^{-2n\mathcal{N}_n}$$

Siegel-Bryuno bound: $\exists\ \nu_0$ s.t. for $-n \ge \nu_0$

$$\mathcal{N}_n(\theta) \le \frac{2k}{\varepsilon 2^{-n/\tau}} - 1, \qquad \varepsilon \stackrel{def}{=} \frac{1}{2^{3/\tau}} \qquad \text{if} \quad \mathcal{N}_n > 0$$

$\nu_0 \le -\tau \log_2(\frac{2^{1/\tau}-1}{2^{1+3/\tau}})$. Therefore

$$\sum_{k(\theta)=k} |\mathrm{Val}(\theta) \le (\overline{P}C2^2 \prod_{n\le -\nu_0} 2^{-n\frac{2}{\varepsilon}2^{n/\tau}})^k (2^{\nu_0})^{k\nu_0} \stackrel{def}{=} B^k$$

convergence for $|\zeta| \le B^{-1}$

Simple proof (Pöschel, 1984, [1]).

Heuristic idea:

To obtain a line of scale n need at least $2^{-n/\tau}$ lines by Diophantine property.

Once obtained as many more are needed to obtain a new one etc

Hence we expect that the total number be of the order of $k/2^{-n\tau}$

$$\mathcal{N}_n(\theta) \le \frac{2k}{\varepsilon 2^{-n/\tau}} - 1, \qquad \varepsilon \stackrel{def}{=} \frac{1}{2^{3/\tau}} \qquad \text{if} \quad \mathcal{N}_n > 0$$

a) induction over number k of branches. If $k < \varepsilon 2^{-n/\tau} \;\Rightarrow \mathcal{N}_n = 0$ (Diophantine inequality)

$|k_\lambda| \;\le\; Nk \;\le\; N\varepsilon 2^{-n/\tau} \;=\; 2^{-3/\tau}2^{-n/\tau} \Rightarrow$
$C|\Delta_\lambda| \ge \frac{1}{(k_\lambda)^\tau} \ge 2^{3-n} > 2^{-n}$

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b) If root scale $= n$ and only 1 incoming line is root of a subtree $\overline{\theta}$ with $\mathcal{N}_n(\overline{\theta}) > 0$: look for its first scale-n line calling $\overline{\theta}$ the subtree with it as root.

Then **if** $\theta/\overline{\theta}$ contains $\overline{k} \ge \frac{1}{2}\varepsilon 2^{-n/\tau}$ lines, $\overline{\theta}$ would contain $< k - \frac{1}{2}\varepsilon 2^{-n/\tau}$:

$$\mathcal{N}_n(\theta) \le 1 + \frac{2(k-\overline{k})}{\varepsilon 2^{-n/\tau}} - 1 = \frac{2k}{\varepsilon 2^{-n/\tau}} - 1$$

Or $\theta/\overline{\theta}$ contains $\overline{k} < \frac{1}{2}\varepsilon 2^{-n/\tau}$ lines, using
$\Delta(k) = \Delta(k-\overline{k}) + \Delta(\overline{k}+1)e^{i(k-\overline{k})\,\omega}$

$$C|\Delta(k)| \geq -2^n + C|\Delta(\overline{k}+1)| \geq -2^n + (\frac{1}{2}\varepsilon 2^{-n/\tau}+1)^{-\tau}$$
$$= \frac{2^{\tau+3}2^n}{(1+2^{1+3/\tau}2^{n/\tau})^\tau} > 2^n$$

for $-n \geq \nu_0$ (e.g. $\nu_0 = -\tau\log_2 \frac{2^{1/\tau}-1}{2^{1+3/\tau}}$).

If root scale n and θ has more than one subtree with root scale n, or

if root scale $> n$ trivial cases (because of the -1)

Extensions:

(1) no need of P being a polynomial: holomorphic suffices

(2) no need to restrict to maps in one complex variable:

$$\mathbf{z}' = e^{i\,\boldsymbol{\omega}}\,\mathbf{z} + \mathbf{P}(\mathbf{z})$$

where $\mathbf{z} = (z_1, \ldots, z_\ell)$, $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_\ell)$, $e^{i\,\boldsymbol{\omega}}\,\mathbf{z} \stackrel{def}{=} (e^{i\,\omega_1} z_1, \ldots, e^{i\,\omega_\ell} z_\ell)$

(3) Diophantine property: $\exists\, C, \tau > 0$ s.t.

$$|e^{i\,\boldsymbol{\omega}\cdot\boldsymbol{\nu}} - e^{i\,\omega_j}| \ge \frac{1}{C|\boldsymbol{\nu}|^\tau}$$

for all integers $\boldsymbol{\nu} \in Z^\ell$ if $\boldsymbol{\nu} \neq (0, \ldots, 1, \ldots, 0)$.

Proofs are essentally the same, [2]

Quasi periodic motions: formal perturbation analysis

Simplest KAM problem: $\mathbf{A}=(A_1,\ldots,A_\ell)\in\mathbb{R}^\lambda$ (*actions*), $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_\ell)\in\mathbb{T}^\ell$ (angles) and Hamiltonian

$$H(\mathbf{A},\boldsymbol{\alpha})=\frac{1}{2}\mathbf{A}^2+\varepsilon f(\boldsymbol{\alpha}),\qquad \text{analytic in } \mathbb{R}^\ell\times\mathbb{T}^\ell$$

Let $\boldsymbol{\omega}_0\equiv\mathbf{A}_0$ be a "Diophantine rotation" of $\mathbb{T}^\ell$. Equations:

$$\begin{aligned}
&\dot{\mathbf{A}}=\mathbf{0},\qquad \dot{\boldsymbol{\alpha}}=\boldsymbol{\omega}_0 &&\text{unperturbed}\\
&\dot{\mathbf{A}}=\mathbf{0},\qquad \dot{\boldsymbol{\alpha}}=\boldsymbol{\omega}_0-\boldsymbol{\partial}_{\boldsymbol{\alpha}}f(\alpha) &&\text{perturbed}
\end{aligned}$$

Question: does the unperturbed motion $t\to(\mathbf{A}_0=\boldsymbol{\omega}_0,\,\boldsymbol{\alpha}=\boldsymbol{\psi}+\boldsymbol{\omega}_0t)$ remain at least for ε small? *i.e.* is there a solution like

$$\begin{aligned}
\mathbf{A}=&\boldsymbol{\omega}_0+\mathbf{H}_\varepsilon(\boldsymbol{\psi}),\ \ \boldsymbol{\alpha}=\boldsymbol{\psi}+\mathbf{h}_\varepsilon(\boldsymbol{\psi})\\
t\to&(\mathbf{A}(t),\boldsymbol{\alpha}(t))\ \ \text{with}\ \ \boldsymbol{\psi}(t)=\boldsymbol{\psi}+\boldsymbol{\omega}_0\,t
\end{aligned}$$

$\mathbf{H}_\varepsilon,\mathbf{h}_\varepsilon$ analytic in $\varepsilon,\boldsymbol{\psi}$

In other words: is the unpertubed uniform rotation with velocity ω_0 continued analytically in ε in presence of interaction? at least if ε is small and $\boldsymbol{\omega}_0$ is Diophantine: $\exists\, C > 0, \tau > 0$ such that:

$$|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}| \ge \frac{1}{C\,|\boldsymbol{\nu}|^\tau}, \qquad \forall\, \boldsymbol{\nu} \in \mathbb{Z}^\ell,\ \boldsymbol{\nu} \ne \mathbf{0}$$

Answer: yes KAM theorem:

Kolmogorov, [3], (diophantine, analytic), followed by
Arnold, [4], (resonant, analytic), and
Moser, (diophantine, differentiable), [5]

Lindstedt series: the equations of motion are

$$\ddot{\boldsymbol{\alpha}} = -\varepsilon \boldsymbol{\partial}_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha})$$

and look for $\boldsymbol{\alpha}(t) = \boldsymbol{\psi} + \boldsymbol{\omega}_0\, t + \mathbf{h}_\varepsilon(\boldsymbol{\psi} + \boldsymbol{\omega}_0\, t)$, $\qquad i.e$

$$(\boldsymbol{\omega}_0 \cdot \boldsymbol{\partial}_{\boldsymbol{\psi}})^2 \mathbf{h}_\varepsilon(\boldsymbol{\psi}) = -\varepsilon\, \boldsymbol{\partial}_{\boldsymbol{\alpha}} f(\boldsymbol{\psi} + \mathbf{h}_\varepsilon(\boldsymbol{\psi}))$$

$$(\boldsymbol{\omega}_0 \cdot \boldsymbol{\partial}_{\boldsymbol{\psi}})^2 \mathbf{h}_\varepsilon(\boldsymbol{\psi}) = -\varepsilon\, \boldsymbol{\partial}_{\boldsymbol{\alpha}} f(\boldsymbol{\psi} + \mathbf{h}_\varepsilon(\boldsymbol{\psi}))$$

Let $\mathbf{h}_\varepsilon(\boldsymbol{\psi}) = \sum_{k=1}^{\infty} \varepsilon^k\, \mathbf{h}^{[k]}(\boldsymbol{\psi})$

$$(\boldsymbol{\omega}_0 \cdot \boldsymbol{\partial}_{\boldsymbol{\psi}})^2 h^{[k]}(\boldsymbol{\psi}) = -\sum_{k=1}^{\infty} \left[\boldsymbol{\partial}_{\boldsymbol{\alpha}} f(\boldsymbol{\psi} + \mathbf{h}(\boldsymbol{\psi}))\right]^{[k-1]}$$

Developing by Taylors series ($\mu = 1, \ldots, \ell$)

$$(\boldsymbol{\omega}_0 \cdot \boldsymbol{\partial}_{\boldsymbol{\psi}})^2 h_\mu^{[k]}(\boldsymbol{\psi}) = -\sum_{s=0}^{\infty} \frac{1}{s!} \sum_{\mu_1, \ldots, \mu_s} \partial_{\mu, \mu_1, \ldots, \mu_s} f(\boldsymbol{\psi}) \Big[\prod_{j=1}^{s} h_{\mu_j}(\boldsymbol{\psi}) \Big]^{[k-1]}$$

hence

$$(\boldsymbol{\omega}_0 \cdot \boldsymbol{\partial}_{\boldsymbol{\psi}})^2 h_\mu^{[k]}(\boldsymbol{\psi}) = -\sum_{s=0}^{\infty} \frac{1}{s!} \sum_{\substack{\mu_1, \ldots, \mu_s \\ k_1 + \ldots + k_s = k-1}} \partial_{\mu, \mu_1, \ldots, \mu_s} f(\boldsymbol{\psi}) \Big(\prod_{j=1}^{s} h_{\mu_j}^{[k_j]}(\boldsymbol{\psi}) \Big)$$

$$(\boldsymbol{\omega}_0\cdot\boldsymbol{\partial}_{\boldsymbol{\psi}})^2 h^{[k]}_\mu(\boldsymbol{\psi}) = -\sum_{s=0}^\infty \frac{1}{s!}\sum_{\substack{\mu_1,\ldots,\mu_s\\ k_1+\ldots+k_s=k-1}} \partial_{\mu,\mu_1,\ldots,\mu_s} f(\boldsymbol{\psi})\Big(\prod_{j=1}^s h^{[k_j]}_{\mu_j}(\boldsymbol{\psi})\Big)$$

Use Fourier series $F(\boldsymbol{\psi}) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^\ell} e^{i\boldsymbol{\psi}\cdot\boldsymbol{\nu}}\, F_{\boldsymbol{\nu}}$ and

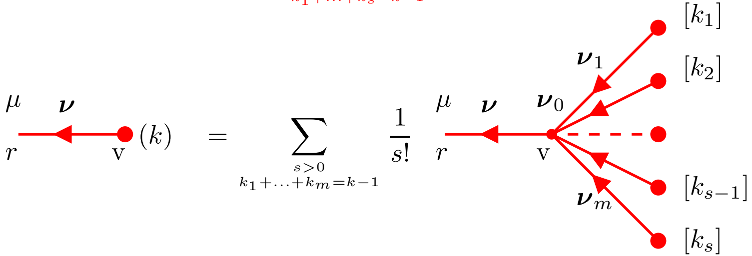
$$((\boldsymbol{\omega}\cdot\boldsymbol{\partial}_{\boldsymbol{\psi}})^2 F)_{\boldsymbol{\nu}} = -(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2 F_{\boldsymbol{\nu}},\quad (\boldsymbol{\partial} F)_{\boldsymbol{\nu}} = i\boldsymbol{\nu}\, F_{\boldsymbol{\nu}}$$

$$(\boldsymbol{\omega}_0\cdot\boldsymbol{\nu})^2 h^{[k]}_{\mu;\boldsymbol{\nu}} = \sum_{s=0}^\infty \frac{1}{s!}\sum_{\substack{\mu_1,\ldots,\mu_s\\ \boldsymbol{\nu}_0+\boldsymbol{\nu}_1+\ldots+\boldsymbol{\nu}_s=\boldsymbol{\nu}\\ k_1+\ldots+k_s=k-1}} i\nu_{0;\mu}\big(\prod_{j=1}^s i\nu_{0;\mu_j}\big) f_{\boldsymbol{\nu}_0}\Big(\prod_{j=1}^s h^{[k_j]}_{\mu_j;\boldsymbol{\nu}_j}\Big)$$

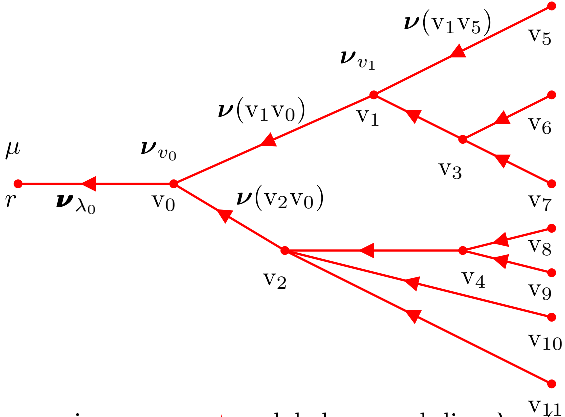
$$(\boldsymbol{\omega}_0\cdot\boldsymbol{\nu})^2 h^{[k]}_{\mu;\boldsymbol{\nu}} = \sum_{s=0}^\infty \frac{1}{s!}\sum_{\substack{\boldsymbol{\nu}_0+\boldsymbol{\nu}_1+\ldots+\boldsymbol{\nu}_s=\boldsymbol{\nu}\\ k_1+\ldots+k_s=k-1}} i\nu_{0;\mu} f_{\boldsymbol{\nu}_0}\Big(\prod_{j=1}^s i\boldsymbol{\nu}_0\cdot\mathbf{h}^{[k_j]}_{\boldsymbol{\nu}_j}\Big)$$

Check (induction): if $\boldsymbol{\nu}=\mathbf{0}$ r.h.s. is 0 (Linstedt,NewcombPoincaré)

$$h_{\mu;\boldsymbol{\nu}}^{[k]} = \frac{1}{(\boldsymbol{\omega}_0\cdot\boldsymbol{\nu})^2}\sum_{s=0}^{\infty}\frac{1}{s!}\sum_{\substack{\boldsymbol{\nu}_0+\boldsymbol{\nu}_1+\ldots+\boldsymbol{\nu}_s=\boldsymbol{\nu}\\ k_1+\ldots+k_s=k-1}} i\nu_{0;\mu} f_{\boldsymbol{\nu}_0}\Big(\prod_{j=1}^{s} i\boldsymbol{\nu}_0\cdot\mathbf{h}_{\boldsymbol{\nu}_j}^{[k_j]}\Big)$$



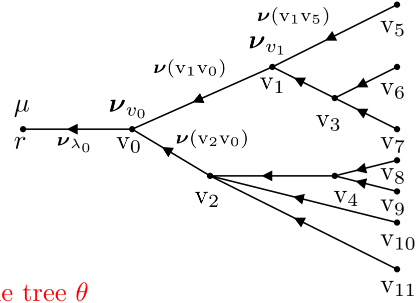
Diagrammatic interpretation. Each line λ with a "fat" origin bearing a label $[k_\lambda]$ represents $\mathbf{h}_{\boldsymbol{\nu}_{v'}}^{[k_\lambda]}$. Iterate.



Each node v carries a momentum label $\boldsymbol{\nu}_v$; each line $\lambda = (v'v)$ from v' to v carries a current $\boldsymbol{\nu}_\lambda \equiv \boldsymbol{\nu}(v'v)$ which equals the sum of all the momentum labels of the nodes that precede v

$$\boldsymbol{\nu}_\lambda = \boldsymbol{\nu}(v'v) = \sum_{w<v} \boldsymbol{\nu}_w = \sum_{w \le v'} \boldsymbol{\nu}_w$$

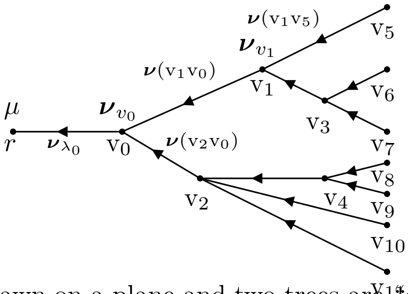
Uniformize notations: imagine the root line end-point (which is not a node) carries a momentum label $\boldsymbol{\nu}_0 = \mathbf{e}_\mu$ = unit vector in direction μ.



define value of the tree θ

$$\mathrm{Val}(\theta) = \prod_{nodes\ v} \frac{-f_{\boldsymbol{\nu}_v}}{s_v!} \prod_{lines\ \lambda=(v'v)} \frac{\boldsymbol{\nu}_v \cdot \boldsymbol{\nu}_{v'}}{(\boldsymbol{\omega} \cdot \boldsymbol{\nu}(v'v))^2}$$

The functions $\mathbf{h}^{[k]}_{\mu;\boldsymbol{\nu}}$ are expressed as sums over all trees with k lines (including the root line).



Trees imagined drawn on a plane and two trees are "equivalent" if can be overlapped by continuously pivoting or deforming in length the branches avoiding any overlapping of branches.

May be convenient to distinguish lines by appending an extra label, number label from 1 to k. Equivalence will again be defined through the overlapping of the branches through pivoting and length deformations. The value of a numbered tree with k nodes will then be

$$\mathrm{Val}(\theta) = \frac{1}{k!} \prod_{nodes\ v} -f_{\boldsymbol{\nu}_v} \prod_{lines\ \lambda=(v'v)} \frac{\boldsymbol{\nu}_v \cdot \boldsymbol{\nu}_{v'}}{(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}(v'v))^2}$$

The Lindstedt cancellation at $\boldsymbol{\nu} = \mathbf{0}$ is inductively obvious here

Conclusion: calling $\boldsymbol{\nu} \stackrel{def}{=} \sum_v \boldsymbol{\nu}_v$

$$\mathbf{h}^{[k]}_{\mu;\boldsymbol{\nu}} = \frac{1}{k!} \sum_{\substack{all\ \ labels \\ k-trees\ \ \theta,\ \boldsymbol{\nu}_r = \mathbf{e}_\mu}} \Big(\prod_{nodes\ \ v} (-f_{\boldsymbol{\nu}_v}) \Big) \Big(\prod_{lines\ \ \lambda=(v'v)} \frac{\boldsymbol{\nu}_v \cdot \boldsymbol{\nu}_{v'}}{(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}(v'v))^2} \Big)$$

This is nowadays called Lindsted series: [6].

Original Lindstedt's work can be found in Poincaré, [7, p.462]: original result is more general. If $\ell = 2$ motions are solved in powers μ and of 4 parameters ω_1, ω_2 representing, in Poincaré's notation, ε and the first order variations of the new rotation vectors (they also depend on two uninteresting arbitrary angles $\overline{\omega}_1, \overline{\omega}_2$). Series yield power series for action and angles of perturbed motion as periodic functions of a uniformly rotating pair of angles with new rotation vector λ_1, λ_2.

Above we fix $\boldsymbol{\lambda} = (\lambda_1, \lambda_2)$ to 0 (*i.e.* fix the rotation vector to the unperturbed one (and $\overline{\omega}_1, \overline{\omega}_2$ also to $\mathbf{0}$)). So quasi periodic solutions are found in powers of $\varepsilon, \omega_1, \omega_2$ and in an open set around the original invariant torus.

Which P. proves to be not convergent, in general, because it would imply 0 Lyapunov exp. for the dense periodic orbits (or also existence of ℓ "uniform", independent, constants of motion).

However the special case $\boldsymbol{\lambda} = \mathbf{0}$ is soluble by the KAM: as Weierstrass seemed to conjecture, [8, p.9]

The convergence problem

Suppose $f_{\boldsymbol{\nu}} = f_{-\boldsymbol{\nu}}$ and f a trigonometric polynomial: $|\boldsymbol{\nu}| \le N < \infty$.

1) number of trees: $\mathcal{A}_k \le k! 2^{2k}$

2) $|f_{\boldsymbol{\nu}}| \le F$

3) Define scale of a line λ the integer $n = 1, 0, -1, -2, \ldots$

$$\begin{array}{lll} n & \text{if} & 2^{n-1} < C|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\lambda| \le 2^n, \text{ and } n \le 0 \\ 1 & \text{if} & C|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\lambda| \ge 1 \end{array}$$

4) $\mathcal{N}_n(k)$ max. number of lines of scale n in a k–lines tree θ

$$\mathrm{Val}(\theta) = \frac{1}{k!} \prod_{nodes\ v} -f_{\boldsymbol{\nu}_v} \prod_{lines\ \lambda=(v'v)} \frac{\boldsymbol{\nu}_v \cdot \boldsymbol{\nu}_{v'}}{(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}(v'v))^2}$$

$$\Rightarrow\ |\mathbf{h}^{[k]}_{\boldsymbol{\nu}}| \le C^{2k} F^k N^{2k} 2^{2k} \prod_{n=1}^{-\infty} 2^{-2n\mathcal{N}_n(k)}\, \delta(|\boldsymbol{\nu} \le kN)$$

Hence need bound on $\mathcal{N}_n(k)$. Siegel-Bryuno? try:

$$\mathcal{N}_n(k) \le \frac{2k}{E2^{-n/\tau}} - 1 \qquad \text{if } \mathcal{N}_n > 0 \qquad ?$$

for the number of lines of scale $\le n$ (notice the $\le$).

The heuristic argument says that $\frac{1}{N}2^{-n\tau}$ lines are needed to build a line of scale n: once build need the same for one more and so on which would imply the bound as in the Siegel problem.

Let $E = \frac{1}{N2^{3/\tau}}$ and proceed by induction over k since for $k \le E2^{-n/\tau}$ the bound certainly holds because need at least $u = \frac{1}{N}2^{-n/\tau}$ lines to build a line of scale n and $E2^{-n/\tau} < \frac{1}{N}2^{-3/t}2^{-n/\tau} < u$ so that $\mathcal{N}_n(k) = 0$.

$$\mathcal{N}_n(k) \le \frac{2k}{E2^{-n/\tau}} - 1 \qquad \text{if } \mathcal{N}_n > 0 \qquad ?$$

ok for $k < E2^{-n/\tau}$

If the root line has scale n and there is only one subtree $\overline{\theta}$ with root line of scale n in the path leading to the root r

either $\theta/\overline{\theta}$ contains $\overline{k}$ "lines" and $\overline{k} > \frac{1}{2E2^{-n\tau}}$ then $\overline{\theta}$ contains $k - \overline{k} < k - \frac{1}{2E2^{-n\tau}}$ then OK because

$$\mathcal{N}_n(k) \le 1 + \frac{2(k-\overline{k})}{E2^{-\nu/\tau}} - 1 < \frac{2k}{E2^{-\nu/\tau}} - 1$$

Or $\theta/\overline{\theta}$ contains "few lines" $\overline{k} \le \frac{1}{2E2^{-n\tau}}$. If the $\boldsymbol{\nu}$ flowing into the root of θ is **different** from the current $\boldsymbol{\nu}'$ through the root line of $\overline{\theta}$ it is

$$|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}| \ge -|\boldsymbol{\omega} \cdot \boldsymbol{\nu}'| + |\boldsymbol{\omega} \cdot (\boldsymbol{\nu} - \boldsymbol{\nu}')| \ge -\frac{1}{C} 2^n + \frac{1}{C} (N2E2^{-n/\tau})^\tau$$

so that $C|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}| \ge -2^n + 2^{\tau+3} 2^n > 2^n$ and this case ie therefore impossible.

Since the cases in which the root line does not have scale n are trivial the only case that needs study is if the root momentum $\boldsymbol{\nu}$ is equal to the root momentum of the only subtree whose root has momentum $\boldsymbol{\nu}$ exactly.

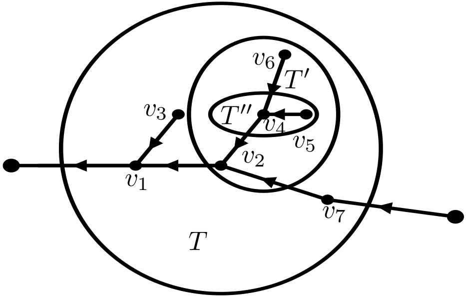
In this case the $\mathcal{N}_n$ can be much larger than wanted.

Remark: in Siegel's case this does not arise because the analogue of the momentum of λ is the number of endpoints of the tree reachable climbing up from $\lambda \Rightarrow$ it is stricly increasing while descending the tree.

Introduce the notion of cluster T of scale n_T in a tree θ as a maximal connected set of lines of scale $\geq n_T$.

$V(T)$ nodes in T,
$\Lambda(T)$ branches connecting them, "contained, or internal" to T,
$\Lambda_1(T)$ branches in $\Lambda(T)$ plus the single exiting one (if any),
$\mathcal{T}(\theta)$ set of all clusters in θ.

Clusters form a hierarchical structure: a cluster can contain subclusters and can be contained in larger clusters: the larger the cluster the smaller its scale n_T



This is an example of a family of clusters T has lower (more negative) scale of T' which has lower scale than T'.

The example is special because each cluster has only one entering line.

A key notion is self energy cluster or resonant cluster.

Self energy clusters: suppose that

1) T has just one entering branch λ_T and one exiting. Its scale $n = n_{\lambda_T}$ is smaller than the smallest scale n_T of the branches inside T

2) let w_1 the node into which the branch λ_T ends inside T.

Then T is a *self-energy subgraph* if

(i) $\sum_{w\in T} \boldsymbol{\nu}_w = \mathbf{0}$: in and out lines carry same momentum.

(ii) If $n = n_{\lambda_T}$, $EEquiv 2^{-3/\tau} N^{-1}$ then $M(T)$ is not too large:

$$M(T) \stackrel{def}{=} \text{number of branches contained in } T \le \frac{1}{2}\varepsilon\, 2^{-n/\tau}$$

Call n_{λ_T} the self-energy-scale of T, and λ_T a self-energy branch

A notion of self energy cluster abridged s.e.c designed so that deleting each self energy clusters and joining the entering and exiting lines a graph without any s.e.c. is obtained. Hence

$$\mathcal{N}_n^* \stackrel{def}{=} \#\ \text{scale-}n \text{ lines not counting the ones exiting a s.e.c.}$$

$$\mathcal{N}_n^* \le \frac{2k}{E 2^{n/\tau}}$$

The s.e.c. do not allow a naive estimate of Siegel-Bryuno type because if $\theta/\overline{\theta}$ contains "few lines" $\overline{k} \le \frac{1}{2} E 2^{-n\tau}$ it can still be that the $\boldsymbol{\nu}$ flowing into the root λ of θ **equals** the current $\boldsymbol{\nu}'$ through the root $\overline{\lambda}$ line of $\overline{\theta}$ and the two lines are the entering and exiting lines of a s.e.c. In fact if $\lambda, \overline{\lambda}$ have equal scale but are not entering and exiting the same cluster the difference $\boldsymbol{\nu} - \boldsymbol{\nu}'$ cannot be 0.

Because all lines outside $\overline{\theta}$ have a scale $> n$ (their momenta are sums of $\leq \overline{k}$ node momenta $\Rightarrow$ their size is $\leq N\frac{1}{2}E2^{-n\tau} \leq (2^{-(1+3/\tau)}2^{-n/\tau})$ plus, maybe, a contribution $\boldsymbol{\nu}$ from the current of $\overline{\lambda}$: so the propagators are $\geq 2^{\tau+3}2^n - 2^n > 2^n \Rightarrow$ have scale $> n+3$.

If $\lambda, \overline{\lambda}$ are part of the same cluster which is not a s.e.c. then the number of lines in this cluster must be large, by the definition of s.e.c. (item (ii)), and the bound is again trivial.

Conclusion, collecting the lines not accounted (*i.e* one per s.e.c.)

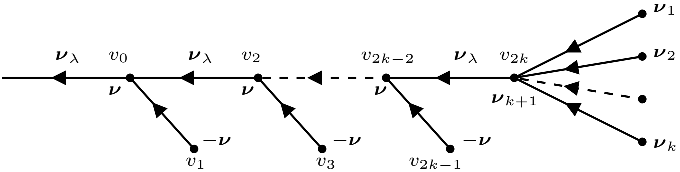
$$\mathcal{N}_n \leq \mathcal{N}_n^* + \sum_{T, n_T = n} m_T = \mathcal{N}_n^* + \sum_{T, n_T = n} (m_T - 1) + p(n,k)$$

where m_T is the number of s.e.c. of scale n in the scale-n cluster T and $p(n,k)$ is the maximum number of scale n clusters that can be in a cluster of scale n.

The number $p(n,k)$ is bounded in the same way by $\frac{2k}{E2^{n/\tau}} - 1$, *i.e.* by the same bound that can be placed on $\mathcal{N}_n^*$, so that the final estimate is

$$\mathcal{N}_n(k) \le \frac{4k}{E2^{n/\tau}} + \sum_{T, n_T = n} (-1 + m_T)$$

Cancellations: (fig5) $3k+1$ lines such that $\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\lambda = O(k^{-\tau})$



$$\mathrm{Val}(\theta) = \frac{1}{k!} \prod_{nodes\ v} (-f_{\boldsymbol{\nu}_v}) \prod_{lines\ \lambda=(v'v)} \frac{\boldsymbol{\nu}_v \cdot \boldsymbol{\nu}_{v'}}{(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}(v'v))^2}$$

after sum over labels is bounded by $N^{3k+1} F^{3k} N^{6k} (Nk)^{\tau k} = O(k!^\tau)$.

This shows the need of cancellations. Cancellations must be relates to s.e.c because values of trees without s.e.c. are bounded by

$$(2N+1)^k F^k N^{2k} \Big(\prod_{n\le 0} 2^{-2n\, 2^{n/\tau}\, 2/E} \Big)^k$$

To treat the values of the trees θ with s.e.c. consider first a tree that contains only one s.e.c. R of scale n and entering line $\lambda_R = (v_R, v_1)$, $(v_R \not\in R, v_1 \in R)$ and inner scale n_R.

It is $n_R > n+2$ as the propagators Δ_λ of the lines $\lambda \subset R$ have a $|\boldsymbol{\nu}_\lambda| \le NE2^{-n/\tau} = 2^{(-3-n)\tau}$ so that $|\Delta_\lambda| \ge 2^{3+n} - 2^n > 2^{2+n}$).

Collect all trees obtained from θ detaching the incoming line λ_T and attaching it successively to the inner nodes v_i of the cluster T and add to the collection the trees obtained by reversing simoultaneously the sign of all nodes momenta $\boldsymbol{\nu}_{v_i}$, $v_i \in R$.

The value of the trees so obtained changes because

1) the factor $\boldsymbol{\nu}_{v_i} \cdot \boldsymbol{\nu}_{v_T}$

2) the current on $\lambda \subset R$ may change because it is $\boldsymbol{\nu}_\lambda = \sum_{w\in R} \boldsymbol{\nu}_w + \eta_i \boldsymbol{\nu}_{\lambda_R}$ with $\eta_i = 0$ if λ is not on the path of the s.e.c. and 1 otherwise

In this process the scale of the inner lines may change: the cluster R remains a s.e.c. cluster (but the clusters inside R may change).

If $\eta = \boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\lambda_R}$ the sum of the values of the considered trees θ_i is obtained by considering the tree θ_0 in which the s.e.c. is deleted and its entering and exiting lines are connected into a single line $\overline{\lambda}$ and multiplying $\mathrm{Val}(\theta_0)$ by a function $\frac{1}{\eta^2} F_{n,R}(\eta)$

If the inner lines of θ do not change scale in all θ_i's no matter which value of η with $|\eta| \le 2^{-n}$ (even if complex) the function $F_{n,R}(\eta)$ is uniformly bounded via the Siegel-Bryuno bound.

$$|F_{n,R}(\eta)| \le (2N+1)^M F^M N^{2k} \Big(\prod_{0\ge p\ge n_R} 2^{-2p\,2^{p/\tau}\,2/E} \Big)^M = B^M$$

where M is the number of lines in R and η is complex $|\eta| \le 2^n$.

Need to understand: what about possible scale change in a s.e.c. ?

Key remark: inside a s.e.c. few lines $\le \overline{k} = \frac{1}{2} E 2^{-n/\tau}$.

Their scale is $> n+3$: their momenta are sums of $\le \overline{k}$ node momenta $\Rightarrow$ size is $\le N\overline{k} \le (2^{-(1+3/\tau)} 2^{-n/\tau})$ plus, maybe, a contribution $\boldsymbol{\nu}$ from the current of λ_R.

So the propagators are $\ge 2^{\tau+3} 2^n - 2^n > 2^{n+3}$ $\Rightarrow$ scale $> n+3$.

Isolate the tree θ_R inside the s.e.c. (delete outside lines and the incoming line).

Let $\boldsymbol{\nu}_\lambda^R$ be the momentum of $\lambda \in \theta_R$: $|\boldsymbol{\nu}_\lambda| \le 2^{-(3+n)/\tau}$: then the true momentum will be either $\boldsymbol{\nu}_\lambda^R$ or $\boldsymbol{\nu}_\lambda^R + \boldsymbol{\nu}$.

If for all $|\boldsymbol{\nu}_0| \le 2^{-(3+n)/\tau}$, and $n \le 0$ it is

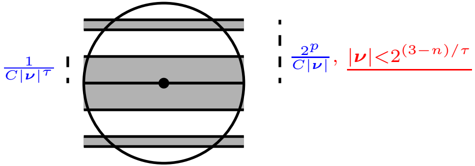
$$\max_{0 \ge p \ge n} |C\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_0 - 2^p| > 2^{n+1}, \qquad \forall |\boldsymbol{\nu}_0| \le 2^{-(3+n)/\tau}$$

then the scale of $\boldsymbol{\nu}_\lambda^R$ and of $\boldsymbol{\nu}_\lambda^R + \boldsymbol{\nu}$ are the same and no line can change scale if the node $v_i \in R$ of $\lambda_R = (v_i v_R)$ is shifted to any $v_j \in R$

Question: can the above strong Diophantine property be imposed?

The set of $\boldsymbol{\omega}_0$ satisfying has full measure in $\mathbb{R}^\ell$

Proof $\forall\boldsymbol{\nu}$ take out of the ball of radius ρ a layer $\lambda_{\boldsymbol{\nu}}$ of width $\frac{1}{C|\boldsymbol{\nu}|^{\tau+1}}$ around the plane $\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}=0$ so that, for $\boldsymbol{\omega}$ out of it, $C|\boldsymbol{\omega}\cdot\boldsymbol{\nu}|>\frac{1}{|\boldsymbol{\nu}|^\tau}$ The total volume extracted from the ball will be $\mu_\ell\sum_{\boldsymbol{\nu}\neq\mathbf{0}}\frac{1}{C|\boldsymbol{\nu}|^{\tau+1}}$.



Furthermore for each $\boldsymbol{\nu}$ with $|\boldsymbol{\nu}|<2^{-(n-3)/\tau}$ and each $p\in[n,0]$ extract a layer of width $\frac{2^{n+1}}{C|\boldsymbol{\nu}|}$ out of the planes orthogonal to $\boldsymbol{\nu}$ at distances $\frac{2^p}{C|\boldsymbol{\nu}|}$ from the plane $\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}=0$; this takes out a further

$$\le\mu'_\ell\sum_{n=0}^{-\infty}2^{n+1}\rho^{\ell-1}(|n|+1)\sum_{|\boldsymbol{\nu}|\le 2^{-(n-3)/\tau}}\frac{1}{|\boldsymbol{\nu}|}$$

$$\le\mu''_\ell\frac{1}{C}(|n|+1)2^{n-(\ell-1)\frac{n}{\tau}}\le\mu'''_\ell\frac{1}{C}$$

Hence (as $\tau>\ell-1$) $C<\infty$ out a zero volume set (Borel-Cantelli)

Going back to the bound of the sum of the values of trees with only one s.e.c. on a line of scale n and momentum $\boldsymbol{\nu}^R$

$$\sum^{*} \frac{1}{k!} \prod_{nodes\ v} -f_{\boldsymbol{\nu}_v} \prod_{lines\ \lambda=(v'v)} \frac{\boldsymbol{\nu}_v \cdot \boldsymbol{\nu}_{v'}}{(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}(v'v))^2}$$

it is for $\eta = \boldsymbol{\omega}_\omega \cdot \boldsymbol{\nu}^R$

$$\leq C^{2k} F^k N^{2k} 2^{2k} \prod_{n=1}^{-\infty} 2^{-2n\mathcal{N}_n(k)} \frac{1}{\eta^2} F_{n,R}(\eta)$$

Since the inner lines of θ do not change scale in all θ_i's no matter which value of η with $|\eta| \leq 2^{-n}$ (even if complex) the function $F_{n,R}(\eta)$ is uniformly bounded via the Siegel-Bryuno bound.

$$|F_{n,R}(\eta)| \leq (2N+1)^M F^M N^{2k} \Big(\prod_{0 \geq p \geq n_R} 2^{-2p\, 2^{p/\tau}\, 2/E} \Big)^M = B^M$$

If there were more than one s.e.c. with no internal s.e.c. the same bound would be

$$\le C^{2k} F^k N^{2k} 2^{2k} \prod_{n=1}^{-\infty} 2^{-2n\mathcal{N}_n^*(k)} \prod_R \frac{1}{(\eta^R)^2} F_{n,R}(\eta^R)$$

and $|F_{n,R}(\eta^R)| \le B_0^{M_R} \le B_0^k B^k \prod_R \frac{1}{(\eta^R)^2}$
with $F_{n,R}(\eta^R)$ holomorphic in η^R for $|\eta^R| \le 2^{n_R}$ (actually for $|\eta^R| \le 2^{n_R+1}$).
So if $F(\eta)$ can be shown to vanish to second order in η the bound can be improved to

$$\le C^{2k} F^k N^{2k} 2^{2k} \prod_{n=1}^{-\infty} 2^{-2n\mathcal{N}_n^*(k)} \prod_R B_0^{M_R} \frac{1}{2^{2n}} (2^{2(n-n_R)})$$

the sum $\sum_R M_R \le k$ and the Siegel-Bryuno bound yields

$$\le B_0^k C^{2k} F^k N^{2k} 2^{2k} \prod_{n=1}^{-\infty} 2^{-2n\mathcal{N}_n^*(k)} \prod_R \frac{1}{2^{2n}} (2^{2(n-n_R)})$$

The function $F_{R,n}$ has a second order 0 in η as it can be seen by remarking that the entering line modifies the propagators when shifted from one inner node to another: hence it does not change if η is set $= 0$ the value of F.

If the s.e.c. contains a $s^0_{v_i}$ lines inner to the cluster R than The value of F contains a combinatorial factor $\frac{1}{s^0_{v_i}!}$ times $\frac{1}{s^0_{v_i}+1}$ due to the node v_i to which the line λ^R is attached.

Therefore we can say that if the line λ^R is attached to v_i besides changing the propagators will contribute a factor

$$\frac{\boldsymbol{\nu}_R \cdot \boldsymbol{\nu}_i}{s^0_{v_i}+1}$$

However the line can be attached in $s^0_{v_i}+1$ ways to v_i.

Thus if different ways are identified we can say that the line attached to v_i gives a factor $\boldsymbol{\nu}_R\cdot\boldsymbol{\nu}_i$ to the value of the graph (neglecting the η changes of the propagators).

Hence if $\eta=0$ summing over i changes the value of the graph by $\boldsymbol{\nu}_R\cdot\sum_i\boldsymbol{\nu}_i=0$ because the sum of the node momenta of nodes internal to R is $\mathbf{0}$ in every s.e.c.

The zero is of order η^2 as $F_{R,n}(\eta)$ is even in η (by the sum over the s.e.c in which all internal node momenta were changed in sign).

Also the s.e.c. internal to larger ones can be treated in the same way: the line λ_R entering a s.e.c. is moved an reattached to the nodes internal to the cluster producing the cancelation for the same reason.

To avoid "over-canceling" attention is paid to avoid reattaching the entering line to nodes that are internal to s.e.c. $R' \subset R$. Which is fine because the sum of the node momenta of $v \in R/R'$ already vanishes.

After collecting the families of graphs associated with the s.e.c. and resumming their values the bound becomes

$$\leq B_0^k B^k \prod_R \frac{1}{(\eta^R)^2}$$

where the product is over all s.e.c.

$$\leq B_0^k C^{2k} F^k N^{2k} 2^{2k} \prod_{n=1}^{-\infty} 2^{-2n\frac{4k}{E}2^{n/\tau}} \prod_T 2^{-2n(m_T-1)} \prod_R (2^{2(n-n_R)})$$

The product of the last two is ≤ 1 and the total contribution of order k is bounded by $\overline{B}^k$ (with $\overline{B}$ computable).

The Siegel-Bryuno lemma in the version of [1] was extended to cover the KAM case in [6]. A stronger bound was derived in [9] who showed

For graphs with no s.e.c., then there is a constant C such that:

$$\prod_{\lambda\in\theta}\frac{1}{|\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\lambda|}\le C^k\,\frac{\prod_{v\in\theta}|\boldsymbol{\nu}_v|^{3\tau}}{(\sum_{v\in\theta}|\boldsymbol{\nu}_v|)^\tau}$$

for a proof see [10, Eq.(5.2)].

The method of Eliasson is quite different from the one here (based on [6]): a careful and detailed comparison can be found in [11].

Differentiable KAM: $f\in C^{(p)}$, $p>2\tau+4$ (Moser).

If $\|\boldsymbol{\nu}\|\stackrel{def}{=}(\sum|\nu_i|^2)^{\frac12}$, $f_{\boldsymbol{\nu}}=\frac{P_j(\boldsymbol{\nu})}{||\boldsymbol{\nu}||^{\ell+p+i}}$ with P_j a harmonic polynomial of degree j (and combinations thereof) tree method OK if $\ell>3+6\tau$: with the extra result of analyticity in ε, [10]. This is so because the Lindstedt series is well defined in spite of the non analyticity of f.

Classic KAM ($\sim$ Arnold's version)

Hamiltonian: $H_0 = h_0(\mathbf{A}) + f_0(\mathbf{A}, \boldsymbol{\alpha})$.

holomorphic in $W_{\rho_0,\xi_0}(\mathbf{A}_0) \stackrel{def}{=} \mathcal{C}_{\rho_0}(\mathbf{A}_0) \times \mathcal{A}_{x_0}$

$\mathcal{C}_{\rho_0}$ = polydisk $\subset \mathbb{C}^\ell$ centered at $\mathbf{A}_0 \in \mathbb{R}^\ell$: $|A_j - A_{0j}| \le \rho_0$

$\mathcal{A}_{\xi_0}$ = polyannulus $\subset \mathbb{C}^\ell$ around unit circle: $e^{-\xi_0} \le z_j \le e^{\xi_0}$, $\xi_0 \le 1$,

$\mathcal{A}_{\xi_0} \equiv \{|\mathrm{Im}\, \alpha_j| \le \xi_0\}$.

Notation $z_j \equiv e^{i\alpha_j}$, $\partial_{\alpha_j} \equiv i z_j \partial_{z_j}$, $\boldsymbol{\omega}_0(\mathbf{A}) \equiv \partial_{\mathbf{A}} h_0(\mathbf{A})$.

$$E_0 \stackrel{def}{=} \max_{W_{\rho_0,\xi_0}(\mathbf{A}_0)} |\partial_{\mathbf{A}} h_0(\mathbf{A})| \equiv \max_{W_{\rho_0,\xi_0}(\mathbf{A}_0)} |\boldsymbol{\omega}(\mathbf{A})| \stackrel{def}{=} ||h_0||_{\rho_0}$$

$$\eta_0 \stackrel{def}{=} \max_{W_{\rho_0,\xi_0}(\mathbf{A}_0)} |(\partial^2_{\mathbf{A}} h(\mathbf{A}))^{-1}|$$

$$\varepsilon_0 \stackrel{def}{=} \max_{W_{\rho_0,\xi_0}(\mathbf{A}_0)} \Big(|\partial_{\mathbf{A}} f_0(\mathbf{A}, \boldsymbol{\alpha})| + \frac{1}{\rho_0} |\partial_{\boldsymbol{\alpha}} f_0(\mathbf{A}, \boldsymbol{\alpha})|\Big) \stackrel{def}{=} ||f_0||_{\rho_0,\xi_0}$$

Suppose $\forall \mathbf{0} \neq \boldsymbol{\nu} \in \mathbb{Z}^\ell$ exist $C_0, \tau > 0$ s..t.

$$|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}| \geq \frac{1}{C_0 |\boldsymbol{\nu}|^\tau}, \qquad \boldsymbol{\omega}_0 \stackrel{def}{=} \boldsymbol{\omega}_0(\mathbf{A}_0)$$

$\exists\ \Gamma, a, b, c > 0$ s.t. $\exists\ \mathbf{H}, \mathbf{h}$ analytic in $\boldsymbol{\psi} \in \mathbb{T}^\ell$ and in f_0 s.t.

$$\mathbf{A}(\boldsymbol{\psi}) = \mathbf{A}_0 + \mathbf{H}(\boldsymbol{\psi}), \qquad \boldsymbol{\alpha}(\boldsymbol{\psi}) = \boldsymbol{\psi} + \mathbf{h}(\boldsymbol{\psi})$$

are parametric eq. of an invariant torus run quasi periodically with spectrum $\boldsymbol{\omega}_0 = \boldsymbol{\omega}_0(\mathbf{A}_0)$: i.e

$$\mathbf{A}(t) = \mathbf{A}(\boldsymbol{\psi} + \boldsymbol{\omega}_o t), \qquad \boldsymbol{\alpha}(t) = \boldsymbol{\alpha}(\boldsymbol{\psi} + \boldsymbol{\omega}_0 t)$$

IF

$$\frac{\varepsilon_0}{E_0} \leq (E_0 C_0)^{-a} (\eta_0 E_0 \rho_0^{-1})^{-b} \xi_0^c\, \Gamma$$

Notice generality: only smallness of $\frac{\varepsilon_0}{E_0}$, expressed via few large dimensionless parameters $E_0 C_0 \geq 1$, $\eta_0 E_0 \rho_0^{-1} \geq 1$, $\xi_0^{-1} \geq 1$

$$f_0(\mathbf{A}, \boldsymbol{\alpha}) = \sum_{\boldsymbol{\nu} \in \mathbb{Z}^\ell} e^{i\boldsymbol{\alpha}\cdot\boldsymbol{\nu}} f_{0,\boldsymbol{\nu}}(\mathbf{A}),$$

$$|\partial_{\mathbf{A}} f_{0,\boldsymbol{\nu}}(\mathbf{A})|_{\rho_0} \leq ||f_0||_{\rho_0,\xi_0} e^{-\xi_0|\boldsymbol{\nu}|}, \quad |i\boldsymbol{\nu} f_{0,\boldsymbol{\nu}}(\mathbf{A})| \leq \rho_0 ||f_0||_{\rho_0,\xi_0} e^{-\xi_0|\boldsymbol{\nu}|}$$

by the maximum principle. Therefore $f_0^{[>N_0]} \stackrel{def}{=} \sum_{|\boldsymbol{\nu}|>N_0} e^{i\boldsymbol{\nu}\cdot\boldsymbol{\alpha}} f_{\boldsymbol{\nu}}(\mathbf{A})$

$$||f_0^{[>N_0]}||_{\rho_0,\xi_0-\delta_0} \leq \sum_{|\boldsymbol{\nu}|>N_0} e^{-\delta_0|\boldsymbol{\nu}|}\big(\rho_0^{-1}|\boldsymbol{\nu}|\,|f_{0,\boldsymbol{\nu}}(\mathbf{A})| + |\partial_{\mathbf{A}} f_{\boldsymbol{\nu}}(\mathbf{A})|\big)$$

$$\leq \gamma_0 \varepsilon_0 \delta_0^{-\ell} e^{-\frac{1}{2}\delta_0 N_0} = \gamma_0 \varepsilon_0^2 C_0$$

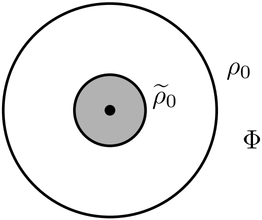
if $N_0 = \frac{2}{\delta_0} \log\,(C_0 \varepsilon_0 \delta_0^l)^{-1}$: dimensional or analyticity loss estimate.

Integrate the Hamilton Jacobi equation to "$O(\varepsilon_0^2)$", setting aside $f_0^{[\geq N_0]}$ in $f_0 = f_0^{[<N_0]} + f_0^{[\geq N_0]}$:

$$h_0(\mathbf{A}' + \partial_{\boldsymbol{\alpha}}\Phi_0(\mathbf{A}', \boldsymbol{\alpha})) + f_0^{[<N_0]}(\mathbf{A}' + \partial_{\boldsymbol{\alpha}}\Phi_0(\mathbf{A}', \boldsymbol{\alpha}), \boldsymbol{\alpha}) = \boldsymbol{\alpha} - \text{indep.}$$

The function Φ_0 is easily determined by solving $\boldsymbol{\omega}(\mathbf{A}') \cdot \boldsymbol{\partial}_{\boldsymbol{\alpha}}\Phi(\mathbf{A}', \boldsymbol{\alpha}) + f_0^{[<N_0]}(\mathbf{A}', \boldsymbol{\alpha}) = \overline{f}(\mathbf{A}')$ in $W(\widetilde{\rho}_0, \xi_0 - \delta_0)$,

δ_0 to be fixed:



$$\Phi_0(\mathbf{A}', \boldsymbol{\alpha}) = -\sum_{0<|\boldsymbol{\nu}|<N_0} \frac{f_{0,\boldsymbol{\nu}}(\mathbf{A}')}{i\boldsymbol{\omega}(\mathbf{A}') \cdot \boldsymbol{\nu}} e^{i\boldsymbol{\alpha}\cdot\boldsymbol{\nu}}$$

provided $\mathbf{A}$ is close to $\mathbf{A}_0$. This is implied by $\widetilde{\rho}_0$ small.

Choose $\widetilde{\rho}_0$ so that $\boldsymbol{\omega}(\mathbf{A}')\cdot\boldsymbol{\nu}| \ge \frac{1}{2C_0|\boldsymbol{\nu}|^\tau}$ for all $|\boldsymbol{\nu}| < N_0$:

$$|\boldsymbol{\omega}(\mathbf{A}')\cdot\boldsymbol{\nu}| = |\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}|\,\Big|1 - \frac{|(\boldsymbol{\omega}(\mathbf{A}') - \boldsymbol{\omega}(\mathbf{A}_0))\cdot\boldsymbol{\nu}}{\boldsymbol{\omega}(\mathbf{A}_0)\cdot\boldsymbol{\nu}}\Big| \ge \frac{1}{2C_0|\boldsymbol{\nu}|^\tau}$$

$\Rightarrow \widetilde{\rho}_0 = \rho_0\frac{1}{4E_0C_0N_0^{\tau+1}}$ (no loss as $E_0C_0 \ge 1$, and $N_0 > 2$ can be) because

$$\tfrac{|(\boldsymbol{\omega}(\mathbf{A}')-\boldsymbol{\omega}(\mathbf{A}_0))\cdot\boldsymbol{\nu}}{\boldsymbol{\omega}(\mathbf{A}_0)\cdot\boldsymbol{\nu}|} \le |\mathbf{A}' - \mathbf{A}_0|\max|\boldsymbol{\partial}^2 h(\mathbf{A}'')| \le \tfrac{E_0C_0N_0^{\ell+1}\widetilde{\rho}_0}{\rho_0-\widetilde{\rho}_0} \le \tfrac{1}{2}$$

Then Φ_0 (dimensionless) is of order $O(\varepsilon_0)$ in the precise sense

$$||\Phi_0||_{\widetilde{\rho}_0,\xi_0-\delta_0} \le \gamma_1\, E_0C_0N_0^{\ell+1}\delta_0^{-(\ell+\tau)}\varepsilon_0C_0$$

(derived from he bound on $\boldsymbol{\partial}_{\boldsymbol{\alpha}} f_{0,\boldsymbol{\nu}} \le \varepsilon_0\rho_0 e^{-\xi_0|\boldsymbol{\nu}|}$)

Motion in $\widetilde{W}(\widetilde{\rho}_0, \xi - \delta_0)$ better described in canonical coordinates $(\mathbf{A}', \boldsymbol{\alpha}')$:

$$\mathbf{A} = \mathbf{A}' + \boldsymbol{\partial}_{\boldsymbol{\alpha}} \Phi_0(\mathbf{A}', \boldsymbol{\alpha}), \qquad \boldsymbol{\alpha}' = \boldsymbol{\alpha} + \boldsymbol{\partial}_{\mathbf{A}'} \Phi(\mathbf{A}', \boldsymbol{\alpha})$$

To find $\mathbf{A} = \mathbf{A}' + \boldsymbol{\Xi}(\mathbf{A}', \boldsymbol{\alpha}')$ and $\boldsymbol{\alpha} = \boldsymbol{\alpha}' + \boldsymbol{\Delta}(\mathbf{A}', \boldsymbol{\alpha}')$ solve the implict functions with equal Jacobians $\boldsymbol{\partial}_{\mathbf{A}'} \boldsymbol{\partial}_{\boldsymbol{\alpha}} \Phi_0(\mathbf{A}', \boldsymbol{\alpha}')$ that can be bounded in $W(\frac{1}{2}\widetilde{\rho}_0, \xi_0 - 2\delta_0)$ (because $\|\Phi_0\|$ is bdd in $W(\widetilde{\rho}_0, \xi_0 - \delta_0)$).

Again by a dimensional bound (Cauchy's estimate), it is for instance

$$\max |\boldsymbol{\partial}_{\boldsymbol{\alpha}} \boldsymbol{\partial}_{A'} \Phi_0|_{\frac{1}{2}\widetilde{\rho}_0, \xi_0 - 2\delta_0} \le \gamma' ||\Phi||_{\widetilde{\rho}_0, \xi_0 - \delta_0} \frac{1}{\delta_0^{\ell}}$$

(here $\frac{1}{2}\widetilde{\rho}_0$ could be $\widetilde{\rho}_0$! and other better bounds possible too). So provided

$$\gamma_2 E_0 C_0 N_0^{\ell+1} \delta_0^{-(\ell+\tau+\ell)} \varepsilon_0 C_0 < 1$$

$\boldsymbol{\Xi}(\mathbf{A}', \boldsymbol{\alpha}') \equiv \boldsymbol{\partial}_{\boldsymbol{\alpha}} \Phi_0(\mathbf{A}', \boldsymbol{\alpha})$ and $\boldsymbol{\Delta}(\mathbf{A}', \boldsymbol{\alpha}') \equiv \boldsymbol{\partial}_{\mathbf{A}'} \Phi_0(\mathbf{A}', \boldsymbol{\alpha})$ defined

$$\max_{W(\frac12\widetilde\rho_0,\xi_0-2\delta_0)}|\Xi|\le\widetilde\rho_0||\Phi||_{\widetilde\rho_0,\xi_0-\delta_0},\qquad \max_{W(\frac12\widetilde\rho_0,\xi_0-2\delta_0)}|\Delta|\le||\Phi||_{\widetilde\rho_0,\xi_0-\delta_0}$$

Therefore if $(\mathbf{A}',\boldsymbol{\alpha}')\in W(\frac14\widetilde\rho_0,\xi_0-3\delta_0)$ the corresponding points $(\mathbf{A},\boldsymbol{\alpha})$ are in $W(\frac12\widetilde\rho_0,\xi_0-2\delta_0)$ provided

$$\widetilde\rho_0||\Phi||_{\widetilde\rho_0,\xi_0-\delta_0}\le\frac14\widetilde\rho_0,\qquad ||\Phi||_{\widetilde\rho_0,\xi_0-\delta_0}<\delta_0$$

which, from $||\Phi||_{\widetilde\rho_0,\xi_0-\delta_0}\le\gamma_1E_0C_0N_0^{\ell+1}\delta_0^{-(\ell+\tau)}\varepsilon_0C_0$, is implied by

$$\gamma_3E_0C_0N_0^{\ell+1}\delta_0^{-(2\ell+\tau)}\varepsilon_0C_0<1$$

if γ_3 is large enough. The last condition implies all the earlier ones.

Hence motions starting in the (much) smaller $W(\frac{1}{4}\widetilde{\rho}_0, \xi_0 - 3\delta_0)$ can be described in the $(\mathbf{A}', \boldsymbol{\alpha}')$ coordinates (as long as they remain in W) by the Hamiltonian

$$h_0(\mathbf{A}' + \Xi(\mathbf{A}', \boldsymbol{\alpha}')) + f_0(\mathbf{A}' + \Xi(\mathbf{A}', \boldsymbol{\alpha}'), \boldsymbol{\alpha}' + \boldsymbol{\Delta}(\mathbf{A}', \boldsymbol{\alpha}'))$$

under the (above) condition

$$\gamma_3 E_0 C_0 N_0^{\ell+1} \delta_0^{-(\ell+\tau+1)} \varepsilon_0 C_0 < 1$$

It is natural to write $h_1(\mathbf{A}') + f_1(\mathbf{A}', \boldsymbol{\alpha}')$ with

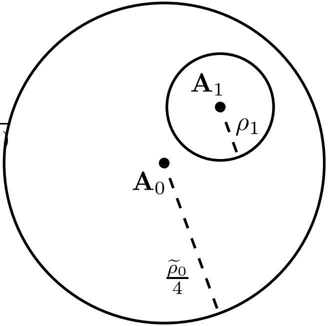
$$h_1(\mathbf{A}') \stackrel{def}{=} h_0(\mathbf{A}') + \overline{f}_0(\mathbf{A}')$$

However $\boldsymbol{\omega}(\mathbf{A}') \stackrel{def}{=} \boldsymbol{\partial}_{\mathbf{A}'} h_1(\mathbf{A}')$ evaluated at $\mathbf{A}_0$ is no longer $\boldsymbol{\omega}_0$.

Look for $\mathbf{A}_1$ s. t. $\boldsymbol{\partial}_{\mathbf{A}_1} h_1(\mathbf{A}_1) = \boldsymbol{\omega}_0$

and for $\rho_1 = \frac{\widetilde{\rho}_0}{8} < \frac{\rho_0}{2}$

$\widetilde{\rho}_0 = \frac{\rho_0}{(E_0 C_0 N_0^{\ell+1} \delta_0^{-(\ell+\tau)})}$



The equation for $\mathbf{A}_1 = \mathbf{A}_0 + \mathbf{a}$ is $\boldsymbol{\omega}_0(\mathbf{A}_1) + \boldsymbol{\partial}_{\mathbf{A}_1} \overline{f}_0(\mathbf{A}_1) = \boldsymbol{\omega}_0$

$$\mathbf{a} + (\boldsymbol{\partial}^2_{\mathbf{A}} h_0(\mathbf{A}_0))^{-1} [\partial_{\mathbf{A}} h_0(\mathbf{A}_0 + \mathbf{a}) - \boldsymbol{\omega}_0 - \boldsymbol{\partial}^2_{\mathbf{A}} h_0(\mathbf{A}_0)\mathbf{a}] + \partial_{\mathbf{A}} \overline{f}_{\mathbf{0}}(\mathbf{A}_0 + \mathbf{a}) = \mathbf{0}$$

$$\mathbf{a} + \mathbf{n}(\mathbf{a}) = \mathbf{0}$$

$$\max |\mathbf{n}(\mathbf{a})| \le \gamma_4 \eta_0 (E_0 \frac{|\mathbf{a}|^2}{\rho_0^2} + \varepsilon_0), \quad \max |\partial_{\mathbf{a}} \mathbf{n}(\mathbf{a})| \le \gamma_4 \eta_0 (E_0 \frac{\widetilde{\rho}_0^2}{\rho_0^2} + \varepsilon_0) \frac{1}{\rho_0}$$

$$\max|\mathbf{n}(\mathbf{a})| \le \gamma_4\eta_0(E_0\frac{\widetilde{\rho}_0^2}{\rho_0^2}+\varepsilon_0), \quad \gamma_4\max|\partial_{\mathbf{a}}\mathbf{n}(\mathbf{a})| \le \eta_0(E_0\frac{|\mathbf{a}|^2}{\rho_0^2}+\varepsilon_0)\frac{1}{\rho_0}$$

Hence $|\mathbf{a}| < \varepsilon_0\eta_0$ if $\gamma_4\frac{E_0}{\rho_0^2}\eta_0^2\varepsilon_0^2 < \varepsilon_0 \Leftarrow \gamma_4\,(\eta_0 E_0\rho_0^{-1})(\eta_0\varepsilon_0\rho_0^{-1}) < 1$ provided Jacobian $|\boldsymbol{\partial}_{\mathbf{a}}\mathbf{n}(\mathbf{a})| < 1$: same condition if γ_4 large enough

$$\gamma_4\,(\eta_0 E_0\rho_0^{-1})(\eta_0\varepsilon_0\rho_0^{-1}) < 1.$$

Under earlier condition $(\gamma_3 E_0 C_0 N_0^{\ell+1}\delta_0^{-(\ell+\tau+1)}\varepsilon_0 C_0 < 1)$ we also need $\varepsilon_0\eta_0 < \frac{1}{8}\widetilde{\rho}_0$. All of them are implied by

$$\gamma_5\frac{\varepsilon_0}{E_0}(E_0\eta_0\rho_0^{-1})(E_0C_0)^2N_0^{\ell+1}\delta_0^{-(\ell+\tau+1)} < 1$$

Let $\rho_1 \stackrel{def}{=} \frac{1}{8}\widetilde{\rho}_0, \xi_1 = \xi_0 - 4\delta_0,\ H_1 = h_1(\mathbf{A}') + f_1(\mathbf{a}', \boldsymbol{\alpha}')$ in $W(\rho_1, \xi_1)$

At $\mathbf{A}_1$ it is $\partial_{\mathbf{A}'} h_1(\mathbf{A}_1) \equiv \boldsymbol{\omega}_0$ and f_1 is

$$
\begin{aligned}
f_1(\mathbf{A}', \boldsymbol{\alpha}') = &\big[h_0(\mathbf{A}' + \Xi(\mathbf{A}', \boldsymbol{\alpha}')) - h_0(\mathbf{A}') - \boldsymbol{\omega}(\mathbf{A}') \cdot \Xi(\mathbf{A}', \boldsymbol{\alpha}')\big] \\
&+ \big\{\boldsymbol{\omega}(\mathbf{A}') \cdot \Xi(\mathbf{A}', \boldsymbol{\alpha}') + f^{[<N_0]}(\mathbf{A}', \boldsymbol{\alpha}' + \boldsymbol{\Delta}(\mathbf{A}', \boldsymbol{\alpha}')) - \overline{f}_0(\mathbf{A}')\big\} \\
&+ \big[f^{[<N_0]}(\mathbf{A}' + \Xi(\mathbf{A}', \boldsymbol{\alpha}'), \boldsymbol{\alpha}' + \boldsymbol{\Delta}(\mathbf{A}', \boldsymbol{\alpha}')) \\
&- f^{[<N_0]}(\mathbf{A}', \boldsymbol{\alpha}' + \Delta(\mathbf{A}', \boldsymbol{\alpha}'))\big] + f^{[\geq N_0]}(\mathbf{A}' + \Xi(\mathbf{A}', \boldsymbol{\alpha}'))
\end{aligned}
$$

the "red" term is 0 by definition of Φ_0, $\boldsymbol{\Xi}, \boldsymbol{\Delta}$:

$$
\boldsymbol{\Xi}(\mathbf{A}', \boldsymbol{\alpha}') \equiv \boldsymbol{\partial}_{\boldsymbol{\alpha}'} \Phi_0(\mathbf{a}', \boldsymbol{\alpha}), \qquad \boldsymbol{\Delta}(\mathbf{A}', \boldsymbol{\alpha}') \equiv \boldsymbol{\partial}_{\mathbf{A}'} \Phi_0(\mathbf{a}', \boldsymbol{\alpha})
$$

with Φ_0 solution of HJ:

$$
\boldsymbol{\omega}_0 \cdot \boldsymbol{\partial}_{\boldsymbol{\alpha}} \Phi_0(\mathbf{A}', \boldsymbol{\alpha}) + f_0^{[<N_0]}(\mathbf{A}', \boldsymbol{\alpha}) - \overline{f}_0(\mathbf{A}') = 0
$$

To iterate it is necessary to estimate the sizes E_1, ε_1 of f_1, E_1

Dimensional estimates possible because $\rho_1 = \frac{1}{8}\widetilde{\rho}_0$, $\xi_1 = \xi_0 - 3\delta_0 - \delta_0$ are smaller than the domains of definition of f_1, h_1

$$E_1 \stackrel{def}{=} \max_{W(\rho_1,\xi_1)} |\boldsymbol{\partial}_{\mathbf{A}'} h_1(\mathbf{A}')| \le E_0 + \varepsilon_0$$

Call f^I, f^{II}, f^{III} the terms remaining in f_1. Then

$$\widetilde{\rho}_0 = \rho_0 \frac{1}{E_0 C_0 N_0^{\ell+1}}, \qquad \rho_1 = \frac{1}{8}\widetilde{\rho}_0$$

$$||\Phi||_{\widetilde{\rho}_0, \xi_0-\delta_0} \le 2E_0 C_0 N_0^{\ell+1} \delta_0^{-(\ell+\tau)} \varepsilon_0 C_0$$

$\Rightarrow ||f^I||_{\rho_1,\xi_1}$

$$\le \gamma_5 \frac{1}{\rho_1} [\frac{E_0}{\rho_0} (\widetilde{\rho}_0 ||\Phi_0\|)^2 \delta_0^{-1}] \le \gamma_5 \varepsilon_0 \frac{\varepsilon_0}{E_0} (E_0 C_0)^4 N_0^{2(\ell+1)} \delta_0^{-2(\ell+\tau)}$$

Likewise $||f^{II}||_{\rho_1,\xi_1} \le \gamma_5 \frac{\varepsilon_0}{\delta_0^\ell} \gamma_1 (E_0 C_0) N_0^{\ell+1} \delta_0^{-(\ell+\tau+1)}$, and $||f^{III}||_{\rho_1,\xi_1} \le \frac{\varepsilon_0^2}{E_0}$

Conclusion: if $N_0 = 2\delta_0^{-1} \log(\varepsilon_0 C_0 \delta_0^{-\ell-1})^{-1}$

$$\begin{aligned}
&H_1 = h_1 + f_1, \qquad \text{analytic in } W_{\rho_1,\xi_1} \\
&E_1 = E_0 + \varepsilon_0 \\
&\varepsilon_1 = \gamma_6 \varepsilon_0 \frac{\varepsilon_0}{E_0} (E_0 C_0)^4 N_0^{2(\ell+1)} \delta_0^{-2(\ell+\tau)} \\
&\eta_1 = \eta_0 (1 + \gamma_6 \sqrt{\frac{\varepsilon_0}{E_0}} (\eta_0 E_0 \rho_0^{-1}) (E_0 C_0)^2 N_0^{\ell+1}) \\
&\xi_1 = \xi_0 - 4\delta_0, \qquad \rho_1 = \frac{1}{8} \rho_0 \frac{1}{\delta_0^{-1} \log(\varepsilon_0 C_0 \delta_0^{-(\ell+1)})}
\end{aligned}$$

provided

$$\gamma_7 \frac{\varepsilon_0}{E_0} (E_0 \eta_0 \rho_0^{-1}) (E_0 C_0)^2 N_0^{\ell+1} \delta_0^{-(\ell+\tau+1)} < 1$$

To iterate choose $\delta_k = \xi_0 (16(1+k^2))^{-1} \Rightarrow \frac{\xi_0}{2} > \xi_0 - \sum_k \delta_k$

Simplify by imposing $\sqrt{\frac{\varepsilon_k}{E_k}}(E_k C_0)^4 N_k^{2(\ell+1)} \delta_k^{-2(\ell+\tau)} < 1$

$$E_{k+1} = E_k + \varepsilon_k, \qquad \varepsilon_{k+1} = \varepsilon_k \sqrt{\frac{\varepsilon_k}{E_k}}$$

Hence (suppose for simpl. $\frac{\varepsilon_0}{E_0} < \frac{1}{2}$)

$$\frac{\varepsilon_k}{E_k} \leq (\frac{\varepsilon_0}{E_0})^{(3k/2)^k}, E_k \leq E_0 \prod_{k=0}^{\infty}(1 + (\frac{\varepsilon_0}{E_0})^{(3k/2)^k}) < 2E_0$$

$$\text{if} \quad \gamma_8 \sqrt{\frac{\varepsilon_k}{E_k}}(E_k \eta_k \rho_k^{-1})(E_k C_0)^4 N_k^{2(\ell+1)} \delta_k^{-2(\ell+\tau+1)} < 1$$

Therefore

$$\rho_{k+1} \geq \frac{\gamma_9 \, \rho_k}{(1+k^2)(-\xi_0^{-2} + \log(16(1+k^2)) - \log(2E_0C_0) - (\frac{3}{2})^k \log \frac{\varepsilon_0}{E_0})}$$

$$\geq \gamma_{10} \frac{\xi_0^2}{-\log \frac{\varepsilon_0}{E_0}} (\frac{2}{3})^{2k} \rho_k \geq (\gamma_{10} \frac{\xi_0^2}{-\log \frac{\varepsilon_0}{E_0}})^k (\frac{2}{3})^{2k^2} \rho_0$$

Therefore if

$$\gamma_{10}\sqrt[4]{\frac{\varepsilon_k}{E_k}}(E_kC_0)^4N_k^{2(\ell+1)}\delta_k^{-2(\ell+\tau)}<1$$

$$\eta_k\le\eta_0\prod_k\sqrt[4]{\frac{\varepsilon_k}{E_k}}\le 2\eta_0$$

Hence

$$E_k\le 2E_0,\ \eta_k\le 2\eta_0,\ \varepsilon_k\le\varepsilon_0\Big(\frac{\varepsilon_0}{E_0}\Big)^{(\frac{3}{2})^k}$$

$$\xi_k>\frac{\xi_0}{2},\ \rho_k\ge\rho_0\big(\frac{\xi_0^2}{-\log\frac{\varepsilon_0}{E_0}}\big)^k(\frac{2}{3})^{k^2}$$

if

$$\gamma_{11}\sqrt[4]{\frac{\varepsilon_k}{E_k}}(E_k\eta_k\rho_k^{-1})(E_kC_0)^4N_k^{2(\ell+1)}\delta_k^{-2(\ell+\tau+1)}<1$$

The condition

$$\gamma_{11}\sqrt[4]{\frac{\varepsilon_k}{E_k}}(E_k\eta_k\rho_k^{-1})(E_kC_0)^4N_k^{2(\ell+1)}\delta_k^{-2(\ell+\tau+1)}<1$$

can fail to be satisfied only for finitely many values of k if $\varepsilon_k, E_k, \eta_k, \rho_k, \xi_k$ are as

$$E_k\le 2E_0,\ \eta_k\le 2\eta_0,\ \varepsilon_k\le\varepsilon_0\Big(\frac{\varepsilon_0}{E_0}\Big)^{(\frac{3}{2})^k}$$

$$\xi_k>\frac{\xi_0}{2},\ \rho_k\ge\rho_0\big(\frac{\xi_0^2}{-\log\frac{\varepsilon_0}{E_0}}\big)^k(\frac{2}{3})^{k^2}$$

Hence given E_0, ρ_0, ξ_0 ε_0 will be imposed to be so small that the condition also holds for $k\le k_0$ $\Rightarrow$ the condition is simply

$$\exists\ \Gamma, a, b, c>0,\qquad \Gamma\,(\frac{\varepsilon_0}{E_0})(E_0C_0)^a(\eta_0E_0\rho_0^{-1})^b\xi_0^{-c}<1$$

is the only condition for the indefinite iteration of the construction.

The successive cannical maps $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_{k-1}$ map W_{ρ_k,ξ_k} into W_{ρ_0,ξ_0} and the composition $\mathcal{C}_0 \circ \cdots \circ \mathcal{C}_{k-1}$ and $\mathcal{C}_k$ is close within $||\Phi_k||$ to the identity, *i.e.* within $\rho_k||\Phi_k||$ in the actions $||\Phi_k||$ in the angles and $||\Phi_k|| = O(\Big(\frac{\varepsilon_0}{E_0}\Big)^{(3/2)^k})$. .

Let initial data $(\mathbf{A}_k, \boldsymbol{\alpha}'_0) \in W_{\rho_k,\xi_k}$ then their motion is up to an error $||f_k||t \sim O(\varepsilon_0^{(3/2)^k})$ given by

$$\mathbf{A}' = \mathbf{A}_k, \boldsymbol{\alpha}' = \boldsymbol{\alpha}'_0 + \boldsymbol{\omega}_0 t$$

therefore fixed t, for k large, the point moves within W_{ρ_k,ξ_k} where the canonical map $\mathcal{C}_0 \circ \cdot \circ \mathcal{C}_{k-1} \xrightarrow[k\to\infty]{} \mathcal{C}$ is defined.

The limit map is defined only for $\mathbf{A}_\infty \times \mathbb{T}^\ell \equiv W_{0,\xi_0/2}$ which in the original variable is a torus of dimension ℓ and motion on it is quasi periodic with spectrum $\boldsymbol{\omega}_0$.

Resummation techniques and renormalized series

Transformations of (possibly divergent) power series in ε into convergent series of functions that depend in a nontrivial way on ε.

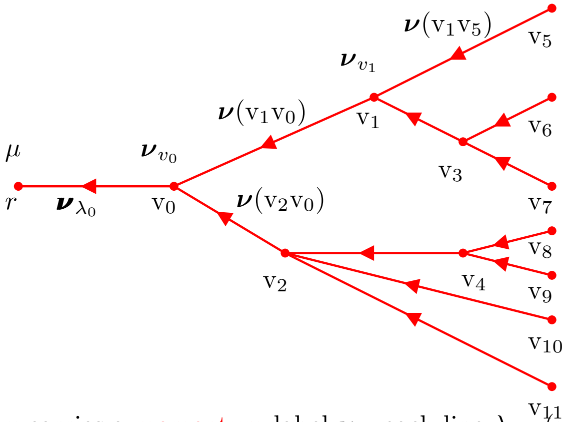
$$\text{Example: tori for } H(\mathbf{A},\boldsymbol{\alpha}) = \frac{1}{2}\mathbf{A}^2 + \varepsilon f(\boldsymbol{\alpha})$$

KAM results are analytic in ε and no resummation is needed. However they provide an interesting arena to explain techniques (*e.g.* like the perturbation series for resonant quasi periodic motions).

Solution "approximated to order k"

$$\mathbf{h}^{(\le k)}(\boldsymbol{\psi},\varepsilon) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^\ell} e^{i\boldsymbol{\nu}\cdot\boldsymbol{\psi}}\mathbf{h}_{\boldsymbol{\nu}}^{(\le k)}(\varepsilon), \qquad \mathbf{h}_{\boldsymbol{\nu}}^{(\le k)}(\varepsilon) = \sum_{k'=1}^{k} \varepsilon^{k'}\mathbf{h}_{\boldsymbol{\nu}}^{(k')},$$

where $\mathbf{h}_{\mathbf{0}}^{(k)} = \mathbf{0}$ and $\mathbf{h}_{\boldsymbol{\nu}}^{(k)}$ are the Lindstedt series coefficients in tree representation.



Each node v carries a momentum label $\boldsymbol{\nu}_v$; each line $\lambda = (v'v)$ from v' to v carries a current $\boldsymbol{\nu}_\lambda \equiv \boldsymbol{\nu}(v'v) = \sum_{w<v} \boldsymbol{\nu}_w = \sum_{w\le v'} \boldsymbol{\nu}_w$

$$\mathrm{Val}(\theta) = \prod_{nodes\ v} \frac{-f_{\boldsymbol{\nu}_v}}{s_v!} \prod_{lines\ \lambda=(v'v)} \frac{\boldsymbol{\nu}_v \cdot \boldsymbol{\nu}_{v'}}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu}(v'v))^2}$$

The functions $\mathbf{h}_{\boldsymbol{\nu}}^{[k]}$ are expressed as sums over all trees with k lines

A line factor is regarded as $\boldsymbol{\nu}_v G(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\lambda)\boldsymbol{\nu}_{v'}$ where $G_\lambda(x)$ is a matrix $G_\lambda(x)_{i,i'} = \frac{\delta_{i,i'}}{x^2}$, "propagator".

The propagator has (trivially) the properties $G^T(-x) = G^*(x) = G(x)$

Scales and clusters: as in the theory of the Lindstedt series

If T is a self-energy cluster in a tree θ, $V(T)$ is the set of nodes in T, $\Lambda(T)$ the set of lines in T, k_T is the number of nodes in T (ie $k_T = |V(T)|$), and λ_T^1 and λ_T^2 the lines exiting and entering T.

θ be a tree $\theta \in \Theta_{k,\boldsymbol{\nu}}$, of order k and momentum $\boldsymbol{\nu}$, with a s.e.c T.

Let θ be a tree $\theta \in \Theta_{k,\boldsymbol{\nu},j}$, of order k and momentum $\boldsymbol{\nu}$, with a s.e.c T.

Let $\theta_0 = \theta \setminus T$ the nodes and branches of θ outside T (of course θ_0 is not a tree), we define $V(\theta_0) = V(\theta) \setminus V(T)$ and $\Lambda(\theta_0) = \Lambda(\theta) \setminus \Lambda(T)$.

Consider all trees s.t. θ_0 outside the s.e.c. is the same, while the s.e.c. itself can be arbitrary, *i.e.* T can be replaced by any other s.e.c. T'.

Define, as a formal power series the matrix,

$$M(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}; \varepsilon) = \sum_{\theta = \theta_0 \cup T'} \mathcal{V}_{T'}(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}), \qquad \text{where}$$

$$\mathcal{V}_T(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}) \stackrel{def}{=} \varepsilon^{k_T} \Big(\prod_{v \in V(T)} F_v \Big) \Big(\prod_{\lambda \in \Lambda(T)} G_\lambda \Big),$$

F_v are tensors of nodes v: $F_v = \frac{1}{s_v!} f_{\boldsymbol{\nu}_v} \nu_{v,j_v} \prod_{i=1}^{s_v} \nu_{v_i,j_i}$; sum is over θ s.t. $\theta \setminus T$ fixed to θ_0 and $\boldsymbol{\nu}_v$, $v \in V(T)$ satisfy conditions defining s.e.c. Tensor labels and corresponding propagartors labels are contracted, Algebraic identity of formal power series (propagators symmetries)

(1) $\left(M(x;\varepsilon)\right)^T = M(-x;\varepsilon)$, *and*
(2) $\left(M(x;\varepsilon)\right)^* = M(x;\varepsilon)$*;*

Check: Consider a graph computed with propagators verifying the properties (1),(2), trivially valid in our case.

Given a s.e.c. T with momentum $\boldsymbol{\nu}$ on entering branch λ_T^2, call $\mathcal{P}$ the path connecting the exiting branch λ_T^1 to the entering branch λ_T^2.

Consider the s.e.c. T' obtained by taking λ_T^1 as entering branch and λ_T^2 as exiting branch and by taking $-\boldsymbol{\nu}$ as momentum flowing through the (new) entering branch λ_T^1.

$\Rightarrow$ arrows of branches along $\mathcal{P}$ change orientation, while subtrees (internal to T) rooted in $\mathcal{P}$ are unchanged.

Momenta of branches in $\mathcal{P}$ change sign, while the others do not.

All propagators G_λ of branches $\lambda \in \mathcal{P}$ are transformed into G_λ^T, hence the ij entry of $M(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}; \varepsilon)$ equals the entry ji of $M(-\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}; \varepsilon)$ $\Rightarrow$ property (1).

Given a self-energy graph T, consider also the self-energy graph T' obtained by reversing the sign of the mode labels of the nodes $v \in V(T)$, and by swapping the entering branch with the exiting one.

Again arrows of branches along $\mathcal{P}$ are reversed, while all the subtrees (internal to T) rooted in $\mathcal{P}$ are unchanged.

The complex conjugate of $\mathcal{V}_{T'}(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu})$ equals $\mathcal{V}_T(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu})$, by using the form of the node factors, and the fact that one has $f_{\boldsymbol{\nu}}^* = f_{-\boldsymbol{\nu}}$ (as $f(\boldsymbol{\alpha})$ is real) and $G^\dagger(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}) = G(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu})$ $\Rightarrow$ property (2).

The symmetries have been obtained without using the exact form of the naked propagator: only exploiting that it enjoys properties (1),(2). Thus it has more general validity.

The function $M(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}; \varepsilon)$ depends on ε but, by construction, it is independent of θ_0: hence we can rewrite as

$$M(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}; \varepsilon) = \sum_{T'} \mathcal{V}_{T'}(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}),$$

where the sum is over all self-egnergy graphs of order $k \geq 1$ with external branches with momentum $\boldsymbol{\nu}$.

By definition of s.e.c., if $2^{n-1} < C|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}| \leq 2^n$, the sum is restricted to s.e.c. T' on scale $n_{T'} \geq n + 3$.

In other words: for $\lambda \in T$ define n_λ^0 s.t., if $\boldsymbol{\nu}_\lambda^0$ *would* flow on the line λ *after setting to* $\boldsymbol{\nu} = \mathbf{0}$ the entering line momentum it is

$$2^{n_\lambda^0 - 1} \le C \left| \boldsymbol{\omega} \cdot \boldsymbol{\nu}_\lambda^0 \right| < 2^{n_\lambda^0},$$

then, for all branches $\lambda \in \Lambda(\theta)$ one has $n_\lambda = n_\lambda^0$ because, for the same reasons discussed in the theory of the Lindstedt series (*i.e.* shifting the branches external to the s.e.c. of a θ, scale labels n_λ of all lines $\lambda \in \Lambda(\theta)$ do not change).

Denote $\Theta_{k,\boldsymbol{\nu}}^{\mathcal{R}}$ trees of order k without s.e.c. and root current $\boldsymbol{\nu}$: renormalized trees.

Dressed propagators will be the matrices ($d \ge 1$)

$$\overline{G}_\lambda^{[0]} = \left(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\lambda\right)^{-2}, \qquad \overline{G}_\lambda^{[d]} = \left[\left(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\lambda\right)^2 - M^{[d]}(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\lambda; \varepsilon) \right]^{-1},$$

defined recursively

$\{M^{[d]}(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}; \varepsilon)\}_{d \in \mathbb{N}}$ is sum of the values of all renormalized s.e.c that can be inserted on a line of momentum $\boldsymbol{\nu}$ computed via the propagators $\overline{G}_\lambda^{[d-1]}$, *i.e.* as (set $M^{[0]}(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}; \varepsilon) \equiv 0$)

$$M^{[d]}(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}; \varepsilon) = \sum_{\text{renormalized}\, T} \mathcal{V}_T^{[d]}(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}),$$

$$\mathcal{V}_T^{[d]}(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}) = \varepsilon^{k_T} \Big(\prod_{v \in V(T)} F_v \Big) \Big(\prod_{\lambda \in \Lambda(T)} \overline{G}_\lambda^{[d-1]} \Big);$$

By construction evaluating renormalized trees with the propagators $\overline{G}_\lambda^{[d-1]}$ and summing, the sum $\mathbf{h}_{\boldsymbol{\nu}}^{[d]}(\varepsilon)$ of the values of all s.e.c. containing only $0 \le p < d$ s.e.c. is reconstructed.

Hence the power series defining the functions $\mathbf{h}_{\boldsymbol{\nu}}^{[d]}(\varepsilon)$, truncated at order $k < d$, coincide with the functions $\mathbf{h}_{\boldsymbol{\nu}}^{(\le k)}(\varepsilon)$ obtained by truncating to order k the (formal) series for $\mathbf{h}_{\boldsymbol{\nu}}(\varepsilon)$.

By induction $\overline{G}^{[d]}(x; \varepsilon)^T = \overline{G}^{[d]}(-x; \varepsilon)$.

If $\overline{G}^{[d]}(x;\varepsilon) \le \frac{B}{x^2}$ it follows from the Siegel-Bryuno estimates that the series defining $\mathbf{h}_{\boldsymbol{\nu}}^{[d]}$ is uniformly convergent for $|\varepsilon|$ small.

The analysis of the cancellations can be reinterpreted to yield

Lemma: (Symmetry and cancellations properties)
(1) $M^{[d]}(\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu};\varepsilon)$ satisfy $M^{[d]}(x;\varepsilon)^T = M^{[d]}(-x;\varepsilon)$.

(2) $M^{[-q]}(x;\varepsilon)$ is restriction to $x = \boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}$ with $\boldsymbol{\nu}$ of scale $\le q$ of a function analytic in $|x| \le 2^q$ if ε is small enough, and satisfies

$$\left\| M^{[d]}(x;\varepsilon) \right\| \le D x^2 |\varepsilon|^2,$$

for all $d \in \mathbb{N}$ and for a d–independent constant D.

(3) Hence $\overline{G}_\lambda^{[d]}$ verify $|\overline{G}^{[d]}(x;\varepsilon)| \le \frac{B}{x^2}$, $\forall d \ge 1$.

(4) $|M^{[d+1]}(x;\varepsilon) - M^{[d]}(x;\varepsilon)| \le \widehat{B}_1\,(\widehat{B}_2)^d\,\varepsilon^{2d}\,x^2$, for $0 < \hat{B}_1, \hat{B}_2 < \infty$, $|\varepsilon| < \varepsilon_0$ small enough.

Hence $\exists\ M^{[\infty]}(x;\varepsilon) \stackrel{def}{=} \lim_{d\to\infty} M^{[d]}(x;\varepsilon)$ and the fully dressed propagators

$$G^{[\infty]}(x;\varepsilon)\overline{(}x^2 - M^{[\infty]}(x;\varepsilon))^{-1}$$

If $\overline{\mathbf{h}}^{[\infty]}(\boldsymbol{\psi},\varepsilon)$ is the sum of the renormalized trees evaluated with the fully renormalized propagators, then order by order in ε

$$\overline{\mathbf{h}}^{[\infty]}(\boldsymbol{\psi},\varepsilon) \equiv \lim_{d\to\infty} \mathbf{h}^{[d]}(\boldsymbol{\psi};\varepsilon) = \mathbf{h}(\boldsymbol{\psi};\varepsilon)$$

where $\mathbf{h}(\boldsymbol{\psi};\varepsilon)$ is the formal Lindstedt power series. The function $\overline{\mathbf{h}}^{[\infty]}(\boldsymbol{\psi},\varepsilon)$ solves, therefore, the equations of motion.

Resonances and low dimensional tori

$$H(\mathbf{A},\boldsymbol{\alpha}) = \boldsymbol{\omega}_0\cdot\mathbf{A} + \frac{1}{2}\mathbf{A}\cdot\mathbf{A} + \frac{1}{2}\mathbf{B}\cdot\mathbf{B} + \varepsilon f(\boldsymbol{\alpha},\boldsymbol{\beta}),$$

$(\boldsymbol{\alpha},\mathbf{A}) \in \mathbb{T}^r\times\mathbb{R}^r$ and $(\boldsymbol{\beta},\mathbf{B}) \in \mathbb{T}^s\times\mathbb{R}^s$ conjugated variables, $\boldsymbol{\omega}_0$ in $\mathbb{R}^r$ satisfies $C|\boldsymbol{\omega}_0\cdot\boldsymbol{\nu}| > |\boldsymbol{\nu}|^{-\tau},\ \forall\boldsymbol{\nu}\in\mathbb{Z}^r\setminus\{\mathbf{0}\}$, $C>0,\tau\ge r-1$

Resonant motion = quasi periodic motion with $r<\ell$ frequencies

Example $\boldsymbol{\alpha}(t) = \boldsymbol{\alpha}_0 + \boldsymbol{\omega}_0 t,\ \boldsymbol{\beta}(t) = \boldsymbol{\beta}_0$ for $\varepsilon = 0$.

For $\varepsilon \neq 0$

$$\begin{aligned}\boldsymbol{\alpha}(t) &= \boldsymbol{\psi}(t) + \mathbf{a}(\boldsymbol{\psi}(t),\boldsymbol{\beta}_0;\varepsilon),\\ \boldsymbol{\beta}(t) &= \boldsymbol{\beta}_0 + \mathbf{b}(\boldsymbol{\psi}(t),\boldsymbol{\beta}_0;\varepsilon),\end{aligned}\qquad \boldsymbol{\psi}(t) = \boldsymbol{\psi}_0 + \boldsymbol{\omega}_0 t$$

Necessary: $\boldsymbol{\beta}_0$ s.t. $\boldsymbol{\partial}_{\boldsymbol{\beta}}\overline{f}(\boldsymbol{\beta}_0) = \mathbf{0}$

$\overline{f}(\boldsymbol{\beta}) \stackrel{def}{=} \frac{1}{(2\pi)^r}\int f(\boldsymbol{\alpha},\boldsymbol{\beta})d^r\boldsymbol{\alpha}.$

Consider $\ddot{\boldsymbol{\alpha}} = -\boldsymbol{\partial}_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha}, \boldsymbol{\beta})$, $\ddot{\boldsymbol{\beta}} = -\boldsymbol{\partial}_{\boldsymbol{\beta}} f(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and $|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}| > \frac{1}{C|\boldsymbol{\nu}|^\tau}$ and $\boldsymbol{\beta}_0$ such that

$$\partial_{\boldsymbol{\beta}} f_{\mathbf{0}}(\boldsymbol{\beta}_0) = \mathbf{0}, \qquad \partial^2_{\boldsymbol{\beta}} f_{\mathbf{0}}(\boldsymbol{\beta}_0) \textit{ is negative definite.}$$

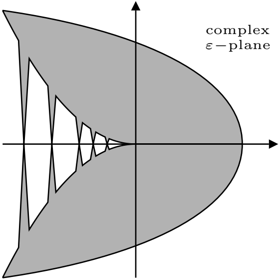
$\exists$ and, $\forall \varepsilon \in (0, \varepsilon_0)$,

$$\begin{aligned} \boldsymbol{\alpha}(t) &= \boldsymbol{\psi}(t) + \mathbf{a}(\boldsymbol{\psi}(t), \boldsymbol{\beta}_0; \varepsilon), \\ \boldsymbol{\beta}(t) &= \boldsymbol{\beta}_0 + \mathbf{b}(\boldsymbol{\psi}(t), \boldsymbol{\beta}_0; \varepsilon), \end{aligned} \qquad \boldsymbol{\psi}(t) = \boldsymbol{\psi}_0 + \boldsymbol{\omega}_0 t$$

two functions $\mathbf{a}(\boldsymbol{\psi}, \boldsymbol{\beta}_0; \varepsilon)$ and $\mathbf{b}(\boldsymbol{\psi}, \boldsymbol{\beta}_0; \varepsilon)$, real analytic in $\boldsymbol{\psi} \in \mathbb{T}^r$, such that is a solution with $\dot{\boldsymbol{\psi}} = \boldsymbol{\omega}_0$. Moreover $\mathbf{a}(\boldsymbol{\psi}, \boldsymbol{\beta}_0; \varepsilon)$ and $\mathbf{b}(\boldsymbol{\psi}, \boldsymbol{\beta}_0; \varepsilon)$ are analytic in ε for $\varepsilon \in (0, \varepsilon_0)$ and divisible by ε.

The functions $\mathbf{a}, \mathbf{b}$ exist and are abalytic in a comlex domain which touches the negative ε axis on a “Cantor set” with $\varepsilon = 0$ as a density point: at its points $\mathbf{a}, \mathbf{b}$ are real and give a solution.

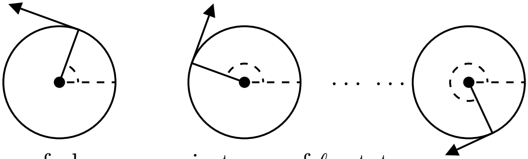
The solutions are called hyperbolic ($\varepsilon > 0$) and elliptic ($\varepsilon < 0$) resonances.



The technique is based on the resummations method just described.

However there is no proof, so far, that the functions $\mathbf{a}, \mathbf{b}$ are not analytic at the origin!

Hyperbolic resonances



Representation of phase space in terms of ℓ rotators.

Existence of a formal solution as a power series in ε:

$$(\mathbf{a}, \mathbf{b}) = \mathbf{h} = \sum_{k=1}^{\infty} \varepsilon^k \mathbf{h}^{(k)}$$

No convergence in general (?): **however**

Idea: "There are no divergent series". Hence look for **sum rules**

Split $\mathbf{h}^{(k)}$ as a sum of many terms and recombine them to obtain an absolutely convergent series.

In doing this we shall be forced to sum divergent series by giving their sum by a prescription. A typical exemple

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, \qquad z \neq 1$$

even when $|z| > 1$!.

Not "harmless": for instance it means that we are going to use:

$$\sum_{k=0}^{\infty} 2^k = 1 + 2 + 4 + 8 + 16 + \ldots = -1 \qquad !!$$

Repeating the algorithm for the nonresonant quasi periodic motions a graphical representation for the functions $\mathbf{a}, \mathbf{b}$ is easily found.

Equations of motion are

$$(\boldsymbol{\omega}_0 \cdot \boldsymbol{\partial}_{\boldsymbol{\psi}})^2 \mathbf{h}(\boldsymbol{\psi}) = -\varepsilon \boldsymbol{\partial} f(\boldsymbol{\psi} + \mathbf{a}(\boldsymbol{\psi}), \boldsymbol{\beta}_0 + \mathbf{b}(\boldsymbol{\psi}))$$

with $\mathbf{h}(\boldsymbol{\psi}) = (\mathbf{a}(\boldsymbol{\psi}), \mathbf{b}(\boldsymbol{\psi}))$

Resonant motions represented as power series

$$\sum_{k\geq 1} \varepsilon^k(\mathbf{a}^{(k)}(\boldsymbol{\psi}), \mathbf{b}^{(k)}(\boldsymbol{\psi}))$$

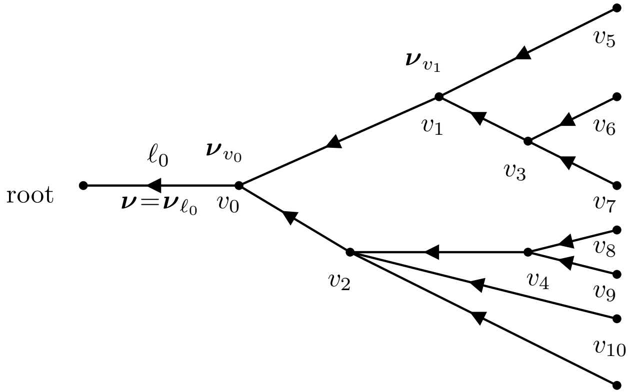
To order k the equations of motion become

$$\begin{aligned}(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2\,\mathbf{a}^{(k)}_{\boldsymbol{\nu}} &= [\partial_{\boldsymbol{\alpha}} f]^{(k-1)}_{\boldsymbol{\nu}}\,,\\ (\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2\,\mathbf{b}^{(k)}_{\boldsymbol{\nu}} &= [\partial_{\boldsymbol{\beta}} f]^{(k-1)}_{\boldsymbol{\nu}}\,,\end{aligned}$$

and $[\partial_{\boldsymbol{\alpha}} f]^{(k-1)}_{\boldsymbol{\nu}}$ is

$$\sum_{p\geq 0}\sum_{q\geq 0}\frac{1}{p!}\frac{1}{q!}\sum\nolimits^{*}(i\boldsymbol{\nu}_0)^{p+1}\,\partial^q_{\boldsymbol{\beta}} f_{\boldsymbol{\nu}_0}(\boldsymbol{\beta}_0)\Big(\prod_{j=1}^{p}\mathbf{a}^{(k_j)}_{\boldsymbol{\nu}_j}\Big)\Big(\prod_{j=p+1}^{p+q}\mathbf{b}^{(k_j)}_{\boldsymbol{\nu}_j}\Big),$$

$0<k_j<k\ \forall\, j=1,\ldots,p+q$; $*\longleftrightarrow 1+\sum_{j=1}^{p+q}k_j=k$, $\boldsymbol{\nu}_0+\sum_{j=1}^{p+q}\boldsymbol{\nu}_j=\boldsymbol{\nu}$



A tree θ with 12 nodes; one has $s_{v_0}=2, s_{v_1}=2, s_{v_2}=3, s_{v_3}=2, s_{v_4}=2$.

The rules for the graphical representation will be a bit different to account for the two types of actions-angles. Changes marked in red

There will be two kinds of vertices v: *nodes* and *leaves*. Leaves can only be endpoints, *i.e.* no lines entering them, nodes can be endpoints or internal vertices.

Lines exiting from leaves play a very different role with respect to the lines exiting from the nodes.

v_0 will be the last (*i.e.* leftmost) node, ℓ_0 the root line; v' denotes the node following v (a different convention with respect to the earlier discussion). $v'_0 = r$ but r will not be considered a node.

$V(\theta) =$ nodes, $L(\theta) =$ leaves and $\Lambda(\theta) =$ lines.

Any $\ell_v \in \theta$ is root to subtree $\theta_{\ell_v} \subset \theta$.

With each node v associate a *mode label* $\boldsymbol{\nu}_v \in \mathbb{Z}^r$, and to each leaf v a *leaf label* $\kappa_v \in \mathbb{N}$. The *order* of the tree θ is

$$k = |V(\theta)| + \sum_{v \in L(\theta)} \kappa_v$$

With a line $\ell = (v'v)$ exiting a node v associate labels $\gamma'_\ell, \gamma_\ell$, assuming the symbolic values α or β and imagined affixed close to v', resp., v and a *momentum label* $\boldsymbol{\nu}_\ell \in \mathbb{Z}^r$, as

$$\boldsymbol{\nu}_\ell \equiv \boldsymbol{\nu}_{\ell_v} = \sum_{\substack{w \in V(\theta) \\ w \preceq v}} \boldsymbol{\nu}_w,$$

while with a line ℓ exiting from a leaf v we associate only one label $\gamma_\ell = \beta$.

On node v labels are also associated: *branching labels* p_v, q_v, denoting how many α or, resp., β-labeled lines enter v, and a label δ_v, as

$$\delta_v = \begin{cases} 1, \, if \;\; \gamma_{\ell_v} = \beta, \\ 0, \, if \;\; \gamma_{\ell_v} = \alpha. \end{cases}$$

Then with each node v associate *node factor*

$$F_v = \frac{1}{p_v!}\frac{1}{q_v!}\Big(i\boldsymbol{\nu}_v\Big)^{p_v+(1-\delta_v)}\partial_{\boldsymbol{\beta}}^{q_v+\delta_v} f_{\boldsymbol{\nu}_v}(\boldsymbol{\beta}_0),$$

a tensor of rank $p_v + q_v + 1$.
With each leaf v we associate a *leaf factor*

$$L_v = \mathbf{b}_{\mathbf{0}}^{(\kappa_v)},$$

a tensor of rank 1 (a vector) to be defined.

To line ℓ exiting from node v associate a *propagator*

$$G_\ell \stackrel{def}{=} \delta_{\gamma_\ell,\gamma_{\ell'}} \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)^2},$$

a (diagonal) $r \times r$ matrix

No small divisor is associated to lines ℓ exiting leaves; let

$$G_\ell \stackrel{def}{=} \delta_{\gamma_\ell,\gamma_{\ell'}}\,\delta_{\gamma_{\ell'},\beta}$$

Define the value and the reduced value as

$$\mathrm{Val}\,(\theta) = \Big(\prod_{v\in V(\theta)} F_v\Big)\Big(\prod_{v\in L(\theta)} L_v\Big)\Big(\prod_{\ell\in\Lambda(\theta)} G_\ell\Big),$$

$$\mathrm{Val}\,'(\theta) = \Big(\prod_{v\in V(\theta)} F_v\Big)\Big(\prod_{v\in L(\theta)} L_v\Big)\Big(\prod_{\ell\in\Lambda(\theta)\setminus\ell_0} G_\ell\Big),$$

where $\ell_0 =$ root line. Formally, $\mathbf{a}_{\mathbf{0}}^{(k)} = \mathbf{0}$ and for $\boldsymbol{\nu} \neq \mathbf{0}$,

$$\mathbf{a}_{\boldsymbol{\nu}}^{(k)} = \sum_{\theta\in\Theta_{k,\boldsymbol{\nu},\alpha}} \mathrm{Val}\,(\theta), \qquad \mathbf{b}_{\boldsymbol{\nu}}^{(k)} = \sum_{\theta\in\Theta_{k,\boldsymbol{\nu},\beta}} \mathrm{Val}\,(\theta),$$

$$\mathbf{b}_{\mathbf{0}}^{(k)} = -\Big[\partial_{\boldsymbol{\beta}}^2 f_{\mathbf{0}}(\boldsymbol{\beta}_0)\Big]^{-1} \sum_{\theta\in\Theta^*_{k+1,\mathbf{0},\beta}} \mathrm{Val}\,'(\theta),$$

where $*$ means that the tree whose reduced value is given by $\partial_{\boldsymbol{\beta}}^2 f_{\mathbf{0}}(\boldsymbol{\beta}_0)\,\mathbf{b}_{\mathbf{0}}^{(k)}$ has to be discarded from the set $\Theta_{k+1,\mathbf{0},\beta}$.

If $\partial_{\boldsymbol{\beta}} f_{\mathbf{0}}(\boldsymbol{\beta}_0) = \mathbf{0}, \quad \det \partial^2_{\boldsymbol{\beta}} f_{\mathbf{0}}(\boldsymbol{\beta}_0) \neq 0$ then are finite for all $\boldsymbol{\nu} \in \mathbb{Z}^p \setminus \{\mathbf{0}\}$ to all orders k.

Summarizing: θ with k lines (and **without** nodes with **0** harmonic and just one entering line carrying a **0** current) we define its **value**

$$\mathrm{Val}\,(\theta) = \frac{\varepsilon^k}{k!}\Big(\prod_{v\in V(\theta)} F_v\Big)\Big(\prod_{\ell\in\Lambda(\theta)} G_\ell\Big),$$

$$F_v = \prod_j \partial_{\gamma_j} f_{\boldsymbol{\nu}_v}(\boldsymbol{\beta}_0),$$

$$G_\ell \equiv \delta_{\gamma_\ell,\gamma'_\ell} \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)^2}, \qquad \text{if } \boldsymbol{\nu}_\ell \neq \mathbf{0},$$

$$G_\ell \equiv -\varepsilon^{-1}\,(\partial^2_{\boldsymbol{\beta}} f_{\mathbf{0}}(\boldsymbol{\beta}_0))^{-1}_{\gamma_\ell,\gamma'_\ell}, \qquad \text{if } \boldsymbol{\nu}_\ell = \mathbf{0},\ \text{and } \gamma_\ell,\gamma'_\ell > r$$

$$G_\ell \equiv 0, \qquad \text{if } \boldsymbol{\nu}_\ell = \mathbf{0},\ \text{and } \gamma_\ell \text{ with } \gamma'_\ell \le r$$

hence division by 0 is forbidden: (*Poincaré*); .

If $\Theta^o_{q,\boldsymbol{\nu},\gamma} = \theta$'s with degree q and no divsion by 0 $\Rightarrow$ **Lindstedt series**

$$\varepsilon^q h^{(q)}_{\boldsymbol{\nu},\gamma} = \sum_{\theta\in\Theta^o_{q,\boldsymbol{\nu},\gamma}} \mathrm{Val}\,(\theta)$$

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